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$$c = \sum_{\substack{v=1 \\ vz^m \equiv l}}^{\infty} \frac{1}{v^{\frac{1}{m}}} \sum_{l_1} \sum_{\substack{v_1 v_2 = v \\ v_2 z_1^m \equiv l_1}} f_1(v_1) f_2(v_2)$$

$$= \frac{h}{b} \sum_{\substack{v=1 \\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}}$$

and consequently

$$\sum_{\substack{d_1 d_2 \leq x \\ d_1 d_2 \equiv l}} F_1(d_1) F_2(d_2) = \frac{bm(n_1+n_2) x^{\frac{1}{m}} x_2^{n_1+n_2-1}}{h n_1! n_2! \log x} \sum_{\substack{v=1 \\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{x^{\frac{1}{m}} x_2^{n_1+n_2-2}}{\log x}\right),$$

which was to be proved.

Mathematics. — "*On an arithmetical function connected with the decomposition of the positive integers into prime factors.*" II. (*Continued and concluded.*) By J. G. VAN DER CORPUT. (Communicated by Prof. J. C. KLUYVER).

(Communicated in the meeting of June 24, 1916).

Lemma. [1]) The number of (positive integral) divisors of the positive integer v satisfies the relation

$$\sum_{d|v} 1 = O(v^{\mu}).$$

for every $\mu > 0$.

Proof. If $v \geq 2$ decomposed into prime factors be equal to

$$v = \prod_{p|v} p^{\alpha}$$

we have

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[1]) This proposition occurs for the first time in RUNGE: Ueber die auflösbaren Gleichungen von der Form $x^5 + ux + v = 0$ [Acta mathematica, Bd. VII (1885), pages 173—186], pages 181—183, with a proof similar to this one. This proof has been borrowed of E. LANDAU. Ueber die Anzahl der Gitterpunkte in gewissen Bereichen [Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse (1912), pages 687—771], page 716. In his "Handbuch der Lehre von der Verteilung der Primzahlen," I. p. 220, he gives the by far sharper relation:

If δ be positive, $\xi = \xi(\delta)$ fitly chosen and x an integer $\geq \xi$, we have

$$\sum_{d|x} 1 < 2^{\frac{(1+\delta)\log x}{\log\log x}}.$$

$$\sum_{d|v} 1 = \Pi_{p|v} (\alpha + 1),$$

$$\frac{\sum_{d|v} 1}{v^{\mu}} = \Pi_{p|v} \frac{\alpha+1}{p^{\alpha\mu}}.$$

The quantity $\frac{\alpha+1}{p^{\alpha\mu}}$ is limited (μ taken fixed!) for an invariable value of p and variable $\alpha = 1, 2, \ldots$, since it is equal to nothing for $\alpha = \infty$; for any $p \geqq 2^{\frac{1}{\mu}}$ and any value $\alpha \geqq 1$ it is even $\leqq 1$, existing in that case the inequalities

$$\frac{\alpha+1}{p^{\alpha\mu}} \leqq \frac{\alpha+1}{2^{\alpha}} \leqq 1.$$

Therefore, if v contains one or more prime factors $\geqq 2^{\frac{1}{\mu}}$, we have

$$\Pi_{\substack{p|v \\ p \geqq 2^{\frac{1}{\mu}}}} \frac{\alpha+1}{p^{\alpha\mu}} \leqq 1$$

and as there exist only a finite number of prime numbers $p < 2^{\frac{1}{\mu}}$ $\frac{\sum_{d|v} 1}{v^{\mu}}$ is limited, i.e. smaller than a number independent of v.

Lemma. Let n_1 and n_2 be two arbitrary positive integers, whose sum $n_1 + n_2$ is equal to n and suppose F to be an arbitrary function with the parameters m and n; three functions F_1, F_2, and F_3 may be found then in such a way that the parameters of F_1 are equal to m and n_1, of F_2 equal to m and n_2 and of F_3 equal to m and $n-1$ with the relation

$$F(u) = \frac{n_1! n_2!}{(n_1+n_2)!} \sum_{d|u} F_1(d) F_2\left(\frac{u}{d}\right) + O\{F_3(u)\} \quad . \quad . \quad . \quad (5)$$

Proof. Introduce the functions F_1, F_2, and F_3 by means of the following relations:

$F_1(u) = f(v_u)$ for $e_u = m$, $a_u = n_1$,
$\quad = 0$ in the other cases,
$F_2(u) = 1$ for $e_u = m$, $a_u = n_2$, $v_u = 1$,
$\quad = 0$ in the other cases,
$F_3(u) = v_u^{\mu}$ for $e_u > m$ and also for $e_u = m$, $a_u \leqq n-1$,
$\quad = 0$ in the other cases, i.e. for $e_u < m$ and also for $e_u = m$, $a_u > n-1$.

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From these definitions it appears that the parameters of F_1 are equal to m and n_1, of F_2 to m and n_2 and of F_3 to m and $n-1$, so that now only relation (5) is to be proved and in order to do this, we distinguish 5 cases:

1. $$e_u > m;$$

then we have

$$u = v_u$$

and the quantity

$$\sum_{d|u} F_1(d) F_2\left(\frac{u}{d}\right) = \sum_{d|v_u} O(1)$$
$$= O \sum_{d|v_u} 1$$

is according to the preceding lemma equal to $O(v_u^\mu)$ and therefore

$$F(u) - \frac{n_1!\, n_2!}{(n_1+n_2)!} \sum_{d|u} F_1(d)\, F_2\left(\frac{u}{d}\right) = O(v_u^\mu) - O(v_u^\mu). \quad . \quad (6)$$
$$= O(F_3(u)).$$

2. $$e_u = m, \qquad a_u \leq n-1;$$

then we have

$$u = p_1^m\, p_2^m \ldots p_{a_u}^m\, v_u$$

and

$$\sum_{d|u} F_1(d) F_2\left(\frac{u}{d}\right) = O \sum_{d|u} 1$$
$$= O \sum_{d|p_1^m p_2^m \ldots p_{a_u}^m} 1 \,.\, \sum_{d|v_u} 1$$
$$= O(1) \,.\, O(v_u^\mu),$$

so that the relation (6) holds good in this case as well.

3. $$e_u = m. \qquad a_u = n;$$

in this case we have

$$u = p_1^m\, p_2^m \ldots p_n^m\, v_u$$

and

$$F(u) = f(v_u),$$

where at least one of the following conditions is satisfied, d representing any arbitrary divisor of u and d' being subsiituted for $\frac{u}{d}$:

a) $e_d < m$,
b) $e_{d'} < m$,
c) $e_d = m, \qquad a_d > n_1$,
d) $e_{d'} = m, \qquad a_{d'} > n_2$,
e) $e_d = e_{d'} = m, \qquad a_d = n_1, \qquad a_{d'} = n_2$.

In the first four cases we have

$$F_1(d)\,F_2(d') = 0$$

and the last case appears only if

$$d = q^m v_d \qquad d' = q'^m v_{d'}$$

where q is a divisor of

$$P = p_1 p_2 \dots p_n$$

consisting of n_1 prime factors, where $q' = \frac{P}{q}$ is therefore composed of $n - n_1 = n_2$ prime factors and where the product of the integers v_d and $v_{d'}$ is equal to v_u. In case (e) we have therefore

$$F_2(d') = 0, \qquad \text{except for} \qquad v_{d'} = 1,$$

consequently

$$\begin{aligned} F_1(d)\,F_2(d') &= f(v_d)\,F_2(d') \\ &= f(v_u) \text{ for } v_{d'} = 1, \\ &= 0, \text{ for } v_{d'} \neq 1, \end{aligned}$$

hence

$$\begin{aligned} \sum_{d \mid u} F_1(d)\,F_2(d') &= \sum_{q \mid P} f(v_u) \\ &= f(v_u) \sum_{q \mid P} 1. \end{aligned}$$

P containing exactly $\frac{(n_1+n_2)!}{n_1!n_2!}$ different divisors composed of n_1 prime factors, we have

$$\sum_{q \mid P} 1 = \frac{(n_1+n_2)!}{n_1!n_2!}$$

and therefore

$$\begin{aligned} \frac{n_1!n_2!}{(n_1+n_2)!} \sum_{d \mid u} F_1(d)F_2(d') &= f(v_u) \\ &= F(u), \end{aligned}$$

from which relation (5) ensues at once.

4. $e_u = m, \quad a_u > n;$

one of the following conditions at least is in this case satisfied

$$\begin{aligned} &a) \quad e_d < m, \\ &b) \quad e_{d'} < m, \\ &c) \quad e_d = m, \quad a_d > n_1, \\ &d) \quad e_{d'} = m, \quad a_{d'} > n_2, \end{aligned}$$

so we have

$$F(u) = 0 \quad \text{and} \quad F_1(d)\,F_2\left(\frac{u}{d}\right) = 0$$

5. $e_u < m;$

in this case one of the numbers e_d and $e_{d'}$ at least is smaller than m, so that again the relations

$$F(u) = 0 \text{ and } F_1(d)\,F_2\left(\frac{u}{d}\right) = 0$$

hold good.

These lemmas having been demonstrated, the proof of formula (1) for any arbitrary value of n will be easy, viz.: we shall demonstrate the proposition for $n = n_1 + n_2$, supposing that it has been proved for $n = n_1$, for $n = n_2$ and for $n = n_1 + n_2 - 1$, where n_1 and n_2 represent two arbitrary positive integers; as the proposition in § 2 has been proved for $n = 1$, the validity for $n = 2, 3, 4 \ldots$ etc. respectively, follows from this argument.

Let $F(u)$ be the function with parameters m and $n_1 + n_2$, for which relation (1) has to be proved; we introduce (and according to the preceding lemma this is possible) the function $F_1(u)$ with parameters m and n_1, the function $F_2(u)$ with parameters m and n_2 and the function $F_3(u)$ with parameters m and $n_1 + n_2 - 1$, so that we have

$$F(u) = \frac{n_1!\,n_2!}{(n_1+n_2)!} \sum_{d \mid u} F_1(d) F_2\left(\frac{u}{d}\right) + O\{F_3(u)\}$$

and consequently

$$\sum_{\substack{u=2 \\ u \equiv l}}^{x} F(u) = \frac{n_1!\,n_2!}{(n_1+n_2)!} \sum_{\substack{dd' \leq x \\ dd' \equiv l}} F_1(d) F_2(d') + O \sum_{\substack{u=2 \\ u \equiv l}}^{x} |F_3(u)|$$

As relation (1) holds good for $n = n_1 + n_2 - 1$, consequently for the function $|F_3(u)|$, we have

$$\sum_{\substack{u=2 \\ u \equiv l}}^{x} |F_3(u)| = O\left\{\frac{x^{\frac{1}{m}} x_2^{\,n_1+n_2-2}}{\log x}\right\}$$

and as according to our proposition, (1) holds good also for $n = n_1$ and for $n = n_2$, i. e. for the functions F_1 and F_2, we have, according to the second lemma of this paragraph

$$\sum_{\substack{dd' \leq x \\ dd' \equiv l}} F_1(d) F_2(d') = \frac{bm(n_1+n_2) x^{\frac{1}{m}} x_2^{\,n_1+n_2-1}}{h n_1!\, n_2! \log x} \sum_{\substack{v=1 \\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{x^{\frac{1}{m}} x_2^{\,n_1+n_2-2}}{\log x}\right),$$

so that we conclude

$$\sum_{\substack{u=2 \\ u \equiv l}}^{x} F(u) = \frac{bm x^{\frac{1}{m}} x_2^{\,n_1+n_2-1}}{h(n_1+n_2-1)!\log x} \sum_{\substack{v=1 \\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left\{\frac{x^{\frac{1}{m}} x_2^{\,n_1+n_2-2}}{\log x}\right\};$$

therefore formula (1) has been proved for all positive integers n.

§ 4. In this last paragraph we have to consider the proof and the significance of the formulae (1) and (3), which have been demonstrated in §§ 2 and 3. As to the proof, we see that relation (1) has been deduced from (2) in an elementary way and as has been observed at the beginning of the preceding paragraph, some other formulae, e. g. (3) may be proved by means of (1). Relation (3) may also be demonstrated directly, viz. without the round-about way along formula (1), by not starting from formula (2) but from the relation

$$\sum_{\substack{p \leq x \\ p \equiv l}} \frac{1}{p} = \frac{1}{h} \log \log x + O(1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)^{1)}$$

This proof is analogous to the one used in order to demonstrate relation (1); on executing it, it will appear that in that case the proof is even simpler. Yet, that proof has not been given in this paper, because (1) lies deeper that (3), i. e. (3) is to be deduced from (1) and the reverse is not possible, so that it would not do to prove formula (3) first, as it is not possible then to conclude to formula (1) and as will be seen it is principally this formula that we want. The question, however, is somewhat different for $k = 1$, as (7) in that case is to be deduced[2]) quite elementarily from the identity

$$\sum_{p \leq x} \log p \left(\left[\frac{x}{p} \right] + \left[\frac{x}{p^2} \right] + \ldots \right) = \sum_{u=2}^{[x]} \log u \qquad ^{3)}$$

$$= x \log x + O(x)$$

so that relation (3) may be proved quite elementarily for $k = 1$.

Formula (1) is also to be proved directly, i. e. without using (2); it is namely possible to prove (1) with propositions in the theory of functions in a way, analogous to the one, used to demonstrate formula (2); it is clear, however, that, in that case, an elementary proof is not to be thought of and we have succeeded in deducing (1) from (2) by means of elementary methods.

If in (1) and (3) μ is taken equal to nothing, we have this

Proposition. If the finite arithmetical function $F(u)$ is equal to nothing for $e_u < m$ and also for $e_u = m, a_u > n$, and the function $f(u)$ equals nothing for $e_u \gtreqless m$, $F(u)$ being equal to $f(v_u)$ for $e_u = m, a_u = n$, the formulae (1) and (3) hold good, if l and k are prime to each other.

1) E. Landau. Handbuch I. p. 450.

2) E. Landau. Handbuch I. p. 98—102.

3) E. Landau. Handbuch I. p. 77, (formula 4).

In order to bring out the significance of this proposition four applications are given as follows.

Application I. Any integer >1, resolved into prime factors, has a series of exponents and the question arises how many integers below a given limit are to be found with a given series of exponents and how many of these integers are to be met in a given arithmetical series, of which the first term and the difference are prime to each other. It is clear that the first question is a special case of the second. If the given series of exponents consists of one number and this number is equal to one, the second question is identical with the question how many prime numbers are to be found in that arithmetical series below a certain limit and the answer is given by formula (2); if the given series of the exponents is composed of one number $m>1$, it is sought how many numbers equal to the m^{th} power of a prime number occur in the arithmetical series, below a given limit and this is easy to calculate by means of formula (2). The question, however, becomes more intricate, as soon as the series of exponents consists of more than one number, but in that case the answer may be found by means of the proposition, for any series of exponents. Take e.g. the smallest number, occurring in the given series of the exponents, equal to m and suppose that this number occurs n times in this series, so that the given series of the exponents is equal to

$$a_1, a_2, \ldots, a_\sigma, m, m, \ldots, m,$$

where

$$a_\varrho > m \quad \text{for} \quad \sigma \geq \varrho \geq 1.$$

Take $F(u)=1$, if the integer u, resolved into prime factors, has a series of exponents, equal to the given series and take $F(u)=0$ in the other cases; take $f(u)=1$, if the series of the exponents of the prime factors of the integer u is equal to $a_1, a_2, \ldots, a$ and $f(u)=0$ in the other cases. The conditions, laid down in the proposition are then satisfied, viz.

1. $F(u)=0$, for $e_u < m$ and also for $e_u = m$, $a_u > n$,
2. $f(v)=0$, for $e_u \leq m$,
3. $F(u)=f(v_u)$, for $e_u = m$, $a_u = n$,

for if $e_u = m$, $a_u = n$ and the given series of the exponents is (not) corresponding to that of u, the series $a_1, a_2, \ldots, a_\sigma$ is (not) corresponding to the series of the exponents of v_n, so that both the functions $F(u)$ and $f(v_u)$ are in that case equal to one (nothing).

The proposition may therefore be applied and formula (1) gives the sums

$$\sum_{u=2}^{x} F(u) \qquad \text{and} \qquad \sum_{\substack{u=2 \\ u \equiv l}}^{x} F(u),$$

which exactly represent the numbers sought. So we find e.g.

The number of positive integers $\leq x$, composed of two different prime factors, occurring in these numbers respectively in the degree α and β, is for $\alpha > \beta$ equal to

$$\frac{x^{\frac{1}{\beta}}}{\log x} \sum_{p} \frac{1}{p^{\frac{\alpha}{\beta}}} + O\left(\frac{x^{\frac{1}{\beta}}}{(\log x)^2}\right).$$

The number of positive integers $\leq x$, composed of one quintuple and three double prime factors (these prime factors are thought different from each other) is equal to

$$\frac{2\sqrt{x}}{\log x} (\log \log x)^2 \sum_{p} \frac{1}{p^{\frac{5}{2}}} + O\left(\frac{\sqrt{x}}{\log x} . \log \log x\right).$$

and among these numbers

$$\frac{2\sqrt{x}}{\log x} (\log \log x)^2 \sum_{p \equiv l \,(\mathrm{mod.}\, 8)} \frac{1}{p^{\frac{5}{2}}} + O\left(\frac{\sqrt{x}}{\log x} . \log \log x\right)$$

integers are to be found, which are congruent to l, with regard to the modulus 8 ($l = 1$, 3, 5 or 7) and

$$\frac{\sqrt{x}}{\log x} (\log \log x)^2 \sum_{p \equiv \pm l \,(\mathrm{mod.}\, 10)} \frac{1}{p^{\frac{5}{2}}} + O\left(\frac{\sqrt{x}}{\log x} . \log \log x\right).$$

integers, which are congruent with l, with regard to the modulus 10 ($l = 1$, 3, 7 or 9).

In the following application, viz. with the function $\pi_n(x)$ defined there, the case will be treated that the given series of exponents consists of n numbers, each equal to 1.

Application II. We introduce the following well-known notation[1]:

$\pi_n(x)$ represents the number of squareless integers $< x$, composed of n prime factors, $\varrho_n(x)$ the number of integers $\leq x$, of which the number of different prime factors is equal to n, and $\sigma_n(x)$ the number of integers $\leq x$, for which the total number of prime factors equals n.

Gauss surmised in 1796

$$\pi_n(x) \sim \frac{1}{(n-1)!} \cdot \frac{x (\log \log x)^{n-1}}{\log x}.$$

[1] See for this notation e.g. E. Landau, Handbuch I, pages 205, 208, 211.

This relation has been first proved by E. LANDAU; from the proposition

$$\sum_{p \leq x} 1 \sim \frac{x}{\log x}$$

he deduced viz. in an elementary way these relations [1] [2])

$$\pi_n(x) \sim \frac{x (\log \log x)^{n-1}}{(n-1)! \log x},$$

$$\sigma_n(x) \sim \frac{x (\log \log x)^{n-1}}{(n-1)! \log x},$$

and

$$\rho_n(x) \sim \frac{x (\log \log x)^{n-1}}{(n-1)! \log x}.$$

By using the deeper lying relation

$$\sum_{p \leq x} 1 = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$$

he proves, also elementarily [1])

$$\pi_n(x) = \frac{x (\log \log x)^{n-1}}{(n-1)! \log x} + O\left(\frac{x (\log \log x)^{n-2}}{\log x}\right),$$

$$\sigma_n(x) = \frac{x (\log \log x)^{n-1}}{(n-1)! \log x} + O\left(\frac{x (\log \log x)^{n-2}}{\log x}\right)$$

and

$$\rho_n(x) = \frac{x (\log \log x)^{n-1}}{(n-1)! \log x} + O\left(\frac{x (\log \log x)^{n-2}}{\log x}\right).$$

What I want to prove now is that these formulae are only special cases of the proposition. Take $F(u) = 1$, if u be equal to a squareless number composed of n prime factors and take $F(u) = 0$ in the other cases; then we have

$$\sum_{u=2}^{x} F(u) = \pi_n(x);$$

if $F(u)$ be equal to 1 or 0 according as the total number of prime factors of u is equal to n or not, we have

$$\sum_{u=2}^{x} F(u) = \sigma_n(x)$$

and finally, by giving to $F(u)$ the value 1 or 0, according as the number of different prime factors of u is equal to n or not, we have

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[1]) E. LANDAU. Sur quelques problèmes relatifs à la distribution des nombres premiers. [Bulletin de la Société mathématique de France. Vol. 28 (1900) pg. 25—28].

[2]) E. LANDAU. Handbuch I. p. 205—213.

$$\sum_{u=2}^{x} F(u) = \varrho_n(x).$$

In each of these three cases the function $F(u)$ satisfies the conditions stated, if in them

$$m = 1 \quad \text{and consequently } b = 1,$$
$$f(1) = 1$$

and $$f(v) = 0, \quad \text{for} \quad v > 1$$

so that a possesses the value $\frac{1}{h.(n-1)!}$, and we conclude, that the relations (1) and (3) are modified to the formulae

$$\sum_{\substack{u=2 \\ u \equiv l}}^{x} F(u) = \frac{x(\log\log x)^{n-1}}{h\,(n-1)!\log x} + O\left\{\frac{x(\log\log x)^{n-2}}{\log x}\right\}$$

and

$$\sum_{\substack{u=2 \\ u \equiv l}}^{x} \frac{F(u)}{u} = \frac{(\log\log x)^n}{h.n!} + O(\log\log x)^{n-1}.$$

For $k = 1$ and consequently $h = 1$ the first of these relations passes into the formulae written down for $\pi_n(x)$, $\sigma_n(x)$ and $\varrho_n(x)$, and the second relation produces an asymptotical expression, not of the number but of the sum of the reciprocals of the integers considered, e.g. the sum of the reciprocals of all squareless numbers composed of n prime factors $\leqq x$, is equal to

$$\frac{(\log\log x)^n}{n!} + O(\log\log x)^{n-1}$$

and the same holds good for the numbers that are mentioned in the definition of $\varrho_n(x)$ or $\sigma_n(x)$. These formulae concerning the sum of the reciprocals being special cases of formula (3), where k has the value 1, may be proved by means of a merely elementary reasoning, as has been observed at the beginning of this paragraph.

By giving an arbitrary value to k in the formula, however, we find that the number of squareless numbers $\leqq x$, composed of n prime factors and congruent to l, with regard to the modulus k, is equal to

$$\frac{x(\log\log x)^{n-1}}{h.(n-1)!\log x} + O\left(\frac{x(\log\log x)^{n-2}}{\log x}\right)$$

and that the sum of the reciprocals of these numbers is equal to

$$\frac{(\log\log x)^n}{h.n!} + O(\log\log x)^{n-1},$$

while again for the integers that are mentioned with the definition of the functions $\varrho_n(x)$ and $\sigma_n(x)$, perfectly analogous formulae hold good.

For the very reason that the function $F(u)$ is general it will not be difficult to deduce other corresponding relations; so we find the same results if we consider the squareless numbers composed of not more than n prime factors, or the integers for which the total number of prime factors is $\leq n$, or the integers for which the number of different prime factors is not greater than n, etc.

Application III. In an arithmetical series, the difference of which is k and the first term of which is prime to k, occur

$$\frac{\pi^2 x}{6\,k \log x} \prod_{p|k}\left(1+\frac{1}{p}\right) + O\left(\frac{x}{(\log x)^2}\right)$$

numbers $\leq x$, equal to a square multiplied by a prime number.

That this is again a special case of our proposition appears by taking $F(u)$ equal to 1 or 0, according to u being equal or not to a square multiplied by a prime number.

We have

$$m=1 \quad \text{hence} \quad b=1,$$
$$n=1$$

and

$$f(v)=1, \text{ if } v \text{ is a square},$$
$$=0, \text{ if } v \text{ is not a square},$$

consequently

$$\frac{b}{h} \sum_{\substack{u=1\\ uz^m \equiv l}}^{\infty} \frac{f(u)}{u^{\frac{1}{m}}} = \frac{1}{k \prod_{p|k}\left(1-\frac{1}{p}\right)} \sum_{\substack{u=1\\ (u,k)=1}}^{\infty} \frac{f(u)}{u} \text{ [1]}$$

$$= \frac{1}{k \prod_{p|k}\left(1-\frac{1}{p}\right)} \sum_{\substack{v=1\\ (v,k)=1}}^{\infty} \frac{1}{v^2}$$

$$= \frac{1}{k \prod_{p|k}\left(1-\frac{1}{p}\right)} \prod_{p|k}\left(1-\frac{1}{p^2}\right) \sum_{v=1}^{\infty} \frac{1}{v^2}$$

$$= \frac{\pi^2}{6k} \prod_{p|k}\left(1+\frac{1}{p}\right)$$

and we have only to substitute these values in (1), in order to find the relation sought.

Application IV. If all the prime factors of the positive integer

[1] (u, k) represents the greatest common divisor of u and k, so that the number u in this sum assumes respectively each integral positive value prime to k.

q are greater than the prime number p, and

$$w = p^{\alpha} q ,$$

the number of positive integers $\leq x$, congruent to l, with regard to the modulus k, (l and k prime to each other) for which the number of divisors is exactly equal to w, is given by

$$\frac{a x^{\frac{1}{p-1}}}{\log x} + O\left(\frac{x^{\frac{1}{p-1}}}{(\log x)^2}\right) \qquad \text{for } \alpha = 1$$

and by

$$\frac{a x^{\frac{1}{p-1}} (\log \log x)^{\alpha-1}}{\log x} + O\left\{\frac{x^{\frac{1}{p-1}} (\log \log x)^{\alpha-2}}{\log x}\right\}$$

for any arbitrary positive integral value of α, where

$$a = \frac{b\,(p-1)}{h.(\alpha-1)!} \sum_{u=1}^{\infty}{}' \frac{1}{u^{\frac{1}{p-1}}}$$

extended over all the positive integers u, of which the number of divisors is exactly equal to q and for which the congruence

$$u z^{p-1} \equiv l \qquad (\text{mod. } k)$$

has roots z; b represents the number of incongruent roots of the congruence

$$z^{p-1} \equiv 1 \qquad (\text{mod. } k).$$

In order to prove this, we take the number of divisors of u equal to τ_u, and

$$F(u) = 1 \quad \text{for} \quad \tau_u = w ,$$
$$= 0 \quad \text{for} \quad \tau_u \neq w .$$

We have to prove first that this function satisfies the conditions written in the proposition, if

$$m = p - 1,$$
$$n = \alpha ,$$
$$f(v) = 1 \quad \text{for} \quad \tau_v = q ,$$
$$= 0 \quad \text{for} \quad \tau \neq q ;$$

In order to give this demonstration, we distinguish four cases:

1. Let e_u be smaller than $p-1$;

for

$$u = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_\sigma^{\alpha}$$

the number of divisors of u

$$\tau_u = (\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_\sigma + 1)$$

is divisible by $e_u + 1$, consequently by a number $< p$, hence

$$\tau_u \neq w ,$$
$$F(u) = 0.$$

2. Take $e_u = p - 1,\ a_u > \alpha$;
τ_u is then divisible by

$$(e_u + 1)(e_u + 1) \ldots (e_u + 1) = (e_u + 1)^{a_u} = p^{a_u},$$

therefore by $p^{\alpha+1}$, so that in this case too, we have

$$\tau_u \neq w ,$$
$$F(u) = 0.$$

3. Take $e_u = p - 1,\ a_u = \alpha$;
we have then

$$u = p_1^{p-1} p_2^{p-1} \ldots p_\alpha^{p-1} v_u$$

and from

$$\tau_u = p^\alpha \tau_{v_u} \qquad w = p^\alpha q$$

it follows that there are only two possibilities, viz.

$a)\quad F(u) = 1, \quad \tau_u = w, \quad \tau_{v_u} = q, \quad f(v_u) = 1,$

$b)\quad F(u) = 0, \quad \tau_u \neq w, \quad \tau_{v_u} \neq q, \quad f(v_u) = 0,$

hence in this case

$$F(u) = f(v_u)$$

4. Take $e_v \leqq p - 1$;
as τ_v is divisible by $e_v + 1$, consequently by a number $\leqq p$, τ_v is in this case unequal to q, hence

$$f(v) = 0.$$

Now that it has been proved that the conditions stated are satisfied, we are allowed to apply the proposition and formula (1) gives at once the relation sought.

Finally we observe: in application II some asymptotical expressions have been written for $\pi_n(x)$, $\sigma_n(x)$ and $\varrho_n(x)$, but LANDAU deduces still sharper formulae for these functions. He proves[1]) that for each positive integral value of q, constant numbers $A_{a,b}$, $B_{a,b}$ and $C_{a,b}$ are to be found, for which the relations

$$\left.\begin{aligned} \pi_n(x) &= x \sum_{a=1}^{q} \sum_{b=0}^{n-1} A_{a,b} \frac{(\log\log x)^b}{(\log x)^a} + o\left(\frac{x}{(\log x)^q}\right), \\ \varrho_n(x) &= x \sum_{a=1}^{q} \sum_{b=0}^{n-1} B_{a,b} \frac{(\log\log x)^b}{(\log x)^a} + o\left(\frac{x}{(\log x)^q}\right) \\ \text{and}\quad \sigma_n(x) &= x \sum_{a=1}^{q} \sum_{b=0}^{n-1} C_{a,b} \frac{(\log\log x)^b}{(\log x)^a} + o\left(\frac{x}{(\log x)^q}\right) \end{aligned}\right\} \quad \ldots \quad (8)$$

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[1]) E. LANDAU. Ueber die Verteilung der Zahlen, welche aus ν Primfaktoren zusammengesetzt sind. [Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Math.-physikalische Klasse. (1911). pages 361—381].

hold good. It is a matter of course that such a relation does not hold good for any function satisfying the condition stated in § 1. It appears, however, that we have only to modify this condition a little to be sure that such a relation does hold good, viz.:

If the arithmetical function $F(u)$ of the integer $u > 1$ satisfies the conditions:

1. for $e_u < m$, and also for $e_u = m$, $a_u > n$, we have
$$F(u) = 0;$$
2. for $e_u = m$, $a_u \leq n$ we have
$$F(u) = f(v_u, a_u),$$

where $f(v,a)$ represents an arithmetical function of the positive integers v and a, and

3. $$F(u) = O(v_u^{\mu}), \quad \text{where} \quad \mu < \frac{1}{m(m+1)};$$

then there are constant values $D_{a,b}$ for $q \geq a \geq 1$, $n-1 \geq b \geq 0$ to be found for any positive integral value of q, satisfying the relation

$$\sum_{\substack{u=2 \\ u \equiv l}}^{x} F(u) = x^{\frac{1}{m}} \sum_{a=1}^{q} \sum_{b=0}^{n-1} D_{a,b} \frac{(\log \log x)^b}{(\log x)^a} + o\left(\frac{x^{\frac{1}{m}}}{(\log x)^q}\right)$$

This proposition is again very general; this appears obviously by the observation that the functions which occur in the four applications of this paragraph and which have been substituted for $F(u)$ also satisfy this condition, so that the formulae deduced in those applications are also to be intensified with this proposition. And the formulae obtained in application II are exactly the formulae (8).

The proposition is elementarily, i. e. without using considerations belonging to the theory of functions to be deduced from the well-known relation

$$\sum_{\substack{p \leq x \\ p \equiv l}} 1 = \frac{1}{h} \int_2^x \frac{du}{\log u} + o\left(\frac{x}{(\log x)^q}\right),^{1)}$$

according to a reasoning somewhat similar to the one followed here in order to prove formula (1); it goes, however, without saying that the proof is not so simple.

[1]) E. Landau. Handbuch. I. p. 468.