Mathematics. — *"On an arithmetical function connected with the decomposition of the positive integers into prime factors."* I. By J. G. VAN DER CORPUT. (Communicated by Prof. J. C. KLUYVER).

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Let u be any arbitrary integer > 1 and resolve u into prime factors; let e_u represent the smallest exponent of these factors and let a_u indicate how many times e_u occurs in the series of this exponents. Moreover we take $e_1 = 0$ and v_u represents the greatest divisor of u, for which $e_{v_u} > m$, m being any arbitrary positive integer. The object of this paper is to deduce a formula obtaining two general arithmetical functions F and f, satisfying four relations, n representing any positive integer, viz.

1st. for $e_u < m$ and also for $e_u = m,\ a_u > n$

$$F(u) = 0;$$

2nd. if $e_u \leq m$,

$$f(u) = 0;$$

3rd. for $e_u = m,\ a_u = n$,

$$F(u) = f(v_u);$$

4th.

$$F(u) = O(v_u^{\mu}),$$

μ having a constant value $< \dfrac{1}{m(m+1)}$.

The integers m and n are called the parameters of the function F and f the function corresponding to F.

This article, now, is intended to demonstrate the formula

$$\left.\begin{aligned} \sum_{\substack{u=2 \\ u \equiv l}}^{x} F(u) &= \frac{a x^{\frac{1}{m}}}{\log x} + O\left(\frac{x^{\frac{1}{m}}}{(\log x)^2}\right) \qquad \text{for } n = 1, \\ &= \frac{a x^{\frac{1}{m}} (\log\log x)^{n-1}}{\log x} + O\left(\frac{x^{\frac{1}{m}} (\log\log x)^{n-2}}{\log x}\right) \end{aligned}\right\} \quad (1)$$

for any arbitrary integral positive value of n and this proof will be given in § 2 for $n = 1$, in § 3 for the other case. The modulus of the congruences, for which this modulus has not been mentioned, is in this paper the arbitrary positive integer k, x represents a number > 1, l an integer, prime to k, a has a constant value, viz.

$$a = \frac{bm}{h.(n-1)!} \sum_{\substack{u=1 \\ uz^m \equiv l}}^{\infty} \frac{f(u)}{u^{\frac{1}{m}}};$$

h is the number of positive integers $\leq k$, prime to k, b is the number of incongruent roots z of the congruence

$$z^m \equiv 1$$

and the sum

$$\sum_{\substack{u=1 \\ uz^m \equiv l}}^{\infty} \frac{f(u)}{u^{\frac{1}{m}}}$$

is extended over all the positive integers u, for which the congruence

$$uz^m \equiv l$$

has roots in z.

§ 2. **Lemma.**

$$\sum_{v=1}^{\sqrt{x}} \frac{f(v)\log v}{v^{\frac{1}{m}}} = O(1),$$

$$\sum_{v=1}^{x} f(v) = O\left(\frac{x^{\frac{1}{m}}}{(\log x)^2}\right)$$

and

$$\sum_{v=1}^{\sqrt{x}} \frac{f(v)}{v^{\frac{1}{m}}} = \sum_{v=1}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{1}{\log x}\right).$$

Proof. From the relations satisfied by the functions F and f, it follows, u being an integer, for which v_u has the value v, that

$$f(v) = f(v_u) = F(u) = O(v_u^{\mu}) = O(v^{\mu})$$

and for $\frac{1}{m} > s > \frac{1}{m+1} + \mu$ the left member of the identity

$$\prod_p \left\{1 + \frac{1}{p^{(m+1)(s-\mu)}} + \frac{1}{p^{(m+2)(s-\mu)}} + \dots\right\} = 1 + \sum_{\substack{v=2 \\ e_v > m}}^{\infty} \frac{1}{v^{(s-\mu)}}$$

is a convergent product and consequently the right member a convergent sum, therefore

$$g(x) = \sum_{v=1}^{x} \frac{f(v)}{v^s} = f(1) + \sum_{\substack{v=2 \\ e_v > m}}^{x} \frac{O(v^{\mu})}{v^s}$$

$$= O(1) + O \,.\, \sum_{\substack{v=2 \\ e_v > m}}^{\infty} \frac{1}{v^{s-\mu}} = O(1).$$

Hence

$$\sum_{v=1}^{\sqrt{x}} \frac{f(v)\log v}{v^{\frac{1}{m}}} = \sum_{\substack{v=1\\ e_v > m}}^{\sqrt{x}} O\left(\frac{1}{v^{s-\mu}}\right)$$

$$= O(1),$$

$$\sum_{v=1}^{x} f(v) = \sum_{v=1}^{x} v^s \{g(v) - g(v-1)\}$$

$$= [x]^s g[x] - \sum_{v=1}^{x-1} g(v)\{(v+1)^s - v^s\}$$

$$= O(x^s) \,.\, O(1) - \sum_{v=1}^{x-1} O(1)\{(v+1)^s - v^s\}$$

$$= O(x^s) + O \,.\, \sum_{v=1}^{x-1}\{(v+1)^s - v^s\}$$

$$= O(x^s) + O\{[x]^s - 1\}$$

$$= O(x^s)$$

$$= O\left(\frac{x^{\frac{1}{m}}}{(\log x)^3}\right)$$

and

$$\sum_{v=x+1}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} = \sum_{v=x+1}^{\infty} \frac{g(v) - g(v-1)}{v^{\frac{1}{m}-s}}$$

$$= -\frac{g[x]}{[x+1]^{\frac{1}{m}-s}} + \sum_{v=x+1}^{\infty} g(v)\left\{\frac{1}{v^{\frac{1}{m}-s}} - \frac{1}{(v+1)^{\frac{1}{m}-s}}\right\}$$

$$= O\left(\frac{1}{x^{\frac{1}{m}-s}}\right) + \sum_{v=x+1}^{\infty} O(1) \,.\, \left\{\frac{1}{v^{\frac{1}{m}-s}} - \frac{1}{(v+1)^{\frac{1}{m}-s}}\right\}$$

$$= O\left(\frac{1}{x^{\frac{1}{m}-s}}\right) + O \sum_{v=x+1}^{\infty}\left\{\frac{1}{v^{\frac{1}{m}-s}} - \frac{1}{(v+1)^{\frac{1}{m}-s}}\right\}$$

$$= O\left(\frac{1}{x^{\frac{1}{m}-s}}\right) + O\left(\frac{1}{[x+1]^{\frac{1}{m}-s}}\right)$$

$$= O\left(\frac{1}{x^{\frac{1}{m}-s}}\right)$$

$$= O\left(\frac{1}{\log x^3}\right),$$

consequently

$$\sum_{v=1}^{\sqrt{x}} \frac{f(v)}{\frac{1}{v^m}} = \sum_{v=1}^{\infty} \frac{f(v)}{\frac{1}{v^m}} - \sum_{v=\sqrt{x}+1}^{\infty} \frac{f(v)}{\frac{1}{v^m}}$$

$$= \sum_{v=1}^{\infty} \frac{f(v)}{\frac{1}{v^m}} + O\left(\frac{1}{\log x}\right).$$

Identity. If ψ_1 and ψ_2 represent two arbitrary arithmetical functions, the sum

$$\sum_{d_1 d_2 \leq x} \psi_1(d_1)\, \psi_2(d_2),$$

extended over all the positive integers d_1 and d_2, of which the product is not greater than x, is equal to

$$T_1 + T_2 - T_3 T_4$$

where

$$T_1 = \sum_{d_1=1}^{\sqrt{x}} \psi_1(d_1) \sum_{d_2=1}^{\frac{x}{d_1}} \psi_2(d_2),$$

$$T_2 = \sum_{d_2=1}^{\sqrt{x}} \psi_2(d_2) \sum_{d_1=1}^{\frac{x}{d_2}} \psi_1(d_1),$$

$$T_3 = \sum_{d_1=1}^{\sqrt{x}} \psi_1(d_1)$$

and

$$T_4 = \sum_{d_2=1}^{\sqrt{x}} \psi_2(d_2).$$

Proof. A term $\psi_1(d_1)\, \psi_2(d_2)$, occurring in the sum in question, appears in the formula $T_1 + T_2 - T_3 T_4$

for $d_1 \leq \sqrt{x}$ $\quad d_2 \leq \sqrt{x}$ $\quad$ exactly $1 + 1 - 1 = 1$ times,
for $d_1 \leq \sqrt{x}$ $\quad d_2 > \sqrt{x}$ $\quad$ exactly $1 + 0 - 0 = 1$ times,
for $d_1 > \sqrt{x}$ $\quad d_2 \leq \sqrt{x}$ $\quad$ exactly $0 + 1 - 0 = 1$ times.

Lemma. If we take $n = 1$, the sum

$$\sum_{\substack{p^m v \leq x \\ p^m v \equiv l}} f(v),$$

extended over all the positive integers v and all the prime numbers p, for which the relations

$$p^m v \leq x \quad \text{and} \quad p^m v \equiv l$$

exist, is equal to

$$\frac{a x^{\frac{1}{m}}}{\log x} + O\left(\frac{x^{\frac{1}{m}}}{(\log x)^2}\right),$$

where

$$a = \frac{bm}{h} \sum_{\substack{v=1 \\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}}.$$

Proof. Let l_1 and l_2 be two integers, prime to k; if the congruence

$$z^m \equiv l_1$$

has no roots in z, we have

$$\sum_{\substack{p^m v \leq x \\ p^m \equiv l_1 \\ v \equiv l_2}} f(v) = 0,$$

since it is then impossible to find a prime number p, satisfying the congruence

$$p^m \equiv l_1.$$

Let us now, however, consider the case, that the congrence does possess roots and consequently has b incongruent roots $z_1, z_2, \ldots, z_b$. The preceding identity gives

$$\sum_{\substack{p^m v \leq x \\ p^m \equiv l_1 \\ v \equiv l_2}} f(v) = T_1 + T_2 - T_3 T_4,$$

where

$$T_1 = \sum_{\substack{v=1 \\ v \equiv l_2}}^{\sqrt{x}} f(v) \,.\, \sum_{\substack{p^m \leq \frac{x}{v} \\ p^m \equiv l_1}} 1,$$

$$T_2 = \sum_{\substack{p^m \leq \sqrt{x} \\ p^m \equiv l_1}} \sum_{\substack{v=1 \\ v \equiv l_2}}^{\frac{x}{p^m}} f(v),$$

$$T_3 = \sum_{\substack{v=1 \\ v \equiv l_2}}^{\sqrt{x}} f(v)$$

and

$$T_4 = \sum_{\substack{p^m \leq \sqrt{x} \\ p^m \equiv l_1}} 1.$$

From the preceding lemma ensues

$$T_4 = O\left\{\frac{(\sqrt{x})^{\frac{1}{m}}}{(\log \sqrt{x})^2}\right\} = O\left(\frac{x^{\frac{1}{2m}}}{(\log x)^2}\right)$$

and for $p^m \leq \sqrt{x}$

$$\sum_{\substack{r=1 \\ v\equiv l_2}}^{\frac{x}{p^m}} f(v) = O\left\{\frac{\left(\frac{x}{p^m}\right)^{\frac{1}{m}}}{\left(\log\frac{x}{p^m}\right)^3}\right\}$$

$$= O\left\{\frac{x^{\frac{1}{m}}}{p\left(\log\frac{x}{\sqrt{x}}\right)^3}\right\}$$

$$= O\left(\frac{x^{\frac{1}{m}}}{p\,(\log x)^3}\right),$$

hence

$$T_2 = \sum_{\substack{p^m \leq \sqrt{x} \\ p^m \equiv l_1}} O\left(\frac{x^{\frac{1}{m}}}{p\,(\log x)^3}\right)$$

$$= O\left(\frac{x^{\frac{1}{m}}}{(\log x)^3}\right) \sum_{\substack{p^m \leq \sqrt{x} \\ p^m \equiv l_1}} \frac{1}{p}$$

$$= O\left(\frac{x^{\frac{1}{m}}}{(\log x)^3}\right) . \, O \sum_{n=1}^{x^{\frac{1}{2m}}} \frac{1}{n}$$

$$= O\left(\frac{x^{\frac{1}{m}}}{(\log x)^3}\right) . \, O\left(\log x^{\frac{1}{2m}}\right)$$

$$= O\left(\frac{x^{\frac{1}{m}}}{(\log x)^2}\right).$$

From the inequalities

$$0 \leq T_4 \leq \sum_{\substack{p^m \leq \sqrt{x} \\ p \equiv l_1}} 1 \leq \sum_{n=1}^{x^{\frac{1}{2m}}} 1 \leq x^{\frac{1}{2m}}$$

ensues

$$T_4 = O\left(x^{\frac{m}{1}}\right)$$

and now only the term T_1 is to be considered.

From the well-known proposition [1]) that the number of prime numbers $\leq x$ and congruent to l with regard to the modulus k, is equal to

$$\sum_{\substack{p \leq x \\ p \equiv l}} 1 = \frac{1}{h} \cdot \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

ensues

$$\sum_{\substack{p^m \leq \frac{x}{v} \\ p^m \equiv l_1}} 1 = \sum_{\substack{p \leq \left(\frac{x}{v}\right)^{\frac{1}{m}} \\ p \equiv z_1, z_2, \ldots, z_b}} 1$$

$$= \sum_{b \geq \varrho \geq 1} \sum_{\substack{p \leq \left(\frac{x}{v}\right)^{\frac{1}{m}} \\ p \equiv z_\varrho}} 1$$

$$= \sum_{\varrho=1}^{b} \left\{ \frac{1}{h} \cdot \frac{\left(\frac{x}{v}\right)^{\frac{1}{m}}}{\log \left(\frac{x}{v}\right)^{\frac{1}{m}}} + O \cdot \frac{\left(\frac{x}{v}\right)^{\frac{1}{m}}}{\left\{\log \left(\frac{x}{v}\right)^{\frac{1}{m}}\right\}^2} \right\}$$

$$= \frac{bm\, x^{\frac{1}{m}}}{h v^{\frac{1}{m}} \log \frac{x}{v}} + O\left\{ \frac{x^{\frac{1}{m}}}{v^{\frac{1}{m}} \left(\log \frac{x}{v}\right)^2} \right\}.$$

For $v \leq \sqrt{x}$ we have

$$\frac{1}{\log \frac{x}{v}} = O\left(\frac{1}{\log \frac{x}{\sqrt{x}}}\right) = O\left(\frac{1}{\log x}\right),$$

$$\frac{1}{\log \frac{x}{v}} = \frac{1}{\log x} + \frac{\log v}{\log x \cdot \log \frac{x}{v}}$$

$$= \frac{1}{\log x} + O\left\{\frac{\log v}{(\log x)^2}\right\},$$

therefore

[1]) E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, I. p. 468.

$$T_1 = \sum_{\substack{v=1 \\ v\equiv l_2}}^{\sqrt{x}} f(v) \left\{ \frac{bm\, x^{\frac{1}{m}}}{h v^{\frac{1}{m}} \log \frac{x}{v}} + O \cdot \frac{x^{\frac{1}{m}}}{v^{\frac{1}{m}} \left(\log \frac{x}{v} \right)^2} \right\}$$

$$= \sum_{\substack{v=1 \\ v\equiv l_2}}^{\sqrt{x}} f(v) \left\{ \frac{bm\, x^{\frac{1}{m}}}{h v^{\frac{1}{m}} \log x} + O \cdot \frac{x^{\frac{1}{m}} \log v}{v^{\frac{1}{m}} (\log x)^2} \right\}$$

$$= \frac{bm\, x^{\frac{1}{m}}}{h \log x} \sum_{\substack{v=1 \\ v\equiv l_2}}^{\sqrt{x}} \frac{f(v)}{v^{\frac{1}{m}}} + O \cdot \frac{x^{\frac{1}{m}}}{(\log x)^2} \sum_{\substack{v=1 \\ v\equiv l_2}}^{\sqrt{x}} \frac{|f(v)| \log v}{v^{\frac{1}{m}}}$$

and according to the preceding lemma this is equal to

$$\frac{bm\, x^{\frac{1}{m}}}{h \log x} \left\{ \sum_{\substack{v=1 \\ v\equiv l_2}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{1}{\log x}\right) \right\} + O \left\{ \frac{x^{\frac{1}{m}}}{(\log x)^2} \right\}$$

$$= \frac{bm\, x^{\frac{1}{m}}}{h \log x} \sum_{\substack{v=1 \\ v\equiv l_2}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O \left\{ \frac{x^{\frac{1}{m}}}{(\log x)^2} \right\}.$$

By substituting the values found for T_1, T_2, T_3 and T_4, we find the relation

$$\sum_{\substack{p^m v \leq x \\ p^m \equiv l_1 \\ v \equiv l_2}} f(v) = T_1 + T_2 - T_3\, T_4$$

$$= \frac{bm\, x^{\frac{1}{m}}}{h \log x} \sum_{\substack{v=1 \\ v\equiv l_2}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O \left\{ \frac{x^{\frac{1}{m}}}{(\log x)^2} \right\},$$

if the condition that the congruence

$$z^m \equiv l_1$$

has roots, is satisfied.

Write down a series, composed of h integers prime to k and not containing two numbers, which are congruent to each other, with regard to the modulus k; give to l_1 successively each value of this series, satisfying the condition, that the congruence

$$z \equiv l_1$$

has roots and determine for every value of l_1 a number l_2 by the congruence

$$l_1 l_2 \equiv l;$$

the relations thus obtained, added give in the left member

$$\sum_{\substack{p^m v \leq x \\ p^m v \equiv l}} f(v)$$

and in the right member

$$\frac{bmx^{\frac{1}{m}}}{h \log x} \sum_{v=1}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left\{\frac{x^{\frac{1}{m}}}{(\log x)^2}\right\},$$

v assuming all the positive integral values for which the congruences

$$v \equiv l_2 \qquad z^m \equiv l_1 \qquad l_1 l_2 \equiv l$$

are possible, i.e. for which the congruence

$$z^m l \equiv v$$

has roots and we conclude

$$\sum_{\substack{p^m v \leq x \\ p^m v \equiv l}} f(v) = \frac{ax^{\frac{1}{m}}}{\log x} + O\left\{\frac{x^{\frac{1}{m}}}{(\log x)^2}\right\},$$

where

$$a = \frac{bm}{h} \sum_{v=1}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}}.$$

$$z^m l \equiv v.$$

We have got on far enough now to proceed to proving formula (2) for $n = 1$; we observe that for $n = 1$

$$F(u) - \sum_{p^m | u} f\left(\frac{u}{p^m}\right)$$

is a finite function of u, which equals nothing for $e_u \leq m$ and which is equal to $O(u^{\mu_1})$ for $e_u > m$, μ_1 representing a constant number $< \frac{1}{m(m+1)}$.

In order to prove this, we distinguish four cases:

1. $e_u < m$;

if $\frac{u}{p^m}$ is resolved into prime factors, the smallest exponent of these factors is in this case smaller than m, hence

$$F(u) = 0 \qquad \text{and} \qquad f\left(\frac{u}{p^m}\right) = 0,$$

so that the formula considered is equal to nothing.

2. $e_u = m, \qquad a_u > 1$;

if $\frac{u}{p^m} = w$ is resolved into prime factors, the series of the exponents of these prime factors obtains at least one exponent $= m$, consequently

$$e_w \leq m,$$

hence

$$f\left(\frac{u}{p_m}\right) = 0,$$

and n having the value 1,

$$e_u = m, \qquad a_u > n,$$

hence

$$F(u) = 0,$$

consequently

$$F(u) - \sum_{p^m/u} f\left(\frac{u}{p^m}\right) = 0.$$

3. $e_u = m, \qquad a_u = 1;$

consequently

$$u = p_1^m v \qquad\qquad e_v > m,$$

v being not divisible by the prime number p_1. In this case we have

$$F(u) = f(v).$$

As $\frac{u}{p^m}$ contains at least one prime factor, of whom the exponent is equal to m (viz. the prime factor p_1) except for $p = p_1$, we have

$$f\left(\frac{u}{p^m}\right) = 0, \text{ for } p \neq p_1,$$

$$= f(v), \text{ for } p = p_1,$$

consequently

$$F(u) - \sum_{p^m/u} f\left(\frac{u}{p^m}\right) = f(v) - f(v) = 0.$$

4. $e_u > m;$

Suppose $\mu < \mu_1 < \frac{1}{m(m+1)}$ and let

$$u = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_\sigma^{\alpha_\sigma},$$

be resolved into prime factors; hence

$$u \geq 2 \,.\, 2 \ldots . \, 2 = 2^\sigma,$$

$$\sigma \leq \frac{\log u}{\log 2} = O(\log u)$$

$$\sum_{p^m/u} 1 = \sigma = O(\log u)$$

The conditions

$$F(u) = O(v_u{}^{\mu}), \quad \text{and} \quad f(v) = O(v^{\mu}),$$

mentioned in § 1 and at the beginning of § 2 give the relations

$$F(u) = O(u^{\mu}) = O(u^{\mu_1})$$

and

$$\sum_{p^m/u} f\left(\frac{u}{p^m}\right) = \sum_{p^m/u} O\left\{\left(\frac{u}{p^m}\right)^{\mu}\right\}$$

$$= O(u^{\mu}) \sum_{p^m/u} 1$$

$$= O(u^{\mu}) \,.\, O(\log u)$$

$$= O(u^{\mu_1})\,;$$

hence it follows that the function considered is in this case equal to

$$O(u^{\mu_1}) - O(u^{\mu_1}) = O(u^{\mu_1}).$$

According to the first lemma we have

$$\sum_{\substack{u=2\\u\equiv l}}^{x} \left\{F(u) - \sum_{p^m/u} f\left(\frac{u}{p^m}\right)\right\} = O\left(\frac{x^{\frac{1}{m}}}{(\log x)^2}\right)$$

and consequently

$$\sum_{\substack{u=2\\u\equiv l}}^{x} F(u) = \sum_{\substack{u=2\\u\equiv l}}^{x} \sum_{p^m/u} f\left(\frac{u}{p^m}\right) + O\left(\frac{x^{\frac{1}{m}}}{(\log x)^2}\right)$$

$$= \sum_{\substack{p^m v \leq x\\p^m v\equiv l}} f(v) + O\left(\frac{x^{\frac{1}{m}}}{(\log x)^2}\right)$$

and according to the last lemma this may be modified to the formula sought

$$\sum_{\substack{u=2\\u\equiv l}}^{x} F(u) = \frac{a x^{\frac{1}{m}}}{\log x} + O\left(\frac{x^{\frac{1}{m}}}{(\log x)^2}\right), \ldots \text{ for } n = 1.$$

§ 3. By starting from formula (1) by which the mean value of the function $F(u)$ has been given in the interval from 1 to x (the limits included), it is possible, as is known, to determine in an elementary way the mean value in the same interval of a number of other functions, connected with the function F; this we shall however only elaborate for some cases.

Lemma. From (1) ensues

$$\sum_{\substack{u=2\\u\equiv l}}^{x} \frac{F(u)}{u^{\frac{1}{m}}} = \frac{a}{mn} (\log\log x)^n + O(\log\log x)^{n-1} \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\sum_{\substack{u=2\\u\equiv l}}^{x} \frac{F(u)\log u}{u^{\frac{1}{m}}} = O(\log x \,.\, (\log\log x)^{n-1}).$$

Proof. Substituting

$$\log\log x = x_2,$$
$$\log\log u = u_2$$

and

$$g(x) = \sum_{\substack{u=2\\u\equiv l}}^{x} F(u) = \frac{a\, x^{\frac{1}{m}} x_2^{\,n-1}}{\log x} + O\left(\frac{x^{\frac{1}{m}} x_2^{\,n-2}}{\log x}\right)$$

we have

$$g_1(x) = \sum_{\substack{u=2\\u\equiv l}}^{x} \frac{F(u)}{u^{\frac{1}{m}}}$$

$$= \sum_{u=2}^{x} \frac{g(u)-g(u-1)}{u^{\frac{1}{m}}}$$

$$= \frac{g[x]}{[x]^{\frac{1}{m}}} + \sum_{u=2}^{x-1} g(u)\left\{\frac{1}{u^{\frac{1}{m}}} - \frac{1}{(u+1)^{\frac{1}{m}}}\right\}$$

$$= O\left(\frac{x_2^{\,n-1}}{\log x}\right) + \sum_{u=2}^{x-1}\left\{\frac{a u^{\frac{1}{m}} u_2^{\,n-1}}{\log u} + O\left(\frac{u^{\frac{1}{m}}\; u_2^{\,n-2}}{\log u}\right)\right\}\left\{\frac{1}{m u^{\frac{1}{m}+1}} + O\left(\frac{1}{u^{\frac{1}{m}+2}}\right)\right\}$$

$$= O(x_2^{\,n-1}) + \frac{a}{m}\sum_{u=2}^{x-1} \frac{u_2^{\,n-1}}{u\log u} + O\sum_{u=2}^{x-1}\frac{u_2^{\,n-2}}{u\log u}$$

$$= O(x_2^{\,n-1}) + \frac{a}{m}\left\{\frac{x_2^{\,n}}{n} + O(x_2^{\,n-1})\right\} + O(x_2^{\,n-1})$$

$$= \frac{a x_2}{mn} + O(x_2^{\,n-1})$$

and

$$\sum_{\substack{u=2\\u\equiv l}}^{x} \frac{F(v)\log u}{u^{\frac{1}{m}}} = \sum_{u=2}^{x}\{g_1(u) - g_1(u-1)\}\log u$$

$$= g_1[x]\log[x] - \sum_{u=2}^{x-1} g_1(u)\log\left(1+\frac{1}{u}\right)$$

$$= \frac{ax_2^n \log x}{mn} + O(x_2^{n-1} \log x)$$

$$- \sum_{u=2}^{x-1} \left\{ \frac{au_2^n}{mn} + O(u_2^{n-1}) \right\} \left\{ \frac{1}{u} + O\left(\frac{1}{u^2}\right) \right\}$$

$$= \frac{ax_2^n \log x}{mn} + O(x_2^{n-1} \log x) - \frac{a}{mn} \sum_{u=2}^{x-1} \frac{u_2^n}{u} + O \sum_{u=2}^{x-1} \frac{u_2^{n-1}}{u}$$

$$= \frac{ax_2^n \log x}{mn} + O(x_2^{n-1} \log x)$$

$$- \frac{a}{mn} \{x_2^n \log x + O(x_2^{n-1} \log x)\} + O(x_2^{n-1} \log x)$$

$$= O(x_2^{n-1} \log x).$$

Lemma. Suppose that the function F_1 has m and n_1 as parameters and f_1 as corresponding function and that F_2 has m and n_2 as parameters and f_2 as corresponding function. If the formula (1) holds good for $n = n_1$ and for $n = n_2$, we have

$$\sum_{\substack{d_1 d_2 \leq x \\ d_1 d_2 \equiv l}} F_1(d_1) F_2(d_2) = \frac{bm(n_1+n_2) x^{\frac{1}{m}} x_2^{n_1+n_2-1}}{hn_1! n_2! \log x} \sum_{\substack{v=1 \\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{x^{\frac{1}{m}} x_2^{n_1+n_2-2}}{\log x} \right),$$

l being prime to k; in this relation $f(v)$ has been substituted for the formula

$$\sum_{d | v} f_1(d) f_2\left(\frac{v}{d}\right)$$

Proof. Let l_1 and l_2 be two integers, prime to k; it follows from the identity deduced in the preceding paragraph that

$$\sum_{\substack{d_1 d_2 \leq x \\ d_1 \equiv l_1 \\ d_2 \equiv l_2}} F_1(d_1) F_2(d_2) = T_1 + T_2 - T_3 T_4 , \quad . \quad . \quad . \quad . \quad (4)$$

where

$$T_1 = \sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\sqrt{x}} F_1(d_1) \sum_{\substack{d_2=1 \\ d_2 \equiv l_2}}^{\frac{x}{d_1}} F_2(d_2),$$

$$T_2 = \sum_{\substack{d_2=1 \\ d_2 \equiv l_2}}^{\sqrt{x}} F_2(d_2) \sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\frac{x}{d_2}} F_1(d_1),$$

$$T_3 = \sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\sqrt{x}} F_1(d_1)$$

and $$T_4 = \sum_{\substack{d_2=1 \\ d_2 \equiv l_2}}^{\sqrt{x}} F_2(d_2).$$

For $d_1 \leq \sqrt{x}$ we have

$$\log\log\frac{x}{d_1} = \log\log x + \log\left(1 - \frac{\log d_1}{\log x}\right) = x_2 + O(1),$$

if x_2 has been again substituted for $\log\log x$; consequently

$$\left(\log\log\frac{x}{d_1}\right)^{n_2-1} + O\left(\log\log\frac{x}{d_1}\right)^{n_2-2} = x_2^{n_2-1} + O(x_2^{n_2-2}),$$

$$\frac{1}{\log\frac{x}{d_1}} = \frac{1}{\log x} + \frac{\log d_1}{\log x \,.\, \log\frac{x}{d_1}} = \frac{1}{\log x} + O\left(\frac{\log d_1}{(\log x)^2}\right)$$

and

$$\frac{\left(\log\log\frac{x}{d_1}\right)^{n_2-1} + O\left(\log\log\frac{x}{d_1}\right)^{n_2-2}}{\log\frac{x}{d_1}} = \frac{x_2^{n_2-1}}{\log x} + O\left(\frac{x_2^{n_2-2}}{\log x}\right) + O\left(\frac{x_2^{n_2-1}\log d_1}{(\log x)^2}\right).$$

It has been assumed that formula (1) holds good for $n = n_2$, hence

$$\sum_{\substack{d_2=1 \\ d_2 \equiv l_2}}^{\frac{x}{d_1}} F_2(d_2) = \frac{a_2\left(\frac{x}{d_1}\right)^{\frac{1}{m}}\left\{\left(\log\log\frac{x}{d_1}\right)^{n_2-1} + O\left(\log\log\frac{x}{d_1}\right)^{n_2-2}\right\}}{\log\frac{x}{d_1}}$$

$$= \frac{a_2 x^{\frac{1}{m}}}{d_1^{\frac{1}{m}}}\left\{\frac{x_2^{n_2-1} + O(x_2^{n_2-2})}{\log x} + O\left(\frac{x_2^{n_2-1}\log d_1}{(\log x)^2}\right)\right\},$$

where a_2 has been substituted for

$$\frac{bm}{h\,(n_2-1)!} \sum_{\substack{v_2=1 \\ v_2 z^m \equiv l_2}}^{\infty} \frac{f_2(v_2)}{v_2^{\frac{1}{m}}}.$$

If this result is substituted for the value found for T_1, we find

$$T_1 = \frac{a_2 x^{\frac{1}{m}} x_2^{n_2-1}}{\log x} \sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\sqrt{x}} \frac{F_1(d_1)}{d_1^{\frac{1}{m}}} + O\left(\frac{x^{\frac{1}{m}} x_2^{n_2-2}}{\log x}\right) \sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\sqrt{x}} \frac{|F_1(d_1)|}{d_1^{\frac{1}{m}}} +$$

$$+ O\left(\frac{x^{\frac{1}{m}} x_2^{n_2-1}}{(\log x)^2}\right) \sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\sqrt{x}} \frac{|F_1(d_1)| \log d_1}{d_1^{\frac{1}{m}}}.$$

It has been understood that formula (1) holds good for $n = n_1$, consequently for the functions F_1 and $|F_1|$ and according to the first lemma of this paragraph we have

$$\sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\sqrt{x}} \frac{|F_1(d_1)|}{d_1^{\frac{1}{m}}} = O(x_2^{n_1}),$$

$$\sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\sqrt{x}} \frac{|F_1(d_1)| \log d_1}{d_1^{\frac{1}{m}}} = O(x_2^{n_1-1} \log x)$$

and

$$\sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\sqrt{x}} \frac{F_1(d_1)}{d_1^{\frac{1}{m}}} = \frac{a_1 x_2^{n_1}}{m n_1} + O(x_2^{n_1-1}),$$

where

$$a_1 = \frac{bm}{h(n_1-1)!} \sum_{\substack{v_1=1 \\ v_1 z^m \equiv l_1}}^{\infty} \frac{f_1(v_1)}{v_1^{\frac{1}{m}}}.$$

Hence

$$T_1 = \frac{a_1 a_2 x^{\frac{1}{m}} x_2^{n_1+n_2-1}}{m n_1 \log x} + O\left(\frac{x^{\frac{1}{m}} x_2^{n_1+n_2-2}}{\log x}\right)$$

$$= \frac{b^2 m n_2 x^{\frac{1}{m}} x_2^{n_1+n_2-1}}{h^2 n_1!\, n_2! \log x} \sum_{\substack{v_1=1 \\ v_1 z_1^m \equiv l_1}}^{\infty} \sum_{\substack{v_2=1 \\ v_2 z_2^m \equiv l_2}}^{\infty} \frac{f_1(v_1) f_2(v_2)}{(v_1 v_2)^{\frac{1}{m}}} + O\left(\frac{x^{\frac{1}{m}} {}_2^{n_1+n_2-2}}{\log x}\right).$$

The value of T_2 is found by interchanging n_1 and n_2 in this formula and as according to our supposition relation (1) holds good for $n = n_1$ and for $n = n_2$, we have

$$T_3 = \sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\sqrt{x}} F_1(d_1) = O\left(\frac{(\sqrt{x})^{\frac{1}{m}} (\log\log \sqrt{x})^{n_1-1}}{\log \sqrt{x}}\right)$$

$$= O\left(\frac{x^{\frac{1}{2m}} (\log\log x)^{n_1-1}}{\log x}\right)$$

and

$$T_4 = O\left(\frac{x^{\frac{1}{2m}} (\log\log x)^{n_2-1}}{\log x}\right).$$

By substituting these values for T, T_2, T_3 and T_4 in (4), we find the formula

$$\sum_{\substack{d_1 d_2 \leq x \\ d_1 \equiv l_1 \\ d_2 \equiv l_2}} F_1(d_1)\,F_2(d_2) = \frac{b^2 m(n_1+n_2) x^{\frac{1}{m}} x_2^{n_1+n_2-1}}{h^2 n_1!\, n_2!\, \log x} \sum_{\substack{v_1=1 \\ v_1 z_1^m \equiv l_1}}^{\infty} \sum_{\substack{v_2=1 \\ v_2 z_2^m \equiv l_2}}^{\infty} \frac{f_1(v_1)\,f_2(v_2)}{(v_1 v_2)^{\frac{1}{m}}} +$$

$$+\, O\left(\frac{x^{\frac{1}{m}} x_2^{n_1+n_2-2}}{\log x}\right).$$

Write again down a series consisting of h integers prime to k and not containing two numbers, which are congruent to each other with regard to the modulus k; give to l_1 successively each value of this series and determine for every value of l_1 a number l_2 by the congruence

$$l_1 l_2 \equiv l\,;$$

the h relations, thus obtained, added, give in the left member

$$\sum_{\substack{d_1 d_2 \leq x \\ d_1 d_2 \equiv l}} F_1(d_1)\,F_2(d_2)$$

and in the right member

$$\frac{b^2 m(n_1+n_2) x^{\frac{1}{m}} x_2^{n_1+n_2-1}}{h^2 n_1!\, n_2!\, \log x} \,.\, c + O\left(\frac{x^{\frac{1}{m}} x_2^{n_1+n_2-2}}{\log x}\right),$$

where

$$c = \sum_{l_1} \sum_{\substack{v_1=1 \\ v_1 z_1^m \equiv l_1}}^{\infty} \sum_{\substack{v_2=1 \\ v_2 z_2^m \equiv l_2}}^{\infty} \frac{f_1(v_1)\,f_2(v_2)}{(v_1 v_2)^{\frac{1}{m}}}$$

$$= \sum_{\substack{v=1 \\ v z^m \equiv l}}^{\infty} \frac{1}{v^{\frac{1}{m}}} \sum_{\substack{l_1 \\ v_1 z_1^m \equiv l_1}}' \sum_{v_1 v_2 = v} f_1(v_1)\,f_2(v_2).$$

For every value of v_1 exactly $\frac{h}{b}$ incongruent values for l_1 are to be found for which the congruence

$$v_1 z_1^m \equiv l_1$$

has roots, hence

$$\sum_{l_1} \sum_{\substack{v_1 v_2 = v \\ v_1 z_1^m \equiv l_1}} f_1(v_1) f_2(v_2) = \frac{h}{b} \cdot \sum_{v_1 v_2 = v} f_1(v_1) f_2(v_2)$$

$$= \frac{h}{b} \sum_{d \mid v} f_1(d)\, f_2\left(\frac{v}{d}\right)$$

$$= \frac{h}{b} f(v),$$

$$c = \sum_{\substack{v=1 \\ vz^m \equiv l}}^{\infty} \frac{1}{v^{\frac{1}{m}}} \sum_{l_1} \sum_{\substack{v_1 v_2 = v \\ v_2 z_1^m \equiv l_1}} f_1(v_1) f_2(v_2)$$

$$= \frac{h}{b} \sum_{\substack{v=1 \\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}}$$

and consequently

$$\sum_{\substack{d_1 d_2 \leq x \\ d_1 d_2 \equiv l}} F_1(d_1) F_2(d_2) = \frac{bm(n_1+n_2) x^{\frac{1}{m}} x_2^{n_1+n_2-1}}{hn_1! n_2! \log x} \sum_{\substack{v=1 \\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{x^{\frac{1}{m}} x_2^{n_1+n_2-2}}{\log x} \right),$$

which was to be proved.

Mathematics. — "*On an arithmetical function connected with the decomposition of the positive integers into prime factors.*" II. (*Continued and concluded.*) By J. G. VAN DER CORPUT. (Communicated by Prof. J. C. KLUYVER).

(Communicated in the meeting of June 24, 1916).

Lemma. [1]) The number of (positive integral) divisors of the positive integer v satisfies the relation

$$\sum_{d|v} 1 = O(v^{\mu}).$$

for every $\mu > 0$.

Proof. If $v \geq 2$ decomposed into prime factors be equal to

$$v = \prod_{p|v} p^{\alpha}$$

we have

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[1]) This proposition occurs for the first time in RUNGE: Ueber die auflösbaren Gleichungen von der Form $x^5 + ux + v = 0$ [Acta mathematica, Bd. VII (1885), pages 173—186], pages 181—183, with a proof similar to this one. This proof has been borrowed of E. LANDAU. Ueber die Anzahl der Gitterpunkte in gewissen Bereichen [Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse (1912), pages 687—771], page 716. In his "Handbuch der Lehre von der Verteilung der Primzahlen," I. p. 220, he gives the by far sharper relation:

If δ be positive, $\xi = \xi(\delta)$ fitly chosen and x an integer $\geq \xi$, we have

$$\sum_{d|x} 1 < 2^{\frac{(1+\delta)\log x}{\log \log x}}.$$