mit den durch Fermente hervorgerufenen Wirkungen vergleichbar, dass wir mit einiger Bestimmtheit annehmen, dass die Fähigkeiten einiger Erbeinheiten im wesentlichen in der Bildung bestimmter Substanzen bestehen, welche in der Art von Fermenten wirken".

Although the observations on which this statement is based are in accordance with the enzyme theory, it is clear that BATESON's view is quite different from mine.

Physics. — *"Contributions to the kinetic theory of solids. I. The thermal pressure of isotropic solids.* By Prof. L. S. ORNSTEIN and Dr. F. ZERNIKE. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of February 26, 1916).

P. DEBIJE[1]) has in his Wolfskehl-lecture developed a theory of the equation of state of solid matter which has been elaborated by Dr. M. I. M. VAN EVERDINGEN[2]). DEBIJE assumes as a physical principle that the forces between the molecules in solid matter are not quasi-elastic, but depend also on higher powers of the deformations. He points out that only this principle enables us to understand the expansion of solid matter which gains energy under constant pressure. This assumption enables him to give a deduction of the GRÜNEISEN-theorem about the connection between the coefficient of expansion and the specific heat.

DEBIJE calculates the free energy of a solid body with the help of a canonical ensemble, using the method of normal vibrations, and introducing from the beginning the hypothesis of energy-quanta.

We shall indicate in this paper another way to find the equation of state with the aid of the physical principles of DEBIJE. The quantum-theory will be applied to our final result if we wish to use it for low temperatures. DEBIJE has taught us to replace in the calculations the space-lattice of molecules by a continuum, BORN[3]) has shown this artifice to be right. Therefore, in considering the isotropic body, we shall use a continuum as a limiting case. For explanation we shall treat the case of a row of points and for this case we shall perform the transition to a continuous bar. Our method consists in determining the thermal pressure, i.e. the pressure that

1) Vorträge über die kinetische Theorie der Materie, Leipzig 1914. "Zustandsgleichung und Quantenhypothese u. s. w.".

2) De toestandsvergelijkingen van het isotrope vaste lichaam. Diss. Utrecht 1914.

3) M. BORN. Dynamik der Krystallgitter. Teubner. 1915,

is required to keep constant the volume of a solid body when gaining heat.

1. Let us consider a row of n equidistant points. Be the elongation in the direction x (the direction of the row being taken as an axis) for the ν^{th} point ξ_ν. Then the force exerted by the ν^{th} molecule on the $(\nu-1)^{th}$ will be represented by

$$S = f(\xi_\nu - \xi_{\nu-1}) + \frac{g}{2}(\xi_\nu - \xi_{\nu-1})^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The total potential energy, then, can be represented by

$$\varepsilon_q = \frac{f}{2}\Sigma_\nu(\xi_\nu - \xi_{\nu-1})^2 + \frac{g}{6}\Sigma_\nu(\xi_\nu - \xi_{\nu-1})^3 \quad . \quad . \quad . \quad . \quad (2)$$

where the sum has to be extended over all molecules.

Now for a stationary state, $\overline{S}$, the time-average of S, will be equal for all points. Therefore, adding (1) for all points, we get n times the time-average of S. Thus

$$n\overline{S} = \frac{g}{2}\Sigma\overline{(\xi_\nu - \xi_{\nu-1})^2} = \frac{g}{2}\overline{\Sigma(\xi_\nu - \xi_{\nu-1})^2} \quad . \quad . \quad . \quad . \quad (3)$$

the mean of the first term in (1) being zero, as the mean length is invariable, and as the taking of the mean and of the sum may be interchanged.

For the mean value of ε_q we have

$$\overline{\varepsilon_q} = \frac{f}{2}\overline{\Sigma_\nu(\xi_\nu - \xi_{\nu-1})^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the mean value of the second term being zero. We thus find

$$n\overline{S} = \frac{g}{f}\overline{\varepsilon_q} = \frac{g}{2f}\overline{\varepsilon}$$

for $\overline{\varepsilon_q} = \overline{\varepsilon_p} = \frac{1}{2}\overline{\varepsilon}$, where ε_p represents the kinetic and ε the total energy. Putting $g = n^2c_2$, $f = nc_1$, we find

$$\overline{S} = \frac{c_2}{2c_1}\overline{\varepsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For the dilatation taken from the absolute zero, we find

$$e = \frac{c_2}{2c_1^2}\overline{\varepsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

being the relation of GRÜNEISEN.

2. We shall now consider the same problem, approximating this time the problem for a row of points by that of a continuum. Therefore we have to do with a bar in which the elastic qualities depart from HOOKE's law.

The force exercised in this bar by the part to the right of a section on the part to the left, will be represented by

$$S = c_1 \frac{\partial \xi}{\partial x} + \frac{c_2}{2}\left(\frac{\partial \xi}{\partial x}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The total potential energy then amounts to

$$\varepsilon_q = \frac{c_1}{2}\int\left(\frac{\partial \xi}{\partial x}\right)^2 dx + \frac{c_2}{6}\int\left(\frac{\partial \xi}{\partial x}\right)^3 dx \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where the integration has to be extended over the length of the bar, which has been put equal to unity.

For this case the mean value $\overline{S}$ of the force is again equal for all points of the bar, and $\overline{\frac{\partial \xi}{\partial x}}$ being zero, we have

$$\overline{S} = \frac{c_2}{2}\overline{\left(\frac{\partial \xi}{\partial x}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Integrating this result over the bar, we get

$$\overline{S} = \frac{c_2}{2}\int\overline{\left(\frac{\partial \xi}{\partial x}\right)^2} dx = \frac{c_2}{2}\overline{\int\left(\frac{\partial \xi}{\partial x}\right)^2 dx}. \quad . \quad . \quad . \quad . \quad (10)$$

as also in this case the integrating and the taking of the mean value may be interchanged.

Now, just as in the discrete problem, we determine ε_q, and $\overline{\left(\frac{\partial \xi}{\partial x}\right)}$ being zero we find

$$\overline{\varepsilon_q} = \frac{c_1}{2}\overline{\int\left(\frac{\partial \xi}{dx}\right)^2 dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and from it

$$\overline{S} = \frac{c_2}{c_1}\overline{\varepsilon_q} = \frac{c_2}{2c_1}\overline{\varepsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

from which the relation of GRÜNEISEN again follows.

In calculating $\overline{\varepsilon}$ we can use the quantum-theory, but the formula (12) is evidently independent of it.

3. We will show that the same result is obtained by applying the method of normal vibrations. The differential equation for the motion of the bar is expressed by

$$\varrho \frac{\partial^2 \xi}{\partial t_2} = c_1 \frac{\partial^2 \xi}{\partial x^2} + c_2 \frac{\partial \xi}{\partial x}\frac{\partial^2 \xi}{\partial x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where ϱ represents the density. Properly speaking the equation (14), being non-linear, possesses no normal vibrations as a solution for a given bar with given conditions to be satisfied at the ends. But if c_2 is small enough, the normal vibrations of the equation without quadratic term will still have a physical meaning. These vibrations may be called quasi-normal vibrations, and the physical meaning of the constant c_2 is to effect a slow exchange of energy between the quasi-normal vibrations.

Now take the solution

$$\xi = \Sigma_k (P_k \sin kx + Q_k \cos kx) \cos kv (t - \varphi_k) \quad . \quad . \quad . \quad (15)$$

for a bar with the ends 0 and 2π; v being the velocity of propagation. The force in a point is represented by (7). Using (15) and calculating the time-average of S in the point 0, we find

$$\overline{S} = \frac{c^2}{4} \Sigma k^3 P_k^2$$

The potential energy is expressed by

$$\varepsilon_q = \frac{2\pi c_1}{4} \Sigma k^2 (P_k^2 + Q_k^2) \cos^2 kv (t - \varphi_k)$$

its time-average by

$$\overline{\varepsilon_q} = \frac{2\pi c_1}{8} \Sigma k^2 (P_k^2 + Q_k^2)$$

Now the mean value of P_k is the same as that of Q_k; therefore we get

$$\overline{\varepsilon_q} = \frac{2\pi c_1}{4} \Sigma k^2 P_k^2$$

$$\overline{S} = \frac{c_2}{2\pi c_1} \overline{\varepsilon_q} = \frac{c_2}{2c_1} \frac{\overline{\varepsilon}}{2\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

As $\frac{\varepsilon}{2\pi}$ is equal to the energy per unit of length, the result agrees with (5) and (12).

4. We can determine the thermal pressure of an isotropic solid body in the same way as in 2. For this case, we have for the potential energy per unit of volume the expression [1])

$$\varepsilon_q = AI_1^2 + BI_2 + CI_1^3 + DI_1I_2 + EI_3. \quad . \quad . \quad . \quad (17)$$

where the invariants I have the following forms

$$I_1 = e_1 + e_2 + e_3$$

[1]) For the first time indicated by J. FINGER, Wiener Sitzungsberichte **103**, 163 (1894), although in a less simple form (l. c. form (55)). Our notation is the one of v. EVERDINGEN, l. c. p. 11, where no literature is mentioned. Cf. also P. DUHEM, Recherches sur l'Elasticité. Paris 1906.

$$I_2 = e_2e_3 + e_3e_1 + e_1e_2 - \tfrac{1}{4}(e_4^2 + e_5^2 + e_6^2)$$
$$I_3 = e_1e_2e_3 + \tfrac{1}{4}(e_4e_5e_6 - e_1e_4^2 - e_2e_5^2 - e_3e_6^2)$$
$e_1 \ldots e_6$ being the "components" of the strain (changes in length and angle).

The energy of a volume is found by multiplying the expression (17) with the element $d\tau$, and integrating.

From (17) the normal stress in the direction of x can be deduced, using the formula

$$S = \frac{\partial \varepsilon_q}{\partial e_1}\,[1]$$

We can only observe the time-average of this force and, taking the mean value, the linear parts issuing from the terms with A and B will fall out. We obtain therefore for the average value of the tension in the direction of x

$$\overline{S} = \overline{3CI_1^2 + DI_1(e_2 + e_3) + DI_2 + E(e_2e_3 - \tfrac{1}{4}e_4^2)}\,.$$

This force is again equal for all points, we can therefore integrate over the volume 1, and interchange the integration and the taking of the average. Taking into account the isotropy, it is easily seen that $\overline{e_2e_3 - \frac{1}{4}e_4^2}$ is equal to $\frac{1}{3}\overline{I_2}$, so that we get for $\overline{S}$

$$\overline{S} = \overline{\int \left\{(3C + \tfrac{2}{3}D)\,I_1^2 + (D + \tfrac{1}{3}E)\,I_2\right\} d\tau} \quad . \quad . \quad . \quad (18)$$

Now determining the mean value of the potential energy, the terms with C, D, and E will be found to fall out, and we get two parts, relating respectively to the longitudinal and transverse waves, as appears from the meaning of the invariants I_1 and I_2. These parts are

$$\overline{\varepsilon}_{q\,l} = A\overline{\int I_1^2\, d\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19a)$$

and

$$\overline{\varepsilon}_{q\,tr} = B\overline{\int I_2\, d\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19b)$$

In the stationary state the potential energy is distributed in a given way over these waves.

For the thermal pressure we now find

$$-\overline{S} = -\frac{3C+\frac{2}{3}D}{A}\,\overline{\varepsilon}_{q\,l} - \frac{D+\frac{1}{3}E}{B}\,\overline{\varepsilon}_{q\,tr} \quad . \quad . \quad . \quad . \quad (20)$$

---

[1]) This is the usual formula, which is, however, not correct if the second powers of the deformations are taken into account, as has been done in the above by introducing C, D, and E. In the third contribution we shall show that even in case of HOOKE's-law being true, a coefficient of expansion will be found, if the correct formula for S is used. As far as numerical values are known, they seem to indicate that the influence of the terms neglected here could sometimes be sensible.

In the notation of VOIGT we have $A=\frac{1}{2}c_{11}$, $B=-2c_{44}$.

If the temperature is high enough for the theorem of equipartition to be true, then

$$\overline{\varepsilon_{q\,l}}=\tfrac{1}{6}\,\varepsilon \text{ and } \overline{\varepsilon_{q\,tr}}=\tfrac{1}{3}\,\varepsilon$$

where ε is the total energy.

For the thermal pressure we then find

$$-\overline{S}=-\left\{\frac{9C+2D}{18A}+\frac{3D+E}{9B}\right\}\varepsilon \quad . \quad . \quad . \quad . \quad (20a)$$

We shall also use (20) for very low temperatures. According to BORN [1]) the proportion of the energy of the longitudinal and transversal waves can be put in the form

$$\varepsilon_{q\,l}:\varepsilon_{q\,tr}=\frac{1}{v_l^3}:\frac{2}{v_{tr}^3}$$

where v_l and v_{tr} are the velocity of propagation of these waves. Introducing the constants c_{11} and c_{44}, we thus have

$$\varepsilon_{q\,l}:\varepsilon_{q\,tr}=c_{44}^{3/2}:2c_{11}^{3/2}\,.$$

Putting the total energy ε, we find

$$\varepsilon_{q\,l}=\frac{c_{44}^{3/2}}{2\,c_{44}^{3/2}+4\,c_{11}^{3/2}}\,\varepsilon \qquad \varepsilon_{q\,tr}=\frac{2\,c_{11}^{3/2}}{2\,c_{44}^{3/2}+4\,c_{11}^{3/2}}$$

and finally

$$\overline{S}=\frac{(3c+\frac{2}{3}D)\,c_{44}^{5/2}-(\frac{1}{2}D+\frac{1}{6}E)\,c_{11}^{5/2}}{c_{44}c_{11}\,(c_{44}^{3/2}+2c_{11}^{3/2})}\,\varepsilon \quad . \quad . \quad . \quad . \quad (20b)$$

This special result agrees with the expression found by VAN EVERDINGEN [2]). The theorem of GRÜNEISEN can be immediately deduced from it.

The influence of temperature on the elastic constants can be examined in the same way, as we shall show in the third contribution.

Utrecht, Febr. 1916. *Institute for mathematical physics.*

[1]) BORN l. c. p. 75.

[2]) VAN EVERDINGEN, l. c. p. 24 form (20) p. 53 form (37).

Physics. *"Contributions to the kinetic theory of solids. II. The unimpeded spreading of heat even in case of deviations from* HOOKE's *law".* By Prof. L. S. ORNSTEIN and Dr. F. ZERNIKE. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of March 25, 1916.)

1. In a supplement to his lecture on the equation of state of the solid body, DEBIJE[1]) has endeavoured to "make a qualitative theoretical calculation of the coefficient of conduction of heat." There the author points out repeatedly that his estimations are only to be taken very approximately and should serve as a first orientation only. So, as we tried to obtain an accurate calculation of the conduction of heat, it did not seem desirable to us to deal with the problem in exactly the same way and to carry out only here and there some corrections and completions.

Now DEBIJE's principle, which we therefore intended to work out otherwise, runs as follows. In an ideal solid body, i.e. a solid for which the elastic equations would be linear, various progressive waves may exist independently of each other, like the electromagnetic waves in a field of radiation. This implies that a heat-motion occurring on one side of the solid spreads unimpededly through the solid, so that the density of energy becomes equal in all parts of the solid. If the solid is in a stationary state, the temperature will thus be everywhere the same, even if continually a current of energy moves through the solid in a definite direction. Hence DEBIJE emphasizes this dictum: *the coefficient of heat conduction of the ideal solid body is infinitely great* (l.c. § 7, cf. the statement given there). Now in several regards it is preferable to formulate the rule in this way: the ideal solid body does not show any resistance of heat.

That a real solid body does show resistance of heat DEBIJE ascribes to the fact that the elastic equations are not perfectly linear. Therefore various normal vibrations strictly cannot be superposed and it is conceivable that waves running in different directions so to say oppose each other. DEBIJE has indeed succeeded in deducing indirectly a scattering and consequently a suppression of the running waves.

Our endeavours to state more directly the connection between resistance of heat and non-linear terms of the elastic equations of motion have failed. Therefore we will not report our considerations

[1]) Mathematische Vorlesungen an der Universität Göttingen VI (WOLFSKEHL-Vorträge) pg. 19.