curve Δ of JACOBI, which is of order $3(n-1)$. The third group consists of $(18n-33)$ points, where a c^n has *four* consecutive points in common with its tangent. From this it ensues that the *points of undulation* of a net form a curve of order $(18n-33)$. [1])

The curve (P) is of order $3(n-1)$ and has a triple point in P; through P pass consequently $(9n^2-21n)$ of its tangents. They now form two groups: the first consists of base tangents t, the second of tangents u in points of undulation.

(P) now intersects the curve Δ in $3(n-1)(2n-3)$ points D, of which one of the two tangents passes through P (class of the curve of ZEUTHEN) [2]), consequently in $9(n-1)^2 - 3(n-1)(2n-3)$ or $3n(n-1)$ points D, for which the base tangent t passes through P.

From this it then ensues, that P lies on $(6n^2-18n)$ tangents u. The *four-point tangents*, therefore, envelop a curve of class $6n(n-3)$. [3])

Mathematics. *"On a Representation of the Plane Field of Circles on Point-Space".* By Dr. K. W. WALSTRA. (Communicated by Prof. JAN DE VRIES).

(Communicated in the meeting of January 27, 1917).

§ 1. The circles in the plane XOY are represented by

$$C \equiv X^2 + Y^2 - 2aX - 2bY + c = 0.$$

If we consider a, b, and c as the co-ordinates x, y, z of a point, a correspondence (1, 1) is obtained between the circles of a plane and the points of space. The image of a circle is obtained by placing a perpendicular in the centre on the plane and by taking on it as co-ordinate the *power* of the point O with regard to the circle.

For the radius we have $r^2 = a^2 + b^2 - c$.

Circles with equal radii are therefore represented by the points of a paraboloid of revolution, with equation $x^2 + y^2 - z = r^2$.

The images of the point-circles lie on the *limiting surface* G,

$$x^2 + y^2 = z,$$

a *paraboloid of revolution*, touching the plane XOY in O.

§ 2. A pencil of circles is indicated by $C_1 + \lambda C_2 = 0$. For the circle λ we have

[1]) Another deduction of this number is to be found in my paper: "Characteristic numbers for nets of algebraic curves". (These Proceedings XVII, 937).

[2]) Cf. my paper "On nets of algebraic plane curves". (These Proc. VII, 633).

[3]) These Proc. XVII, 936.

$(1+\lambda)a = a_1 + \lambda a_2, (1+\lambda)b = b_1 + \lambda b_2, (1+\lambda)c = c_1 + \lambda c_2.$

From this we find for the images

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} = \frac{z-z_1}{z_1-z_2}.$$

A pencil of circles is therefore represented by a straight line.

Its intersections with G are the images of the point-circles of the pencil. The point at infinity of the line represents the axis of the pencil.

A tangent at G is the image of a pencil of circles of which the limiting points have coincided; *any two points of a tangent are therefore the images of two touching circles.*

This may be confirmed as follows. Let d be the distance of the centres of *two circles* with radii r and r'; we have then $d = r \pm r'$ or $\sqrt{(a-a')^2 + (b-b')^2} = \sqrt{a^2 + b^2 - c} \pm \sqrt{a'^2 - b'^2 - c'}$.

After some reduction we find for the images

$$\left(xx' + yy' - \frac{z+z'}{2}\right)^2 = (x'^2 + y'^2 - z')(x^2 + y^2 - z),$$

which relation expresses that the images lie on a tangent of G.

§ 3. A net of circles is represented by $C_1 + \lambda C_2 + \mu C_3 = 0$.

From this it ensues for the images

$(1+\lambda+\mu)x = x_1 + \lambda x_2 + \mu x_3$ etc. consequently

$$\begin{vmatrix} x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

A net of circles is therefore represented by a plane.

Plane sections of G have circles as horizontal projections. For the section of $x^2 + y^2 = z$ with $z = \alpha x + \beta y + \gamma$ has as projection the figure represented by $x^2 + y^2 - \alpha x - \beta y - \gamma = 0$.

The point-circles of a net of circles lie therefore on a circle; this proposition is reversible.

The net that corresponds to $z = \alpha x + \beta y + \gamma$, has as equation

$$X^2 + Y^2 - 2aX - 2bY + (\alpha a + \beta b + \gamma) = 0,$$

where a and b are variable parameters. If we write for this

$$X^2 + Y^2 + a(\alpha - 2X) + b(\beta - 2Y) + \gamma = 0,$$

it appears that all circles have in the point $(\frac{1}{2}\alpha, \frac{1}{2}\beta)$ equal power viz. $\frac{1}{4}(\alpha^2 + \beta^2) + \gamma$; this point is the centre of the circle that contains the point-circles of the net.

To a tangent plane of G corresponds a net of circles that pass through a fixed point. For, to $2x_1x + 2y_1y = z + z_1$ corresponds a net of which all the circles have in (x_1,y_1) the power $x_1^2 + y_1^2 - z_1 = 0$.

Two pencils of circles are in general represented by two skew straight lines. If, however, they have a circle in common their images lie in a plane and their four point-circles lie on a circle; the pencils belong to a net.

§ 4. For two orthogonal circles we have $d^2 = r_1^2 + r_2^2$, so

$$(a_1 - a_2)^2 + (b_1 - b_2)^2 = (a_1^2 + b_1^2 - c_1) + (a_2^2 + b_2^2 - c_2)$$

or

$$2a_1a_2 + 2b_1b_2 = c_1 + c_2.$$

For the images we have consequently $2x_1x_2 + 2y_1y_2 = z_1 + z_2$, i. e. *the images of two orthogonal circles are harmonically separated by the limiting surface.*

To the connection between pole and polar plane corresponds the fact that all circles intersecting a given circle orthogonally form a net.

To the relation between two associated polar lines corresponds the fact that pencils of circles may be arranged in pairs, so that any circle of a pencil is intersected orthogonally by any circle of the other.

To a polar tetrahedron corresponds a group of four circles that are orthogonal in pairs. (Of them only three are real).

§ 5. If the circle C intersects the circle C_1 diametrically we have $d^2 = r^2 - r_1^2$ or

$$(a_1 - a)^2 + (b_1 - b)^2 = (a^2 + b^2 - c) - (a_1^2 + b_1^2 - c_1).$$

We consequently have for the images

$$2x_1x + 2y_1y - z = 2x_1^2 + 2y_1^2 - z_1.$$

The circles that intersect a given circle diametrically form a net.

According to § 3 this net has as radical centre $\frac{1}{2}\alpha = x_1$, $\frac{1}{2}\beta = y_1$, i. e. the centre of C_1 (which was to be expected), and in that point the power $z_1 - x^2_1 - y^2_1 = - r^2_1$.

§ 6. The circles touching at a given C_1, have their images on the enveloping cone of G, which has the image of C_1 as vertex (§ 2). Three enveloping cones have eight points in common; they are the images of eight circles which touch at three given circles.

The circles touching at two circles C_1 and C_2 are represented by a twisted curve ϱ^4 of the fourth degree, a net of circles consequently contains four circles that touch at C_1 and C_2. The enveloping cones that have the images of C_1 and C_2 as vertices touch at G along conics that have two points in common, viz. the images of the intersections of C_1 and C_2.

The intersections of ϱ^4 with a tangent plane of G are the images of four circles passing through a given point and touching at C_1, C_2 (§ 3).

The circles touching at a given straight line are represented by a cylindrical surface that envelops G and of which the straight lines are perpendicular to the given straight line consequently parallel to the plane XOY.

Mathematics. — "*A Quadruply Infinite System of Point Groups in Space*". By Dr. CHS. H. VAN OS. (Communicated by Prof. JAN DE VRIES).

(Communicated in the meeting of January 27, 1917).

Let a pencil (a^3) be given, consisting of cubic surfaces a^3. An arbitrary straight line l is touched by four surfaces a^3 of the pencil. As the space contains ∞^4 lines l, there are ∞^4 groups of four points of contact. We shall indicate this system of groups of four points by S^4.

§ 1. If we take for the line l a line g lying on *one* of the surfaces a^3, the four surfaces mentioned coincide with this surface a^3, while the points of contact become indefinite. These straight lines g are therefore *singular lines* of S^4. They form a ruled surface R, of which we shall determine the order.

A line g intersects a second surface a^3 in three points lying on the base-curve ϱ^9 of the pencil (a^3); the lines g are therefore *trisecants* of the curve ϱ^9. If on the other hand we consider a trisecant of ϱ^9, the surface a^3, which passes through an arbitrary point of this trisecant will have four, consequently an infinitely great number of points in common with it, so that the trisecant is a straight line g.

Through an arbitrary point pass 18 bisecants of ϱ^9 [1]), the genus of ϱ^9 amounts consequently to $\frac{1}{2} \times 8 \times 7 - 18 = 10$. If we therefore project the curve ϱ^9 out of one of its points, we get as projection a curve of order eight with $\frac{1}{2} \times 7 \times 6 - 10 = 11$ nodes. Through the said point pass therefore 11 trisecants of ϱ^9, so that the surface R has the curve ϱ^9 as 11-fold curve.

A surface a^3 intersects the surface R along the curve ϱ^9 and according to the 27 straight lines g lying on a^3, the *order* of R amounts to 42.

§ 2. Any line l passing through a given point P contains one

[1]) Cf. e.g. ZEUTHEN, *Lehrbuch der abzählenden Geometrie*, page 46.