Mathematics. — *"On the nodal-curve of an algebraic surface".* By Dr. J. WOLFF. (Communicated by Prof. HK. DE VRIES).

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1. We consider a surface F of order n with a nodal curve Δ, and without any other singularity. Suppose that we represent F by means of a birational transformation on another surface F^*, in such a way that Δ passes into a non-singular curve Δ^* of F^*. Δ^* may then be *one* single curve or consist of two parts. The former occurs if the developable surface Ω of the pairs of tangent planes along Δ forms one whole, the latter if Ω consists of different parts. We shall occupy ourselves with the first case. The deficiency π^* of Δ^* is then equal to that of Ω, for the points of Δ^* correspond *one* for *one* with the planes of Ω. In whatever way F is birationally transformed into a surface in which Δ gets a non-singular curve as image, that image will always have the the same deficiency π^*. The value of π^* has been calculated by CLEBSCH in case of F being a rational surface, in other words, may be birationally represented in a plane. He finds $\pi^* = d(n-4) + 1$, in which d is the order of Δ [1]). This is deduced analytically. By means of a geometrical wording the proof is to be simplified. We shall start with this and then prove the proposition for an arbitrary surface, consequently also if it is not rational.

2. Let $F^n(x_1 x_2 x_3 x_4) = 0$ be a rational surface of order n_2, which, by means of the formulae

$$\begin{aligned} \varrho x_1 &= f_1(\xi_1\, \xi_2\, \xi_3) \\ \varrho x_2 &= f_2(\xi_1\, \xi_2\, \xi_3) \\ \varrho x_3 &= f_3(\xi_1\, \xi_2\, \xi_3) \\ \varrho x_4 &= f_4(\xi_1\, \xi_2\, \xi_3) \end{aligned}$$

is represented in a plane F^*, in which ξ_1, ξ_2, ξ_3 stand for the homogeneous coordinates of a point, while the f_i are homogeneous functions of a certain degree ν. Let F have no other singularities but a nodal curve Δ of order d, and let its image on F^* be *one* single curve Δ^*. The plane sections C of F are represented as curves C^* of order ν, forming a linear system on F^*. The sections of F with the ∞^2 planes passing through a point P represent themselves as the ∞^2 curves C^* of a net. The Jacobian J^* of that net, locus of the nodus of the ∞^1 curves provided with them contained in the net, is the image of the curve of contact J of the cone of contact

[1]) Math. Ann. Bd. 1, bl. 270.

laid out of P at F. J is the section of F with the first polar surface of P, apart from Δ. From this it ensues that the sections Σ of F with *arbitrary* surfaces of order $n-1$ represent themselves as a system of curves Σ^*, individuated by the compound curve Δ^*+J^*. As the f_i are of order ν, the order of Δ^*+J^* is equal to $(n-1)\nu$. J^* (as jacobian of a net of curves of order ν) is of order $3(\nu-1)$. Hence Δ^* is of order

$$(n-1)\,\nu-3\,(\nu-1)=\nu(n-4) \qquad 3.$$

The curves C^* may have base points. If B_h is an h-fold base point, in such a way that all C^* pass h-times through B_h, the Jacobian J^*, as is known, passes $3h-1$ times through B_h. The section Σ of F with an arbitrary surface of order $n-1$ is represented on F^* as a curve Σ^*, which is represented by a homogeneous equation of order $n-1$ in the f_i, so that such a curve passes $(n-1)h$ times through B_h. Hence Δ^* passes $(n-1)h-(3h-1)=h(n-4)+1$ times through B_h. The deficiency π^* is easy to calculate now. We have viz.

$$\pi_*=\tfrac{1}{2}\,\{\nu\,(n-4)+2\}\,\{\nu\,(n-4)+1\}-\Sigma\,\tfrac{1}{2}\,h\,(n-4)\,\{h\,(n-4)+1\},$$

in which the summation extends over the various base points B_h. If we consider that the deficiency of a plane section C of F is equal to that of its image C^* on F^*, in other words that we have

$$\tfrac{1}{2}\,(\nu-1)\,(\nu-2)-\tfrac{1}{2}\,\Sigma\,h\,(h-1)=\tfrac{1}{2}\,(n-1)\,(n-2)-d,$$

we find

$$\pi^*=d\,(n-4)+1.$$

3. The above reasoning can be of no service if F is not rational, so that F^* is not a plane. Let F^n be a surface of order n, rational or not, with a double curve Δ of order d and without any other singularity. Let μ be the class of the developable surface Ω formed by the pairs of tangent planes along Δ and let k be the number of points of Δ, where the two tangentplanes coincide (pinch-points).

Suppose that F has been transformed into another surface F by a birational transformation into another surface F^*, in such a way that Δ passes into *one* single curve Δ^*. The plane sections C of F are represented by curves C^*, which form a linear system on F^*. These C^* may have base points B_h so that they all pass h times through B_h. The sections of F with the ∞^2 planes passing through a point P are represented by the curves of a net (C^*). The curve of contact J of the cone of contact laid at F out of P is transformed into the Jacobian J^* of (C^*), from which it follows again that Δ^*+J^* is a curve belonging to the linear system $|\Sigma^*|$ formed by the images of the sections Σ of F with *arbitrary* surfaces of order n—1. Let us now for a moment suppose that Δ^* belongs to a linear system on F^* of which all curves pass as often though the different points B_h as Δ^*, and let Δ_1^* be a curve of that linear system. In that case $\Delta_1^* + J^*$ is also a curve of $|\Sigma^*|$. Let us for convenience' sake represent the number of intersections of two curves *outside the points* B_h by placing the letters we have chosen for those curves, between brackets, we have

$$[\Sigma^*,\ \Delta^*] = [\Delta_1,\ \Delta^*] + [J^*,\ \Delta^*].$$

$[\Delta_1,\ \Delta^*]$ is called the "*degree*" g of the linear system to which Δ^* belongs.[1] J rests in $k + \mu$ points on Δ, consequently

$$[J^*,\ \Delta^*] = k + \mu$$

Σ has $d\,(n-1)$ nodes on Δ, therefore $[\Sigma^*,\ \Delta^*] = 2d\,(n-1)$.

Hence,

$$2d\,(n-1) = g + k + \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We obtain a second relation if for a moment we make a particular supposition: let there exist surfaces φ^{n-4} passing through the nodal curve Δ, consequently *adjuncts* of order n—4 of F. They intersect F apart from Δ in so-called *canonical curves* K, which have the property of being represented on F^* as *canonical curves* K^*, consequently as sections of F^* with adjuncts of order n^*-4.

Two properties of the canonical curves K we have to apply here. An adjunct φ^{n-4} forms with 3 planes an adjunct φ^{n-1} of order n—1. To φ^{n-1} belong also the 1st polar surfaces of arbitrary points of space. So a K forms together with 3 plane sections C a curve of $|J|$. Consequently a K^* forms together with 3 curves C^* a J^*, so that

$$[K^*, \Delta^*] + 3\,[C^*, \Delta^*] = [J^*, \Delta^*]\,.$$

The second property we want, we find by observing that an adjunct φ^{n-4} forms with 1 plane an adjunct φ^{n-3}. A φ^{n-3} intersects the plane of a C in a curve φ^{n-3}, which passes through the d nodes of C so that outside it, it has moreover $2p-2$ points in common with C, where p is the deficiency of C. Hence the canonical curves K intersect C in $2p-2-n$ points, where n is the *degree* of the linear system of the C. But this holds good for any linear system of curves.[2]). Let us apply this to the system to which Δ^* on F^* belongs, we have then $[K^*, \Delta^*] = 2\pi^*-2-g$.

Further is $[C^*, \Delta^*] = 2d$, because C has d nodes on Δ, and

[1]) Cf. e.g. F. ENRIQUES, *Introduzione alle Geometria sopra le superficie algebriche.* (Memorie di mat. e di fis. d. Soc. It. d. Sc., Serie 3, volume 10, p. 14)

[2]) F. ENRIQUES, *Introduzione*, p. 64.

$[J^*, \Delta^*] = k + \mu$, because J has with Δ $k + \mu$ points in common.
We find therefore

$$2\pi^* - 2 - g + 6d = k + \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) it ensues at once

$$\pi^* = d(n-4) + 1.$$

§ 4. The two particular suppositions we have made are superfluous. Let Δ^* *not* belong to a linear system of which all the curves in the points B_h have the multiplicity h. For such an *isolated curve* Δ^* a positive or negative integer g is always to be defined, which is called the *virtual degree* [1]) of Δ^*. An infinite number of linear systems may be construed, in such a way that Δ^* is a part of curves belonging to it. Let $|E^*|$ be such a system and let R be a curve that completes Δ^* into an L^*. It may be seen to that there are an indefinite number of such curves R^*. They form then a linear system $|R^*|$, the restsystem of Δ^* with regard to $|L^*|$. Let g_1 be the degree of $|L^*|$ in other words the number of variable intersections of two L^*, and g_2 the degree of R^*. If now the Δ^* also formed a linear system $|\Delta^*|$, then we should of course have, g being the degree of it: $g_1 = g + g_2 + 2i$, where i represents $[\Delta^*, R^*]$. For g_1 is $[\Delta^* + R^*, \Delta_1^* + R_1^*]$, in which Δ_1^* and R_1^* are arbitrary curves of $|\Delta^*|$ and $|R^*|$.

If Δ^* is isolated then its virtual degree, by definition, is the number g determined by the equation $g_1 = g + g_2 + 2i$.

This virtual degree g is independent of the choice of $|L^*|$. We may further prove that we may calculate with it as if g were "the number of intersections of Δ^* with itself", independent of its positive or negative sign. The formula (1) holds good if Δ^* is isolated: in that case g represents its virtual degree.

The same holds true of the formula (2), not only if Δ^* is isolated, but also if no canonical curves exist. In all cases $2\pi^* - 2 - g$ is called the *immersion constant* of Δ^* and is $[J^*, \Delta^*] - 3[C^*, \Delta^*]$ [2]).

§ 5. If Ω consists of two different developable surfaces Ω_1 and Ω_2, Δ^* consists of two different curves Δ_1^* and Δ_2^*, which both have the same deficiency π as Δ. Δ_1^* and Δ_2^* intersect each other in the k images of the pinch-points on Δ. Without nearer determinations it cannot be said that the formula $\pi^* = d(n-4) + 1$ holds good, because Δ^* is degenerated. But it may be supposed that the

[1]) F. Enriques, *Introduzione*, p. 28.

[2]) F. Severi, *Il genere aritmetico ed il genere lineare*, (Atti della R. Acc. d. Sc. di Torino, vol. 37, 1901—2).

curve $\Delta_1^* + \Delta_2^*$ belongs to a continuous system and in that case the curves of that system are of the deficiency

$$\pi^* = \pi + \pi + k - 1 = 2\pi + k - 1.$$

And also if $\Delta_1^* + \Delta_2^*$ does not belong to such a system $2\pi + k-1$ is called the *virtual deficiency* of this degenerate curve [1]). Let μ_1 be the class of Ω_1 and μ_2 the one of Ω_2. A Σ intersects Δ in its $d(n-1)$ nodes. They are represented in $d(n-1)$ pairs on F^* and of each pair *one* point lies on Δ_1^*, the other on Δ_2^*. Hence

$$[\Sigma^*, \Delta_1^*] = [\Sigma^*, \Delta_2^*] = d(n-1).$$

And as $|\Sigma^*| = |\Delta_1^* + \Delta_2^* + J^*|$, we have

$$d(n-1) = [\Delta_1^* + \Delta_2^* + J^*, \Delta_1^*] = [\Delta_1^* + \Delta_2^* + J^*, \Delta_2^*].$$

Consequently

$$d(n-1) = g_1 + k + k + \mu_1 = g_2 + k + k + \mu_2 . \quad . \quad . \quad (1')$$

where g_1 and g_2 are the virtual degrees of Δ_1^* and Δ_2^*.

The immersion constant $2\pi - 2 - g$ of Δ_1^* is equal to

$$[J^*, \Delta_1^*] - 3[C^*, \Delta_1^*],$$

hence

$$\left.\begin{array}{l} k + \mu_1 - 3d = 2\pi - 2 - g_1 \\ k + \mu_2 - 3d = 2\pi - 2 - g_2 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2')$$

and

From (2') it ensues

$$2\pi + k - 1 = \tfrac{1}{2}(g_1 + \mu_1 + g_2 + \mu_2 + 4k) - 3d + 1.$$

Consequently with regard to (1')

$$2\pi + k - 1 = d(n-4) + 1.$$

The formula $\pi^* = d(n-4) + 1$ holds consequently good if Ω degenerates, provided the virtual deficiency is taken for π^*.

So we have this general proposition:

The order of an algebraic surface that has no other singularity but a nodal curve Δ of order d, along which the pairs of tangent planes form a developable surface Ω of deficiency π^*, is

$$n = 4 + \frac{\pi^* - 1}{d}.$$

1) Cf. e.g. E. PICARD "*Théorie des fonct alg. de 2 var.*" vol. 2, page 106.

Zoology. — "*On an eel, having its left eye in the lower jaw*". By Mrs. C. E. DROOGLEEVER FORTUYN—VAN LEYDEN. (Communicated by Prof. J. BOEKE).

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Through the kindness of Dr. H: C. REDEKE I obtained an eel in which the left eye was lacking in the ordinary place, while on the lower side of the head, somewhat to the left of the medial line, an eye was visible which externally was quite normally shaped.

In order to find out whether this submaxillary eye was the left one and if so, how it had come to occupy such a curious position, and further whether it was also internally of normal structure, two series of transverse sections were made, one of the lower jaw and one of the remainder of the head.

It appeared that the left eye had indeed been shifted downward, that the structure was quite normal and that a well developed optic nerve and strong muscles, attached to the sclerotic in the usual way, rendered it possible and even very likely that the eye had functionated. These nerve and muscles originated from the upper part of the head; the nerve came forth from the brain in the usual manner, perfectly symmetrically with the nerve of the right eye; the muscles proceeded caudally quite symmetrically with the muscles of the right side. Nerve and muscles however followed the normal way over a short distance only, they soon bent downward and descended through the head to the lower jaw, right through the buccal cavity along a stalk connecting the upper and lower jaws and situated just before the tongue. The nerve was surrounded by the four straight eye-muscles; the two oblique ones were situated orally of the first-mentioned complex of muscles and nerve.

From this stalk the whole complex proceeded downward right through the lower jaw to the place where the eye was found. Besides nerve and muscles also a blood-vessel descended, which entered the eye together with the nerve.

Of the bony roof of the mouth, which this complex of muscles and nerve had passed, the entopterygoid, lying between the parasphenoid and the palatine, was laterally and posteriorally shifted, so that it no longer bordered on the parasphenoid. A muscle, the arco-palatine adductor muscle, was much lengthened and behind the muscle-nerve complex bent from the entopterygoid to the parasphenoid.

For the rest little change had occurred in the upper part of the head. The place where the eye should have been, was filled up with