small term $a_2 = a_2 e^{2kt}$ in equation (1) a term which could not, however, originate in the motion of the liquid (an asymmetry of the oscillating system was out of the question and, moreover, a_2 was independent of the nature of the liquid), but probably had to be ascribed to a want of symmetry in the wire [1]).

Physics. — *"The viscosity of liquefied gases.* VII. *The torsional oscillatory motion of a body of revolution in a viscous liquid."* By J. E. VERSCHAFFELT. (Communication N°. 151*e* from the Physical Laboratory at Leiden). (Communicated by Prof. H. KAMERLINGH ONNES.)

(Communicated in the meeting of February 24, 1917).

1. In Comm. N°. 148*b* the theory of the torsional oscillatory motion of a sphere in a viscous liquid was developed to a first approximation; the results of the experimental investigation described in the previous part (VI, Comm. N°. 151*d*) render it advisable to develop the theory to a higher degree of approximation. The present paper is an attempt to a solution of the problem, not only for a sphere but for an arbitrary body of revolution. This attempt was in so far successful as a method of solution is given, in which the motion of the liquid and of the body is put into the form of a series; the terms of these series, however, contain functions of the coordinates which in the mean time owing to the difficulties of the integration remain determined by differential equations, and coefficients the numerical value of which cannot yet be given. In form these series agree with those which were found experimentally (Comm. N°. 151*d*).

The motion of the liquid.

2. We start from the well-known hydrodynamical equations [2])

[1]) An asymmetry of this kind is not improbable, as the wire owing to the method of preparation showed a permanent twist: on the tension being taken off it curls up spirally and the zero changed with the weight suspended from it. During the oscillation of the system the wire obtains a higher twist in the one direction and is untwisted in the other, which might involve a small deviation from HOOKE's law to be expressed by a term Na^2 in the equation of motion of the oscillating body (Comm. N^0. 148*b*, equation 23).

[2]) u, v, w are the components of the velocity at a point x, y, z; p is the pressure in the liquid, its density being ω and its viscosity η; X_0, Y_0, Z_0 are the components of the external field of force, in which the liquid is placed, in our

$$\left.\begin{array}{r} -\dfrac{\partial p}{\partial x} + \eta \Delta u + \mu X_0 = \mu\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}\right) \text{ ,etc} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 , \end{array}\right\} \quad . \quad (1)$$

the last of which we shall replace by a different one which follows from the whole set of four viz.:

$$\Delta p = 2\mu\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial z}\frac{\partial w}{\partial x}\right). \quad (1')$$

If the motion of the liquid is the result of the friction of an immersed body of revolution rotating about its axis (the z-axis) and if, moreover, the boundary of the liquid, if it exists, is also the same in alle meridian planes, we may put

$$u = \varepsilon x - \omega y \quad , \quad v = \varepsilon y + \omega x \, , \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ε and ω are functions of the cylindrical coordinates $\varrho = \sqrt{x^2+y^2}$ and z and of the time.

For small velocities we may write:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \dots \quad , \quad \dots \quad , \quad p = p_1 + p_2 + p_3 + \dots \quad (3)$$

where each successive term of the series is considered as infinitely small with respect to the preceding one. We therefore treat the motion of the liquid as the result of a composition of a series of conditions of motion, the velocities of which diminish very rapidly, the further we go down the series [1]). Consequently the equations (1) can now be separated into a series of sets each of which determines a condition of motion. Putting in the n^{th} set:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -X_n \text{ , etc.} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

X_n, Y_n, Z_n are to be looked upon as the components of a force generated by the inertia of the liquid; they are completely determined by the preceding approximations. Also by (1') in each successive approximation the distribution of pressure is determined by the preceding approximation.

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case the field of gravity ($X_0 = 0$, $Y_0 = 0$, $Z_0 = g$). Δ is the symbol for LAPLACE's operator.

It will be supposed, that neither μ nor η are functions of the coordinates or of the time. In a piece of apparatus of ordinary dimensions this condition is practically fulfilled, even for a gas, when no greater differences of pressure occur than are occasioned by gravity. (Comp. on this point ZEMPLÉN, Ann. d Phys., 38, 81, 1912).

[1]) This method of treatment of the problem was given by A. N. WHITEHEAD (Quarterly Journ. of pure and applied Mathem., 23, 78, 1889) for the purpose of finding a second approximation: he applied it to the case of a uniform rotation of a sphere. Comp. also ZEMPLÉN, Ann. d. Phys., 38, 74, 1912.

3. In the *first approximation* (velocities infinitely small of the *first* order) we have

$$\Delta p_1 = 0 \text{ [1]} \quad , \quad \eta \Delta u_1 = \mu \frac{\partial u_1}{\partial t} , \text{ etc.} \quad . \quad . \quad . \quad (5)$$

If the motion of the body is an oscillation of the damped harmonic kind, the angle of deviation of which can be represented by the real part of

$$\alpha = a e^{kt} , \text{[2]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we must put

$$p_1 = 0 \quad , \quad w_1 = 0 \quad , \quad \varepsilon_1 = 0 \quad , \quad \omega_1 = k \alpha \varphi_1 , \quad . \quad . \quad . \quad . \quad (7)$$

where φ_1 is now only a function of ϱ, z and $b = \sqrt{\frac{\mu}{\eta} k}$ determined by the differential equation

$$\frac{\partial^2 \varphi_1}{\partial \varrho^2} + \frac{3}{\varrho} \frac{\partial \varphi_1}{\partial \varrho} + \frac{\partial^2 \varphi_1}{\partial z^2} - b^2 \varphi_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and by the boundary conditions, that at the surface of the body $\varphi_1 = 1$ and at the external boundary of the liquid (at infinity, if the liquid is infinite) $\varphi_1 = 0$) [3]).

4. In the *second approximation* one finds:

$$\left. \begin{aligned} & -\frac{\partial p_2}{\partial x} + \eta \Delta u_2 + \mu X_2 = \mu \frac{\partial u_2}{\partial t} , \text{ etc.} \\ \text{where} \quad & X_2 = \omega_1{}^2 x \ , \ Y = \omega_1{}^2 y \ , \ Z_2 = 0 \, . \end{aligned} \right\} \quad . \quad . \quad . \quad (9)$$

This represents the first approximation to a motion in the field of the centrifugal force [4]). Moreover:

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[1]) In the first approximation the distribution of pressure is the same as in condition of rest.

[2]) k is a complex imaginary quantity; a may be taken as real. (Comp. Comm. N⁰. 148*b*).

[3]) If the body of revolution is a sphere, φ_1 becomes a function of $r = \sqrt{x^2 + y^2 + z^2}$ only, and equation (8) reduces to equation (11) of Comm. N⁰. 148*b*.

[4]) In general this field of force has no potential; that is why it causes a movement of circulation in the liquid. (Comp. Comm. N⁰. 148*d*, Proc. XVIII, 2, p. 1038). For that reason it is also impossible to put in general $\frac{\partial p_2}{\partial x} = \mu X_2$, etc., as in the distribution of the pressures under the influence of the external field of force. Only in the case of an infinitely long cylinder, where φ_1 is merely a function of ϱ, the field of force of the centrifugal force has a potential; a movement of circulation is absent in that case and the motions of higher order disappear at the same time.

$$\Delta p_2 = \frac{\mu}{\varrho} \frac{\partial (\varrho \omega_1)^2}{\partial \varrho} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9')$$

These equations are satisfied by putting:

$$\varepsilon_2 = 2\,k\,a^2\,\psi_2 \quad , \quad \omega_2 = 0 \quad , \quad w_2 = 2\,k\,a^2\,\gamma_2 \quad , \quad p_2 = 2\,k\,a^2\,\pi_2 \,. \,. \quad (10)$$

where ψ_2, γ_2, π_2 are new functions of ϱ, z, and b. A motion of circulation is thus obtained in the meridian planes; this motion is a damped pulsating one, with twice as high a frequency and degree of damping, as the oscillation in first approximation.

5. In *third approximation* we have again $\Delta p_3 = 0$, or $p_3 = 0$, and

$$\eta \, \Delta \, u_3 + \mu \, X_3 = \mu \frac{\partial u_3}{\partial t}, \quad \text{etc.}$$

where this time

$$X_3 = \Phi_3 \, y \quad , \quad Y_3 = -\,\Phi_3 \, x \quad , \quad Z_3 = 0 \,, \quad . \quad . \quad (11)$$

with

$$\Phi_3 = 2\,k^2\,a^3 \left[2\,\varphi_1\,\psi_2 + \varrho\,\psi_2 \frac{\partial \varphi_1}{\partial \varrho} + \gamma_2 \frac{\partial \varphi_1}{\partial z} \right].$$

In a third approximation we thus have a motion caused by a periodic damped field of force at right angles to the meridian planes and containing the time in the factor e^{3kt}. It follows, that this motion like the one in first approximation consists of an oscillatory rotation of the liquid in shells, each with its own amplitude and phase, but with the same period and degree of damping; i. e. the equations can be satisfied by putting:

$$w_3 = 0 \quad , \quad \varepsilon_3 = 0 \quad , \quad \omega_3 = 3\,k\,a^3\,\varphi_3 \,,$$

φ_3 being a new function of ϱ, z, and b, determined by the differential equation:

$$\frac{\partial^2 \varphi_3}{\partial \varrho^2} + \frac{3}{\varrho}\frac{\partial \varphi_3}{\partial \varrho} + \frac{\partial^2 \varphi_3}{\partial z^2} - 3b^2 \varphi_3 = 2b^2 \left(2\varphi_1 \psi_2 + \varrho \psi_2 \frac{\partial \varphi_1}{\partial \varrho} + \gamma_2 \frac{\partial \varphi_1}{\partial z} \right) \quad (12)$$

and by the condition, that $\varphi_3 = 0$ at the boundaries of the liquid.

6. As one would be inclined to expect and as, moreover, can be easily proved, further approximations yield alternately circulation in meridian planes and oscillations, about the axis, with frequencies and degrees of damping which increase in an arithmetical series. By putting

$$-\,X_n = \Psi_n \, x - \Phi_n \, y \quad , \quad -\,Y_n = \Psi_n \, y + \Phi_n \, x \,, \,. \quad . \quad (13)$$

one finds, using a well known method [1])

[1]) If the relations hold for $n = 1$ to m, it may be proved, that they also hold for $n = m + 1$.

$$\left.\begin{array}{l}\Psi_{2n+1}=0, \Phi_{2n}=0, Z_{2n+1}=0\,, \varepsilon_{2n+1}=0\,, \omega_{2n}=0\,, w_{2n+1}=0\,, p_{2n+1}=0 \\ \varepsilon_{2n}=2nka^{2n}\psi_n\,, \omega_{2n+1}=(2n+1)ka^{2n+1}\varphi_{2n+1}\,, w_{2n}=2nka^{2n}\gamma_{2n}\,, \\ \qquad\qquad p_{2n}=2nka^{2n}\pi_{2n}\end{array}\right\} \quad (14)$$

From the foregoing discussion it appears, that, when a body of revolution in a liquid oscillates about its axis in a simple harmonic damped motion (how the motion is sustained, is of no account), the liquid will assume a motion which consists partly of a compound harmonic damped oscillation of liquid shells, where the amplitude may be represented by

$$\alpha = \alpha_1 + \alpha_3 + \alpha_5 + \ldots = a\varphi_1 e^{kt} + a^3\varphi_3 e^{3kt} + a^5\varphi_5 e^{5kt} + \ldots \text{ [1]} \quad . \quad (15)$$

and for the rest of a motion of circulation in meridian planes.

The above reasoning still holds, if the motion of the oscillating body itself is a compound harmonic one of the form:

$$\alpha = a\,e^{kt} + \sigma_3\,a^3\,e^{3kt} + \sigma_5\,a^5\,e^{5kt} + \ldots; \quad . \quad . \quad . \quad . \quad (16)$$

the functions φ_{2n+1} are, however, not then zero at the surface of the body, but equal to σ_{2n+1}.

The motion of the body.

7. The question now arises: of what nature will the motion be which the body assumes in the liquid, when without friction it would perform a simple harmonic oscillation? Certainly not a simple damped motion, for, even if by some artifice the body was for some time made to swing exactly in the simple damped motion, which it must assume according to the first approximation, the higher terms of the liquid motion would still by friction give rise to forces which would try to disturb the simple motion and which would certainly create this disturbance, as soon as the body was left to itself. It is obvious that they would impart to the body a composite motion, corresponding to equation (16) where the even terms would not occur, seeing that the liquid motions of even order only give friction along the meridians and thus cannot have any influence on the oscillation [2].

[1] The quantities φ are functions of ρ, z and b which are exclusively determined by the boundary conditions. In the case, when the body is an infinitely long cylinder, all φ's are zero, with the exception of φ_1. At the solid boundaries of the liquid the φ's become zero (except $\varphi_1 = 1$). If the liquid is partly bounded by a free surface, a special condition will hold there.

[2] The friction along the meridians can only produce an imperceptible deformation of the body. It might seem as if the circulational motion in the liquid, although it is kept up by the body and damped by friction in the liquid, did not occasion a loss of energy of the body. The explanation of this seeming contradiction may be found in the circumstance, that the motions of different order are not mutually independent and a loss of energy of even order is provided by products of velocities of uneven order.

In this manner it becomes intelligible that the oscillating body, also when it is made to swing freely in the liquid, will assume a motion corresponding to equation (16)[1]), as was first revealed by experiment (comp. Part VI).

In equation (16) the exponent k which contains the time of oscillation T and the logarithmic decrement $\left(k = -\frac{\delta}{T} + \frac{2\pi i}{T}\right)$, as also the coefficients σ_3, σ_5, etc. (a is arbitrary) are determined by the interactions between liquid and body. By the friction to which the body is subject, moments act on it which can be calculated, as was actually done in the first approximation in Comm. N°. 148b for the case of a sphere, as soon as the functions ω_n and γ_n are known; if the moments of the couples caused by the successive sets of motion are represented by C_1, C_3, C_5 etc. (the moments of even order are all zero), the equation of motion of the body is:

$$K\frac{d^2 a}{dt^2} - C + Ma = 0 \; . \; . \; . \; . \; . \; . \; . \; . \qquad (17)$$

(comp. 23, Comm. N°. 148b), where $C = C_1 + C_3 + C_5 + \ldots$ The quantities C are given by

$$\left.\begin{aligned} C_n = -\int_{z_1}^{z_2} \varrho F ds = \eta \int_{z_1}^{z_2} \varrho^2 \frac{\partial \omega_n}{\partial N} ds \text{ [2]}) = \eta n k a_n \int_{z_1}^{z_2} \varrho^2 \frac{\partial \varphi_n}{\partial N} ds = -L_n \frac{d a_n}{dt}, \\ \text{where} \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \\ L_n = -\eta \int_{z_1}^{z_2} \varrho^2 \frac{\partial \varphi_n}{\partial N} ds = -\eta A_n ; \qquad \qquad \qquad \end{aligned}\right\} \quad (18)$$

A_n is a numerical quantity (of the dimensions of a volume) which depends on the shape of the body and further on the quantity b, i.e. on k (time of swing and decrement), on the constants η and μ of the liquid and finally on the coefficients σ up to and including σ_n [3]). Equation (17) can thus be separated into a series of equations

[1]) That is to say after the disturbances, which are due to the starting of the motion, have subsided: these disturbances are not gone into here (comp. Comm. N°. 148b, § 4, note).

[2]) As in Comm. N°. 148b, F represents the tangential force per unit area in the direction of the motion; ds is the area of a circular strip round the body of revolution; $\frac{\partial \omega_n}{\partial N}$ is the gradient of the angular velocity of the n^{th} order in the liquid close to the body; z_1 and z_2 are the z-limits of the body.

[3]) If different parts of the oscillating system are surrounded by different fluids (e.g. a part by a liquid and the other part by air, as was the case in the experiments) L_n itself has to be divided into parts, each of which refers to one of the fluids.

$$\left.\begin{aligned} K\frac{d^2 a_n}{dt^2}+L_n\frac{da_n}{dt}+M\,a_n=0\,, \\ \text{or} \qquad n^2 k^2 K+n\,k\,L_n+M=0\,, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19)$$

the first of which ($n=1$) is the same as equation (26) of Comm. N°. 148*b*, by which k is determined. The other equations determine the quantities σ.[1]).

Herewith the problem is formally completely solved. Numerical application would, however, only be possible, if one succeeded in finding the functions ω_n and γ_n[2])

Physics. — "*The viscosity of liquefied gases.* VIII. *The similarity in the oscillatory rotation of a body of revolution in a viscous liquid*". By J. E. Verschaffelt. (Communication N°. 151*f* from the Physical Laboratory at Leiden). (Communicated by Prof. H. Kamerlingh Onnes).

(Communicated in the meeting of February 24, 1917).

1. In Comm. N°. 148*c* the conditions were derived under which similarity would exist between two different modes of motion of an oscillating sphere in a viscous liquid. The discussion was at that time entirely based on the first approximation of the problem: but even then it was anticipated that the conclusions would prove to hold in general (Comm. N°. 148*c* § 5), not only in nearer approximations, but also for bodies of different shape to the sphere; this will now be shown to be the case.

Returning once more to the general hydrodynamical equations (equation (1) of the previous communication, Comm. N°. 151*e*), we will inquire whether it is possible to introduce units of length, mass and time such that everything specific disappears from the equations. The *external* similarity of the liquid motion of course requires in the first place similarity of the body oscillating in the liquid (the latter condition is of itself satisfied in the case of a sphere); let R be a

1) In all this the supposition is retained that the moment of the torsional couple is proportional to the angle of torsion and that the ordinary laws of friction remain valid.

2) Not till then would it be possible to settle the exact condition for infinite smallness of the velocities (comp. preceding communication § 1 note), i.e. the condition for α_8 (for the body) to remain below a definite fraction of α_1 or, as would be even more useful for our purpose, the condition for the decrement δ not to deviate by more than a definite amount from the limiting value δ_0.