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are not invariant at infinity. Then, of course, inertia is not explained: we must then prefer to leave it unexplained rather than explain it by the undetermined and undeterminable constant λ. It cannot be denied that the introduction of this constant detracts from the symmetry and elegance of EINSTEIN's original theory, one of whose chief attractions was that it explained so much without introducing any new hypothesis or empirical constant.

Postscript.

Prof. EINSTEIN, to whom I had communicated the principal contents of this paper, writes (March 24, 1917): "Es wäre nach meiner Meinung unbefriedigend, wenn es eine denkbare Welt ohne Materie gäbe. Das $g_{\mu\nu}$-Feld soll vielmehr *durch die Materie bedingt sein, ohne dieselbe nicht bestehen können.* Das ist der Kern dessen, was ich unter der Forderung von der Relativität der Trägheit verstehe". He therefore postulates what I called above the logical impossibility of supposing matter not to exist. We can call this the "material postulate" of the relativity of inertia. This can only be satisfied by choosing the system A, with its world-matter, i. e. by introducing the constant λ, and assigning to the time a separate position amongst the four coordinates.

On the other hand we have the "mathematical postulate" of the relativity of inertia, i. e. the postulate that the $g_{\mu\nu}$ shall be invariant at infinity. This postulate, which, as has already been pointed out above, has no real physical meaning, makes no mention of matter. It can be satisfied by choosing the system B, without a world-matter, and with complete relativity of the time. But here also we need the constant λ. The introduction of this constant can only be avoided by abandoning the postulate of the relativity of inertia altogether.

Astronomy. — "*On the Theory of Hyperion, one of Saturn's Satellites.*". By J. WOLTJER JR. (Communicated by Prof. W. DE SITTER).

(Communicated in the meeting of April 27, 1917).

1. Among the peculiar disturbances, which the satellites of Saturn undergo by their mutual attraction, those, produced by Titan in the motion of Hyperion, are of much importance. In this paper I intend to give a short development of the theory of the latter satellite; my dissertation will contain more extensive calculations on this subject.

The ratio of the masses of Hyperion and Titan is only a very small quantity. The inclinations of the orbital planes of Hyperion and Titan to the aequator of Saturn are also small. To simplify the problem I shall neglect these inclinations as well as the influence of the mass of Hyperion, the sun, the other satellites, the ellipticity of Saturn and the rings. I shall, therefore, suppose Hyperion to be a particle with the mass zero, moving in the orbital plane of Titan, while the latter describes an undisturbed elliptical motion around the centre of a sphere with the mass of Saturn.

Notation:

$a =$ semi-major axis, $e =$ excentricity;

$l =$ mean anomaly, $g =$ longitude of pericentre;

$\mathrm{M} =$ mass of Saturn, $m' =$ mass of Titan;

$\mathrm{L} = \sqrt{a\mathrm{M}}$; $\mathrm{G} = \mathrm{L}\sqrt{1-e^2}$.

The accented letters refer to Titan, those without accents to Hyperion. The units are chosen so, that the constant of attraction $= 1$; x and y are coordinates in a system of axes in the orbital plane of Titan, the origin coinciding with the centre of Saturn.

From the fundamental equations, which Delaunay has used in his lunar theory [1]), the following differential equations for the motion of Hyperion result:

$$\frac{d\mathrm{L}}{dt} = \frac{\partial \mathrm{R}}{\partial l}, \quad \frac{d\mathrm{G}}{dt} = \frac{\partial \mathrm{R}}{\partial g},$$

$$\frac{dl}{dt} = -\frac{\partial \mathrm{R}}{\partial \mathrm{L}}, \quad \frac{dg}{dt} = -\frac{\partial \mathrm{R}}{\partial \mathrm{G}},$$

$$\mathrm{R} = \frac{\mathrm{M}^2}{2\mathrm{L}^2} - m'\frac{xx'+yy'}{r'^3} + \frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2}}.$$

The function R is, with regard to the angular elements, a function of $l+g-l'-g'$, l, l' only. The following new quantities are introduced:

$$l + g - l' - g' = \Phi,$$
$$4l - 3l' + 3g - 3g' - 180^\circ = \theta,$$
$$g - g' = \Omega,$$
$$g' = \chi.$$

One sees at once that R, with regard to the angular elements, is a function of Φ, θ, Ω only. The reason why these three quantities are introduced is this: from the observations the mean motion of the argument θ appears to be zero and θ to perform a libration on each side of the value $\theta = 0^\circ$ with an amplitude of about 36°;

1) Théorie du Mouvement de la Lune I, 13.

this argument, therefore, is of much importance; Ω is a secular argument, its introduction is therefore obvious; the argument Φ has a short period and thus leads to the terms in the development of the perturbative function, which are of little importance. From R therefore all terms with arguments containing Φ or a multiple of Φ are omitted; thus instead of R one uses the function $\bar{R}=\frac{1}{2\pi}\int_0^{2\pi} R d\Phi$.

For convenience sake in the rest of this paper the line above the letter R is omitted.

Putting

$$L - G = \Lambda,$$
$$3L - 4G = \Gamma,$$

and n' for the mean motion of Titan, the equations for θ, Φ, Λ, Γ become:

$$\frac{d\Lambda}{dt}=\frac{\partial R}{\partial \theta}-\frac{\partial R}{\partial \Omega}, \quad \frac{d\theta}{dt}=-\frac{\partial R}{\partial \Lambda}-3n',$$
$$\frac{d\Gamma}{dt}=-4\frac{\partial R}{\partial \Omega}, \quad \frac{d\Phi}{dt}=\frac{\partial R}{\partial \Gamma}-n'.$$

2. The excentricity of Titan being a small quantity, I shall try to develop the solution of these equations in powers of this excentricity. Thus, first I put $e'=0$; then R appears to be independent of Ω and to be a function of θ only. The equations then are:

$$\frac{d\Lambda}{dt}=\frac{\partial R\,(e'=0)}{\partial \theta}, \quad \frac{d\theta}{dt}=-\frac{\partial R\,(e'=0)}{\partial \Lambda}-3n',$$
$$\frac{d\Gamma}{dt}=0, \quad \frac{d\Phi}{dt}=\frac{\partial R\,(e'=0)}{\partial \Gamma}-n'.$$

Putting $R_0=\frac{M^2}{2(4\Lambda-\Gamma)^2}$, $R_1=(R-R_0)_{e'=0}$, $R=R_0+R_1+R_2$, the development of R_1 is:

$$R_1=m'\sum_0^{\infty}{}_p A_p \cos p\theta,$$

where $A_0, \ldots A_p \ldots$ are functions of Λ and Γ.

$R\,(e'=0)$ being independent of Φ and Ω, the equations for Λ, Γ and θ form a system apart; after the integration of this system Φ is determined by a quadrature. This system admits the solution: $\theta=0$, $\Lambda=$ const., $\Gamma=$ const.. However, between the constant values of Λ and Γ, which are called Λ_0 and Γ_0, a relation must exist, which is a consequence of the condition that $\frac{d\theta}{dt} = 0$ for $\theta = 0$. This relation is:

$$\frac{4M^2}{(4\Lambda_0 - \Gamma_0)^2} - 3n' = m' \sum_0^\infty p \left[\frac{\partial A_p}{\partial \Lambda}\right]_{\Lambda_0, \Gamma_0}.$$

Let a_0 and e_0 be the values of a and e belonging to $\Lambda = \Lambda_0$, $\Gamma = \Gamma_0$. The relation becomes, with regard to the equation $n' = \frac{\sqrt{M+m'}}{a'^{\frac{3}{2}}}$:

$$4\left(\frac{a'}{a_0}\right)^{\frac{3}{2}} - 3\sqrt{1+\frac{m'}{M}} =$$

$$= \frac{m'}{M} \sum_0^\infty p\, a'^{\frac{3}{2}} \left[8\sqrt{a_0}\frac{\partial A_p}{\partial a} + \frac{4(1-e_0^2) - 3\sqrt{1-e_0^2}}{e_0\sqrt{a_0}}\frac{\partial A_p}{\partial e}\right]_{a_0, e_0}.$$

From this equation the following value of $\frac{a'}{a_0}$ results:

$$\frac{a'}{a_0} = \sum_0^\infty p\, \alpha_p \left(\frac{m'}{M}\right)^p,$$

where α_p is a function of e_0; the value of α_0 is $(\frac{3}{4})^{\frac{2}{3}}$ and thus: $\alpha_0 = 0.825$.

To investigate the nature of this solution of the differential equations, the adjacent solutions are to be examined. Putting $\Lambda = \Lambda_0 + \delta\Lambda$, $\theta = \delta\theta$ and taking account of the first power of these quantities only, the differential equations become:

$$\frac{d\delta\Lambda}{dt} = \frac{\partial^2 R_1}{\partial\theta^2}\delta\theta + \frac{\partial^2 R_1}{\partial\theta\,\partial\Lambda}\delta\Lambda,$$

$$\frac{d\delta\theta}{dt} = -\frac{\partial^2(R_0+R_1)}{\partial\Lambda^2}\delta\Lambda - \frac{\partial^2 R_1}{\partial\Lambda\,\partial\theta}\delta\theta,$$

whence, by elimination of $\delta\Lambda$:

$$\frac{d^2\delta\theta}{dt^2} = -\frac{\partial^2(R_0+R_1)}{\partial\Lambda^2}\frac{\partial^2 R_1}{\partial\theta^2}\delta\theta.$$

I have developed certain portions of the perturbative function numerically for the values $e = 0.1043$ and $\frac{a'}{a} = 0.8250634$. The first value is that which H. Struve[1]) has derived from observations, the

[1]) Beobachtungen der Saturnstrabanten. Publications de l'Observatoire Central Nicolas. Série II. Vol. XI, pg. 290 and 267.

second is the same he has used in the computation of a few coefficients from the perturbative function. From my developments I deduce:

$$\frac{\partial^2 R_1}{\partial \theta^2} = + \frac{m'}{a'} \times 0.0728.$$

Neglecting $\frac{\partial^2 R_1}{\partial \Lambda^2}$ by the side of $\frac{\partial^2 R_0}{\partial \Lambda^2}$ (this is allowed, the first term having m' as factor, the second not), the differential equation for $\delta\theta$, taking account of the relation $\frac{\partial^2 R_0}{\partial \Lambda^2} = + \frac{48}{a_0{}^2}$, becomes:

$$\frac{d^2\delta\theta}{dt^2} + 2.38 \frac{m'}{a'^3}\, \delta\theta = 0,$$

or, taking account of the relation $n'^2 a'^3 = M + m'$ and neglecting the higher powers of m':

$$\frac{d^2\delta\theta}{dt^2} + 2.38\, n'^2 \frac{m'}{M}\, \delta\theta = 0,$$

hence:

$$\delta\theta = q \sin (\nu t + \gamma),$$

q and γ being the constants of integration and

$$\nu = + 1.54\, n' \sqrt{\frac{m'}{M}}.$$

Thus, the stability of the solution $\theta = 0$, $\Lambda =$ const., $\Gamma =$ const. is evident and oscillations about these values are possible. In reality these oscillations are very considerable. STRUVE derives the value 36°.64 for the amplitude of the libration in θ (l. c. pg. 287). However, the value of ν is already a close approximation, as appears from a comparison with observation: taking for $\frac{m'}{M}$ SAMTER's [1]) value $\frac{1}{4125}$, the above-mentioned formula for ν gives: $\nu = 0°.542$, while STRUVE gets (l. c. pg. 287): 0°.562.

3. Starting from the solution for $e' = 0$, viz. $\theta = 0$, $\Lambda =$ const., $\Gamma =$ const., I will construct the development of the solution in powers of e'. Putting $\delta\theta$, $\delta\Lambda$, $\delta\Gamma$ for the first-order terms in θ, Λ, Γ, the differential equations for these quantities are:

[1]) Die Masse des Saturnstrabanten Titan. Sitz. Ber. der K. Preussischen Akad. der Wissenschaften 1912, pg. 1058.

$$\frac{d\delta A}{dt}=\frac{\partial^2 R_1}{\partial\theta\partial A}\delta A+\frac{\partial^2 R_1}{\partial\theta\partial\Gamma}\delta\Gamma+\frac{\partial^2 R_1}{\partial\theta^2}\delta\theta+\frac{\partial R_2}{\partial\theta}-\frac{\partial R_2}{\partial\Omega},$$

$$\frac{d\delta\Gamma}{dt}=-4\frac{\partial R_2}{\partial\Omega},$$

$$\frac{d\delta\theta}{dt}=-\frac{\partial^2(R_0+R_1)}{\partial A^2}\delta A-\frac{\partial^2(R_0+R_1)}{\partial A\partial\Gamma}\delta\Gamma-\frac{\partial^2(R_0+R_1)}{\partial A\partial\theta}\delta\theta-\frac{\partial R_2}{\partial A}.$$

In R_2 the terms of an order higher than the first with respect to e' are to be omitted. Taking account of the solution for $e'=0$, these equations become:

$$\frac{d\delta A}{dt}=+\frac{\partial^2 R_1}{\partial\theta^2}\delta\theta+\frac{\partial R_2}{\partial\theta}-\frac{\partial R_2}{\partial\Omega},$$

$$\frac{d\delta\Gamma}{dt}=\quad 4\frac{\partial R_2}{\partial\Omega},$$

$$\frac{d\delta\theta}{dt}=-\frac{\partial^2(R_0+R_1)}{\partial A^2}\delta A-\frac{\partial^2(R_0+R_1)}{\partial A\partial\Gamma}\delta\Gamma-\frac{\partial R_2}{\partial A}.$$

Eliminating δA and $\delta\Gamma$, one gets:

$$\frac{d^2\delta\theta}{dt^2}+\frac{\partial^2(R_0+R_1)}{\partial A^2}\frac{\partial^2 R_1}{\partial\theta^2}\delta\theta=-\frac{\partial^2(R_0+R_1)}{\partial A^2}\frac{\partial R_2}{\partial\theta}-$$

$$-\frac{d}{dt}\frac{\partial R_2}{\partial A}+\left[\frac{\partial^2(R_0+R_1)}{\partial A^2}+4\frac{\partial^2(R_0+R_1)}{\partial A\partial\Gamma}\right]\frac{\partial R_2}{\partial\Omega}.$$

The development of R_2 (taking account of the terms of the first order with respect to e' only) is this:

$$R_2=\frac{m'e'}{a'}\left[\cos\Omega\sum_0^\infty{}_p B_p\cos p\theta+\sin\Omega\sum_1^\infty{}_p C_p\sin p\theta\right],$$

B_p and C_p being functions of A and Γ.

From my development of certain portions of the perturbative function I deduce:

$$\left[\frac{\partial R_2}{\partial\theta}\right]_{\theta=0}=-0.574\frac{m'e'}{a'}\sin\Omega.$$

From the solution of the differential equations for $e'=0$ we have: $\frac{d\Omega}{dt}=4\frac{\partial R_1}{\partial\Gamma}+\frac{\partial R_1}{\partial A}$; thus $\Delta\Omega$, the mean motion of Ω, is of the order of m'. A consequence of this and of the equation $\frac{\partial^2 R_0}{\partial A^2}+4\frac{\partial^2 R_0}{\partial A\partial\Gamma}=0$ is, that only the term in $\frac{\partial R_2}{\partial\theta}$ in the right member of the differential equation gives a contribution of the order of m'; the remaining terms only give contributions of the order of m'^2. The coefficient of $\delta\theta$ in the left member is the square of the mean motion of the argument of libration and of the order of m'; thus the divisor, which appears at the integration, is of the order of m'. In this divisor neglecting $\overline{\Delta\Omega^2}$, which is only of the order of m'^2, the solution of the differential equation, taking account only of the term in $\frac{\partial R_2}{\partial\theta}$ from the right member, becomes:

$$\delta\theta = -\frac{1}{\frac{\partial^2 R_1}{\partial\theta^2}}\frac{\partial R_2}{\partial\theta}.$$

Substituting the numerical values of the various quantities, we get:

$$\delta\theta = +\,7.89\, e' \sin\Omega$$

and, for e' taking the value 0.0272 (STRUVE, l. c. pg. 172) and expressing the result in degrees:

$$\delta\theta = +\,12^\circ.3 \sin\Omega\,.$$

The value from observation is (STRUVE, l. c. pg. 290): $\delta\theta = +\,14.^\circ0 \sin\Omega$; thus the agreement is very satisfactory, considering the simplifications admitted for the deduction of the theoretical value.

The value of $\delta\Gamma$ is to be determined by a quadrature from the equation:

$$\frac{d\delta\Gamma}{dt} = -\,4\frac{\partial R_2}{\partial\Omega},$$

while $\delta\Lambda$ results from the equation for $\frac{d\delta\theta}{dt}$, without any integration. Considering the fact that R_2 as well as the mean motion of Ω are of the order of m', the values of $\delta\Gamma$ and $\delta\Lambda$ are seen at once to be of the order zero with respect to m'.

The value of $\delta\Phi$ results from the equation:

$$\frac{d\delta\Phi}{dt} = \frac{\partial^2(R_0+R_1)}{\partial\Gamma\partial\Lambda}\delta\Lambda + \frac{\partial^2(R_0+R_1)}{\partial\Gamma^2}\delta\Gamma + \frac{\partial R_2}{\partial\Gamma}.$$

Subtracting the equation for $\frac{d\delta\theta}{dt}$ from four times this equation, we get:

$$4\frac{d\delta\Phi}{dt} - \frac{d\delta\theta}{dt} = \left[4\frac{\partial^2 R_0}{\partial\Gamma\partial\Lambda} + \frac{\partial^2 R_0}{\partial\Lambda^2}\right]\delta\Lambda + \left[4\frac{\partial^2 R_0}{\partial\Gamma^2} + \frac{\partial^2 R_0}{\partial\Lambda\partial\Gamma}\right]\delta\Gamma +$$
$$+\left[4\frac{\partial^2 R_1}{\partial\Gamma\partial\Lambda} + \frac{\partial^2 R_1}{\partial\Lambda^2}\right]\delta\Lambda + \left[4\frac{\partial^2 R_1}{\partial\Gamma^2} + \frac{\partial^2 R_1}{\partial\Lambda\partial\Gamma}\right]\delta\Gamma + 4\frac{\partial R_2}{\partial\Gamma} + \frac{\partial R_2}{\partial\Lambda}.$$

Taking into account the relations:

$$4\frac{\partial^2 R_0}{\partial\Gamma\partial\Lambda} + \frac{\partial^2 R_0}{\partial\Lambda^2} = 4\frac{\partial^2 R_0}{\partial\Gamma^2} + \frac{\partial^2 R_0}{\partial\Lambda\partial\Gamma} = 0,$$

$\delta\Phi$ is seen to be of the order zero with respect to m'.

4. From the preceding developments there is seen to be every reason for the expectation, that the development of the solution in powers of the excentricity of Titan, supposing the free libration of θ to be zero, will meet with no difficulties. This conclusion is at variance with NEWCOMB's opinion in his paper: "On the motion of Hyperion. A new case in Celestial Mechanics". There he reaches the conclusion, that the development in powers of e' is not possible. The incorrect performance of this development by NEWCOMB is the reason of this difference of opinion; he omits the terms in the differential equation, which arise from the part of the perturbative function that does not contain e'; thus he gets a divisor of the order of m'^2 instead of one of the order of m'. In this respect the theory of Hyperion appears to present no difficulty.

In my dissertation I hope to extend the preceding developments by taking into account the amplitude of the free libration, as well as by giving more accurate results as regards the number of decimals.

Chemistry. — "*Vapour pressures in the system: carbon disulphide-methylalcohol*". By Dr. E. H. BÜCHNER and Dr. ADA PRINS. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated in the meeting of March 31, 1917).

With regard to the vapour of partially miscible liquids, we find in many textbooks the following consideration for the case that the composition of the vapour lies between that of the liquid phases. When, on altering the temperature, the concentrations of the two liquids tend to the same value and, finally, become identical in a critical solution point, the vapour also, it is argued, must have the same composition at that temperature. It is then, however, tacitly assumed that the vapour, which lies at any low temperature between the liquids L_1 and L_2, remains between them at all other temperatures. This is, however, not at all the case, as KUENEN[1]) already showed some years ago with the help of VAN DER WAALS' theory. In an analytical way he proved, on the contrary, that at the critical point the vapour must have a different composition.

Also from general considerations it is easily seen that a vapour lying within the region of the two liquids must pass without, before the critical point is reached. If it did not, there would exist a point where three phases had the same composition. Now, it is already

[1]) These Proc. 6, Oct. 1903.