noticing that in the deduction of (8) the fact that we have a homogeneous system has not been used. The given relation holds also in a capillary layer. However $f_{\tau\tau}$ will depend on a parameter in the direction of the layer $\left(\text{for } \frac{dp}{d\varrho} \text{ depends on it}\right)$. The consideration may easily be extended to the case of a mixture and the capillary layers in a mixture. It will be possible then to develop MANDELSTAMM's [1]) considerations on the diffuse reflection at the layer of contact between two liquid phases in the critical point of mixture more exactly than he himself has done.

Utrecht, Mei 1916.

Physics. — *"The dilatation of solid bodies by heat."* By Prof. H. A. LORENTZ.

(Communicated in the meeting of October 30, 1915.)

When in the theory of specific heat the idea had been worked out that the heat motion of solid bodies consists in vibrations of the particles under the influence of the same forces that give rise to the phenomena of elasticity, DEBYE [2]) successfully attacked the problem of thermal dilatation. In his theory, which has been further developed by M. J. M. VAN EVERDINGEN [3]), it is shown that this phenomenon may be accounted for in a satisfactory way by adding in the expression for the potential energy of the body terms which are of the third order with respect to the displacements of the particles.

In the present paper considerations similar to those of DEBYE and VAN EVERDINGEN are presented in a form that is perhaps somewhat simpler.

§ 1. We shall suppose the body to be isotropic or crystallized in the regular system. Let S be its surface and v its volume at the temperature T and under a uniform pressure p. We can imagine that the particles lying on the surface are kept fixed in the positions about which they vibrate and that, when this has been done, the

[1]) Ann. der Phys. 42.

[2]) P. DEBYE, Zustandsgleichung und Quantenhypothese, Wolfskehl-Vorträge, Göttingen, 1913, p. 17; Leipzig, Teubner.

[3]) M. I. M. VAN EVERDINGEN, De toestandsvergelijking van het isotrope, vaste lichaam. Proefschrift, Utrecht, 1914.

inner particles are likewise deprived of their heat motion. Then they will take definite positions of equilibrium P, P', P'' etc. This configuration of the system may be considered as the result of a dilatation equal in all directions, starting from the configuration that would exist at the absolute zero and in the absence of external pressure.

Now, always keeping fixed the outer particles, we may investigate the vibrations of the inner ones about their just mentioned positions of equilibrium $P, P', P'' \ldots$ This is a perfectly definite problem. It can be simplified in the well known way by the introduction of a certain number of normal coordinates $q_1, q_2 \ldots q_s$, which we shall choose in such a manner that they are 0 in the position of equilibrium, so that they determine the deviation from that position. The corresponding velocities are $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_s$ and if the values of the coordinates are not too great we have for the potential energy U and the kinetic energy T expressions of the form

$$U = \tfrac{1}{2}(a_1 q_1^2 + a_2 q_2^2 + \ldots + a_s q_s^2), \quad \ldots\ldots (1)$$

$$T = \tfrac{1}{2}(c_1 \dot{q}_1^2 + c_2 \dot{q}_2^2 + \ldots + c_s \dot{q}_s^2), \quad \ldots\ldots (2)$$

with positive, constant coefficients. Further there are s fundamental modes of vibration. In the first of these all the normal coordinates except q_1 are 0, in the second all except q_2 and so on. The frequencies of these fundamental vibrations are determined by

$$n_1 = \sqrt{\frac{a_1}{c_1}}, \quad n_2 = \sqrt{\frac{a_2}{c_2}}, \ldots \quad n_s = \sqrt{\frac{a_s}{c_s}}.$$

As to the deviation of a particle from the position of equilibrium, its components for the first particle may be represented by

$$\left.\begin{aligned} \xi &= \alpha_1 q_1 + \alpha_2 q_2 + \ldots + \alpha_s q_s, \\ \eta &= \beta_1 q_1 + \beta_2 q_2 + \ldots + \beta_s q_s, \\ \zeta &= \gamma_1 q_1 + \gamma_2 q_2 + \ldots + \gamma_s q_s, \end{aligned}\right\}$$

for the second by

$$\left.\begin{aligned} \xi' &= \alpha'_1 q_1 + \alpha'_2 q_2 + \ldots + \alpha'_s q_s, \\ \eta' &= \beta'_1 q_1 + \beta'_2 q_2 + \ldots + \beta'_s q_s, \\ \zeta' &= \gamma'_1 q_1 + \gamma'_2 q_2 + \ldots + \gamma'_s q_s, \end{aligned}\right\} \quad \ldots\ldots (3)$$

and so on.

Here the coefficients α, β, γ have definite constant values. As the number s of the normal coordinates is equal to that of the degrees of freedom of the system, viz. to three times the number of the particles that are assumed to be movable, all possible displacements $\xi, \eta, \zeta, \xi', \eta', \zeta', \ldots$ may be represented by suitably chosen values of $q_1, q_2, \ldots q_s$.

§ 2. We shall now ascribe a greater mobility to the system by imagining that the outer particles too can be displaced, with the restriction, however, that for any instant their coordinates can be found by multiplying by the same factor $1+q$ the coordinates which they had in the case considered in the preceding paragraph. Then, the quantity q, which we shall suppose to be very small compared with 1, will determine the position of the outer particles and by suitably extending the meaning of $q_1, \ldots . q_s$, these parameters may be made, together with q, to determine the position of the entire system.

Indeed, let $\overline{P}$, $\overline{P}'$, $\overline{P}''$, ... be the points which are found if the coordinates of P, P', P'', ... (§ 1) are altered in ratio of 1 to $1+q$ simultaneously with the coordinates of the outer points, let $\xi, \eta, \zeta, \xi', \eta', \zeta' \ldots$ be the components of the deviations of the particles from *these* positions $\overline{P}$, $\overline{P}'$, $\overline{P}''$, ... and let q_1, $q_2 \ldots q_s$ be quantities connected with $\xi, \eta, \zeta, \xi', \eta', \zeta'$ in the way shown by equations (3), if we continue to assign to α, β, γ the values we had to give them in the preceding paragraph. Then it is clear that the configuration of the whole system is really determined by $q, q_1 \ldots q_s$. The quantity q being now considered as variable, so that, though the places of the outer particles *in* the surface S be prescribed, this surface, keeping the same form, may dilate or contract as a whole, a constant value of q, i.e. a constant volume, can in general be maintained only by the application of an external force Q corresponding to that coordinate. It is precisely this force which we want to know, especially for the case $q=0$, i. e. for the configuration of the body with which we began in § 1.

The value of Q is connected with that of the external pressure, for Q is defined by the condition that, for an infinitesimal variation δq of the coordinate q, the other coordinates remaining unchanged, the work of the external forces is $Q\delta q$. If now this change takes place starting from the value $q=0$, all dimensions of the surface increase in ratio of 1 to $1+\delta q$. The volume increases by $3v.\delta q$ and the work of the external pressure is $-3pv.\delta q$. Hence

$$Q = -3pv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

§ 3. The force Q may be determined by means of the equations of LAGRANGE, as soon as the potential energy U and the kinetic energy T are known as functions of all the coordinates q and the corresponding velocities.

Then we have

$$Q = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) \quad \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q},$$

where we must remark that the first term may be omitted. Indeed, whatever be the value of

$$\frac{\partial T}{\partial \dot{q}},$$

it certainly will be determined by the state of the body and its variations will therefore be limited to small changes on both sides of a certain mean value when the state is stationary. Then, however, the mean value of the differential coefficient $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right)$ taken over a time of sufficient length, will be equal to zero. We hardly need remark that it is such a mean value of the pressure p and therefore of the force Q, which we want to find.

So we may write

$$Q = -\frac{\partial T}{\partial q} + \frac{\partial U}{\partial q},$$

and, as we are seeking the value for the case $q = 0$, $\dot{q} = 0$, we may directly introduce this latter value into T and confine ourselves to terms with the first power of q when we represent U by a series.

By putting $\dot{q} = 0$, the points $\overline{P}, \overline{P}', \ldots$ of which we spoke in § 2 become immovable, so that we shall find the velocities of the particles by differentiating with respect to the time their deviations $\xi, \eta, \zeta, \xi', \eta', \zeta' \ldots$ from the positions $\overline{P}, \overline{P}', \ldots$ As now the coefficients α, β, γ in equations (3) are constants, the coordinate q does not appear in the expressions for the velocities and neither in T. This leads to a further simplification, viz.

$$Q = \frac{\partial U}{\partial q} \quad \ldots\ldots\ldots \quad (5)$$

§ 4. If, in the series for the potential energy, we confine ourselves, as we did in § 1, to the terms that are of the second order with respect to $q_1, q_2 \ldots q_s$, we may put

$$U = \tfrac{1}{2}(a_1 q_1^2 + a_2 q_2^2 + \ldots + a_s q_s^2) + (A_0 + A_1 + A_2)\, q \quad . \quad (6)$$

It is evident that the first term, which is to represent the potential energy for $q = 0$, must be the expression (1). Further A_0 is a constant, A_1 a homogeneous linear function of the coordinates $q_1, q_2 \ldots q_s$ and A_2 a homogeneous quadratic function of these same variables.

We have therefore, by (5)

$$Q = A_0 + A_1 + A_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In this equation we must take for $q_1, q_2, \ldots q_s$ the values as they are in the heat motion such as it really is. As now in the case of oscillations the mean value of each coördinate q over a long interval of time is 0, the term A_1 may be omitted.

As to A_0, this term represents the value of the force Q that would be required for maintaining the assumed value v of the volume, in case all the coordinates $q_1, \ldots q_s$ were 0, so that there would be no heat motion. For this force we may find an expression if we introduce the volume v_0 that the body would have at the absolute zero if it were free from external forces. In order to maintain at this same temperature the volume v, which we shall suppose to be greater than v_0, a negative pressure would have to be exerted on the body. It may be represented by

$$p = -\frac{v - v_0}{\varkappa v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $\varkappa$ is a certain mean coefficient of cubical compressibility.

Substituting this in (4) we find

$$Q = \frac{3(v - v_0)}{\varkappa},$$

and this is the value of A_0. Thus, if there is a heat motion, we have according to (7)

$$Q = \frac{3(v - v_0)}{\varkappa} + A_2.$$

If finally we want to know what volume the body will occupy in the case of a heat motion, and in the absence of an external pressure, we have only to put $Q = 0$. We then find

$$v - v_0 = -\tfrac{1}{3}\varkappa A_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

for the connection between the heat motion and the volume, which it was our object to deduce.

§ 5. As to the meaning of A_2 we must remember that the part of the potential energy which contains terms of the second order with respect to $q_1, q_2 \ldots q_s$, will be

$$\tfrac{1}{2}(a_1 q_1^2 + a_2 q_2^2 + \ldots + a_s q_s^2) + qA_2$$

when the volume has increased to the extent determined by q.

After this expansion to the volume $(1 + 3q)v$ the coordinates $q_1, q_2, \ldots q_s$ need no longer be normal coordinates as they were for the volume v; so that A_2 may also contain products $q\, q_j$. As however the fundamental vibrations which constitute the heat motion,

must be regarded as incoherent in phase, products of this kind will vanish from the mean value of A_2. So we obtain the right result, if we put

$$A_2 = \tfrac{1}{2}(a'_1 q_1^2 + a'_2 q_2^2 + \ldots + a'_s q^2 s).$$

Therefore, according to (9), each harmonic mode of vibration contributes its part to the dilatation $v - v_0$, independently of the other modes.

The first of these parts is

$$-\tfrac{1}{6}\varkappa a'_1 q_1^2,$$

for which we may write

$$-\tfrac{1}{3}\varkappa \frac{\partial}{\partial q}(\tfrac{1}{2} a_1 q_1^2 + \tfrac{1}{2} q a'_1 q_1^2)$$

or on account of the connection between q and the volume

$$-\varkappa v \cdot \frac{\partial}{\partial v}(\tfrac{1}{2} a_1 q_1^2 + \tfrac{1}{2} q a'_1 q_1^2) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now $\frac{1}{2} a_1 q_1^2$ is the value of the potential energy u_1 that belongs to the first coordinate during the heat motion in the state considered and $\frac{1}{2} a_1 q_1^2 + \frac{1}{2} q a'_1 q_1^2$ the value which this potential energy would have, if after the increase in volume determined by q the particles had the same deviations determined by q_1, from the positions $\overline{P}, \overline{P'}, \overline{P''}, \ldots$ specified in § 2.

Thus we may write for (10)

$$-\varkappa v \frac{\partial u_1}{\partial v}.$$

To calculate the differential coefficient we must attend only to the first coordinate q_1, putting 0 for all the others.

Further, in performing the differentiation we must imagine that in the original volume v the particles have the deviations from their positions of equilibrium which, in the real heat motion, correspond to the first mode of vibration and that, after an infinitesimal increase of the volume they have the same deviations from the new positions of equilibrium $\overline{P}, \overline{P'}, \overline{P''}, \ldots$

Proceeding in the same way with respect to the other coordinates, we obtain

$$v - v_0 = -\varkappa v\left(\frac{\partial u_1}{\partial v} + \frac{\partial u_2}{\partial v} + \ldots + \frac{\partial u_s}{\partial v}\right) \quad . \quad . \quad . \quad . \quad (11)$$

§ 6. The calculation of the thermal dilatation by means of this formula will necessarily be a rather rough one. In the first place it is very questionable whether for somewhat high temperatures we

may confine ourselves to terms of the second order with respect to the inner coordinates, and even if this were allowed, the difficulty would remain that we do not know enough about the forces acting between the particles to calculate the differential coefficients $\frac{\partial u}{\partial v}$. For the modes of vibration in which the wave-length is many times greater than the distances between neighbouring particles, these forces, so far as they have to be considered here, are determined by the ordinary elastic constants. If, however, the wave-length becomes of the same order of magnitude as those distances, this of course will no longer be so and unfortunately these very short waves are most prominent in the heat motion.

In his theory of specific heat however DEBYE, not withheld by this consideration, has applied the ordinary formulae of the theory of elasticity to all the modes of motion with which he was concerned, down to the shortest waves. Encouraged by his success we may avail ourselves of the same simplification in the theory of dilatation as has been done already by him and VAN EVERDINGEN. This enables us to continue the calculation of the right hand side of (11).

§ 7. We shall introduce the two constants of elasticity λ and μ, which are also used by DEBYE and which have been chosen in such a way that the potential energy per unit of volume is represented by the expression

$$\mu (x_x^2 + y_y^2 + z_z^2) + \tfrac{1}{2} \lambda (x_x + y_y + z_z)^2 + \tfrac{1}{2} \mu (x_y^2 + y_z^2 + z_x^2) \quad (12)$$

where

$$x_x = \frac{\partial \xi}{\partial x}, \ldots \qquad x_y = \frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}, \ldots$$

We remark that, if χ is any one of these six components of strain, or a homogeneous linear function of some of them, we may write

$$\frac{\partial \log \chi}{\partial \log v} = -\tfrac{1}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This is evident, if we keep in mind that, in the infinitesimal expansion determined by q, the quantities ξ, η, ζ are kept constant, so that their differential coefficients with respect to the coordinates are changed in ratio of 1 to $(1 + q)^{-1}$.

The modes of vibration of which the heat motion consists, may be divided into two groups, that of the longitudinal and that of the transverse vibrations.

Now, if ω is an element of volume, the potential energy v_l contained

in it and proper to a mode of motion of the first group, is proportional to an expression of the form

$$(\lambda + 2\mu)\,\chi^2\omega,$$

while the potential energy v_t belonging to a mode of the second group is proportional to

$$\mu\,\chi^2\,\omega.$$

As ω changes proportionally to v, we have in virtue of (13)

$$\frac{\partial\, log\, v_l}{\partial\, log\, v} = \frac{d\, log\,(\lambda+2\mu)}{d\, log\, v} + \tfrac{1}{3},$$

$$\frac{\partial\, log\, v_t}{\partial\, log\, v} = \frac{d\, log\,\mu}{d\, log\, v} + \tfrac{1}{3},$$

and this leads to similar relations for the potential energy u_l, u_t contained in the whole body. We may write them in the form

$$\frac{\partial u_l}{\partial\, log\, v} = \left\{\frac{d\, log\,(\lambda+2\mu)}{d\, log\, v} + \tfrac{1}{3}\right\} u_l\,, \quad \ldots\ldots \quad (14)$$

$$\frac{\partial u_t}{\partial\, log\, v} = \left\{\frac{d\, log\,\mu}{d\, log\, v} + \tfrac{1}{3}\right\} u_t \quad \ldots\ldots\ldots \quad (15)$$

These formulae, of which the first may be used for all the terms in (11) that correspond to longitudinal motions and the second for all those that refer to transverse motions, also hold for the mean values which we have to take on the right hand side of (11). The mean values both of u_l and of u_t however are each half the total energy, and to this latter we must assign, both for the longitudinal and the transverse vibrations, the value ε_ν which depends on the frequency ν in the way specified in PLANCK's formula.

§ 8. Let us now first consider the terms on the right hand side of (11) that belong to modes of motion with frequencies between ν and $\nu + d\nu$. Let N be the total number of these modes, gN the number of those in which the vibrations are longitudinal and hN the number of those which consist in transverse vibrations, so that $g + h = 1$. To obtain

$$\Sigma \frac{\partial u}{\partial\, log\, v} \quad \ldots\ldots\ldots \quad (16)$$

for this group of terms we must multiply (14) and (15) by gN and hN respectively and then take the sum, replacing at the same time u_l and u_t by their common value $\frac{1}{2}\varepsilon_\nu$. We shall also substitute for g and h the values that follow from DEBYE's calculations. He has found that the number of the longitudinal and that of the transverse modes of motion for which the frequency lies below an arbitrarily

chosen limit are to each other in ratio of $(\lambda + 2\mu)^{-3/2}$ to $2\mu^{-3/2}$. As this is independent of v it is also the ratio between the fractions g and h. Performing the calculations indicated we find for the sum (16)

$$\Sigma \frac{\partial u}{\partial \log v} = \left[-\tfrac{1}{3} \frac{d \log \{(\lambda+2\mu)^{-3/2} + 2\mu^{-3/2}\}}{d \log v} + \tfrac{1}{6} \right] N\varepsilon_v .$$

To derive from this the sum

$$\frac{\partial u_1}{\partial \log v} + \frac{\partial u_2}{\partial \log v} + \ldots + \frac{\partial u_s}{\partial \log v}$$

which occurs in (11) we have still to extend the summation to all the modes of motion of different frequencies. As now $\Sigma N\varepsilon_v$ is the total energy E of the heat motion, (11) becomes

$$v - v_0 = \varkappa \left[\tfrac{1}{3} \frac{d \log \{(\lambda+2\mu)^{-3/2} + 2\mu^{-3/2}\}}{d \log v} - \tfrac{1}{6} \right] E \quad . \quad . \quad (17)$$

In this formula we must give as well to $\varkappa$ as to the elastic constants λ and μ the values they would have if there were no heat motion, the volume being v, and strictly speaking it ought to be taken into account that these quantities and therefore the coefficient by which E is multiplied are more or less dependent on that volume; by this the equation becomes rather complicated. The simplest results will be obtained for very low temperatures. For these E is proportional to T^4. Hence, if we assume that the coefficient of E may be represented by a series

$$C_1 + C_2 (v - v_0) + \ldots$$

we may conclude that quite near the absolute zero $v - v_0$ is proportional to T^4 and the coefficient of dilatation $\frac{1}{v_0} \frac{dv}{dT}$ to T^3.

§ 9. The equation obtained for the dilatation can be still further simplified if one makes the assumption, rather arbitrary of course, that by an isotropic dilatation the coefficients λ and μ are made to change proportionally to each other. The coefficient of compressibility (for an infinitesimal change of volume) which has the value

$$\frac{3}{3\lambda + 2\mu}$$

and with which, in a rough approximation, we may identify the coefficient $\varkappa$ occurring in our formula, will then change in the inverse ratio to μ. We may also say that the quantity of which the logarithm appears in the numerator of the first fraction in (17) changes proportionally to $\varkappa^{3/2}$. Hence, denoting the pressure by p and using the relation

$$d \log v = - \varkappa \, dp$$

we have

$$v - v_0 = \left[- \tfrac{1}{2} \frac{d \log \varkappa}{dp} - \tfrac{1}{6} \varkappa \right] E,$$

and if the coefficient is treated as independent of the temperature,

$$\frac{dv}{dT} = \left[- \tfrac{1}{2} \frac{d \log \varkappa}{dp} - \tfrac{1}{6} \varkappa \right] \frac{dE}{dT}.$$

If now ϱ is the density of the body, c the specific heat (the difference between c and c_v being considered as immaterial) expressed in calories and A the mechanical equivalent of heat, we have $\frac{dE}{dT} = A c \varrho v$, and for the coefficient of cubical expansion

$$\alpha = \frac{1}{v} \frac{dv}{dT} = \left[- \tfrac{1}{2} \frac{d \log \varkappa}{dp} - \tfrac{1}{6} \varkappa \right] A c \varrho \, , \quad . \quad . \quad . \quad . \quad (18)$$

a value that can well be positive, as the compressibility decreases with increasing pressure.

§ 10. An example may teach us, whether this result agrees with observation, at least as to the order of magnitude.

According to the measurements of LUSSANA [1]) the compressibility of lead decreases by about $\frac{1}{30}$ of its value when the pressure is raised to 1000 atmospheres. Therefore we have, taking the atmosphere as unit of pressure

$$\frac{d \log \varkappa}{dp} = - 3{,}3.10^{-5}$$

and if p is expressed in dynes per cm^2

$$\frac{d \log \varkappa}{dp} = - 3{,}3.10^{-11}.$$

For the compressibility itself LUSSANA's value is $\varkappa = 3{,}9.10^{-12}$, so that the coefficient of $Ac\varrho$ in (18) becomes equal to $1{,}6.10^{-11}$. With $A = 4{,}18.10^7$; $c = 0{,}03$ and $\varrho = 11$ we find $\alpha = 0{,}00022$, while in reality the coefficient of expansion is 0,00008.

For tin LUSSANA's observations lead to the numbers

$$\frac{d \log \varkappa}{dp} = - 3{,}7.10^{-11}, \qquad \varkappa = 2{,}7.10^{-12}.$$

Here $c = 0{,}05$ and $\varrho = 7{,}3$. This gives $\alpha = 0{,}00027$. The coefficient of expansion is 0,00006.

It is seen that the agreement is scarcely satisfactory.

[1]) Taken from W. SCHUT, Piëzochemie der gecondenseerde systemen, p. 72. Proefschrift, Utrecht, 1912.

§ 11. For a few metals the value of $\frac{d\mu}{d \log v}$ can be derived from measurements made by POYNTING.[1]) This physicist has investigated the changes in length and diameter caused by the torsion of a wire. We shall shortly discuss this phenomenon, not only with a view to the numerical value that follows from it, but also because the theory shows a certain analogy with that of the dilatation by heat.

Let us consider a cylindrical wire, the axis of which we take for the z-axis, and let us suppose that, starting from the unstrained state, it is subjected to the following three deformations: 1. a homogeneous stretch in the direction of the length, 2. a displacement of the particles in radial direction, so that the distance r of a particle from the axis changes by sr, s being a function of r, and 3. a torsion, by which each cross-section normal to the axis is turned over an angle ϑz about its point of intersection with that line; then ϑ is the angle of torsion per unit of length.

Supposing the temperature to be kept constant we shall seek the free energy of the body in the final state reached by these three steps. Assuming it to be 0 in the original state we can calculate its changes by means of (12) or of similar expressions.

As the second of the three changes produces a stretch $\frac{d(sr)}{dr}$ in radial and a stretch s in tangential direction, we obtain the free energy that exists per unit of volume after the first two steps if we replace x_x, y_y, z_z in the first two terms of (12) by $s, s + r\frac{ds}{dr}$ and q.

The result is

$$\mu\left\{2s^2 + 2rs\frac{ds}{dr} + r^2\left(\frac{ds}{dr}\right)^2 + q^2\right\} + \tfrac{1}{2}\lambda\left(2s + r\frac{ds}{dr} + q\right)^2 . \quad (19)$$

A point that originally was at a distance r from the axis, has now shifted to the distance $r' = (1 + s)r$, while an element of the length dl has become $dl' = (1 + q)\,dl$.

By the first two changes an annular element between two cylindric surfaces described about the axis with the radii r and $r + dr$, and further limited by two cross sections at a distance dl from each other, will have taken a volume for which with the approximation required for our calculation we may write

[1]) POYNTING, On the changes in the dimensions of a steel wire when twisted, and on the pressure of distortional waves in steel, Proc. Royal Soc. (A) **86** (1912), p. 534.

$$2\pi\left(1 + 2s + r\frac{ds}{dr} + q\right) r\,dr\,dl \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now to obtain the free energy in the state S that is reached by the first two deformations we should have to multiply (19) by this expression (20), and then to integrate it with respect to r and l. For *this* calculation however we may replace (20) by $2\pi r\,dr\,dl$, because in the expression for the free energy we shall omit terms that are of an order higher than the second with respect to q and s.

§ 12. To calculate now the change of the free energy accompanying the third deformation specified in § 11, we shall consider the state S as the original one and introduce elastic constants referring to it. On account of the preceding deformations determined by q and s, these constants are a little different from the values λ and μ introduced into (12). To find an expression for them we regard the quantities q and s as infinitely small and neglect their second and higher powers. The change we are investigating being proportional to ϑ^2, we obtain in this way terms with $q\vartheta^2$ and $s\vartheta^2$.

A point which in the state S has the coordinates x, y, z and lies at a distance r' from the axis, is displaced by the torsion ϑ over the distances

$$\xi = -\vartheta yz, \qquad \eta = +\vartheta xz, \qquad \zeta = 0,$$

to which correspond the components of strain

$$x_x = 0, \quad y_y = 0, \quad z_z = 0, \quad x_y = 0, \quad x_z = -\vartheta y, \quad y_z = +\vartheta x.$$

Let us now consider an element of volume which in the state S lies at a distance r' from the axis and for which $x=0$, $y=r'$. The preceding changes have given to this element the stretches $\mathbf{x} = s$, $\mathbf{y} = s + r\frac{ds}{dr}$ and $\mathbf{z} = q$ in the direction of the axes, without other changes of form. By the torsion it is now further subjected to a shear $x_z = -\vartheta r'$.

It is evident that the change in free energy per unit of volume caused by this shear will be obtained by multiplying $\frac{1}{2}x_z^2$ by a coefficient μ', which is the coefficient of rigidity μ as it has been modified by the dilatations $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. In calculating this modification we may treat $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ as infinitely small. It can be shown that

$$\mu' = \mu(1 + 2\mathbf{z}) + a(\mathbf{x} + \mathbf{z}) + b\mathbf{y}, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where a and b are two constants depending on the nature of the material [1]). In this way we find for the change of the free energy

[1]) In my original paper I had used a wrong formula, in which the term $2\mu\mathbf{z}$ was wanting, an error that has been pointed out by Mr. Tresling in his paper: On the use of third degree terms in the energy of a deformed elastic body. (These

per unit of volume caused by the torsion

$$\tfrac{1}{2}\left\{\mu(1+2q+2s)+a(q+s)+b\left(s+r\frac{ds}{dr}\right)\right\}\vartheta^2 r^2,$$

Proceedings. **19** (1916), p. 281). I shall avail myself of the occasion of this translation for introducing the corrections necessitated by his remark.

In deducing the new equation (21) we need not occupy ourselves with the term $b\,\mathbf{y}$; we have only to show that

$$\mu' = \mu(1+2\mathbf{z}) + a(\mathbf{x}+\mathbf{z}),$$

if $\mathbf{y}=0$. By this latter assumption the problem is reduced to one in two dimensions, which may be treated as follows.

Let x, z be the coordinates of a point in the original state and $x+\xi$, $z+\zeta$ its coordinates in the strained state, the displacements ξ, ζ being functions of x, z. We shall consider the free energy per unit of volume at the point $x+\xi$, $z+\zeta$, as compared with the free energy which we had originally in unit of volume.

The difference ψ must be a function of the quantities

$$\frac{\partial\xi}{\partial x}=a_1,\ \frac{\partial\zeta}{\partial z}=a_2,\ \frac{\partial\xi}{\partial z}=b_1,\ \frac{\partial\zeta}{\partial x}=b_2$$

and can be developed in ascending powers of these, the series beginning with quantities of the second order and terms of the third order being necessary for our purpose.

As we may assume that the free energy is the same in the body considered and in a second body that is the image of the first with respect to a plane perpendicular to one of the axes of coordinates, the expansion can contain no terms that are of an odd order with respect to b_1 and b_2. Moreover the value of ψ must remain the same when the axes are rotated in their plane. These considerations lead to the formula

$$\psi = f(a_1+a_2)^2 + g(b_1-b_2)^2 + h(a_1a_2-b_1b_2) + k(a_1+a_2)^3 + \\ + l(a_1+a_2)(b_1-b_2)^2 + m(a_1+a_2)(a_1a_2-b_1b_2)$$

with six constants f, g, h, k, l, m, which can be easily verified. Indeed it can be shown that the values of a_1+a_2, b_1-b_2 and $a_1a_2-b_1b_2$ are not altered by a rotation of the axes.

Let us next suppose the body, strained already in the way determined by a_1, a_2, b_1, b_2, to be rotated about OY through an infinitely small angle ω. This rotation, which must leave the value of ψ unchanged, leads to the variations

$$\delta\xi = -\omega(z+\zeta),\quad \delta\zeta = \omega(x+\xi),$$

$$\delta(a_1+a_2)=\omega(b_1-b_2),\quad \delta(b_1-b_2)=-2\omega-\omega(a_1+a_2),$$
$$\delta(a_1a_2-b_1b_2)=-\omega(b_1-b_2).$$

Substituting these values in $\delta\psi$ and putting equal to 0 the coefficients of the terms that are of the first and the second order with respect to a_1, a_2, b_1, b_2, one is led to the relations

$$h=-4g,\quad m=2f-2g-4l,$$

so that

where r' has been replaced by $(1+s)\,r$ and where, as to q and s we have neglected terms of orders higher than the first.

Multiplying this by (20), integrating over the cylinder and adding the result to the free energy in the state S, of which the value has been found already, we obtain for the free energy in the final state

$$\psi = 2\pi l \int_0^R \left[\mu \left\{ 2s^2 + 2rs\frac{ds}{dr} + r^2\left(\frac{ds}{dr}\right)^2 + q^2 \right\} + \tfrac{1}{2}\lambda\left(2s + r\frac{ds}{dr} + q\right)^2 \right] r\,dr$$

$$+ \frac{\pi\Theta^2}{l}\int_0^R \left[\mu\left(1 + 4s + r\frac{ds}{dr} + q\right) + a(q+s) + b\left(s + r\frac{ds}{dr}\right)\right] r^3\,dr \quad . \quad (22)$$

where the original length and radius are denoted by l and R, and the total angle of torsion by Θ, so that $\vartheta = \dfrac{\Theta}{(1+q)l}$.

§ 13. Now Θ, q and the value $\mathfrak{s}$, which s assumes for $r = R$, may be regarded as the parameters upon which external forces can act directly. If these parameters are kept fixed, we can determine the values of s within the wire by means of the condition that, for an arbitrary infinitesimal variation δs given to them, $\delta\psi$ must be 0.

For constant values of Θ and q we have by (22)

$$\delta\psi = \int_0^R G\,\delta s\,dr + \int_0^R F\,\frac{d\delta s}{dr}\,dr, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where

$\psi = f(a_1 + a_2)^2 + g(b_1 - b_2)^2 - 4g(a_1 a_2 - b_1 b_2) + k(a_1 + a_2)^3 + l(a_1 + a_2)(b_1 - b_2)^2 + (2f - 2g - 4l)(a_1 + a_2)(a_1 a_2 - b_1 b_2)$.

In the case considered in the text the final values of ξ and ζ (after the application of the torsion) are

$$\xi = \mathfrak{x}\,x - \vartheta(1+\mathfrak{z})\,r'z = \mathfrak{x}\,x + x_z(1+\mathfrak{z})\,z, \quad \zeta = \mathfrak{z}\,z,$$

if x, z are the coordinates in the original state (before the application of the dilatations $\mathfrak{x}$, $\mathfrak{z}$). Hence

$$a_1 = \mathfrak{x}, \; a_2 = \mathfrak{z}, \; b_1 = (1+\mathfrak{z})\,x_z, \; b_2 = 0.$$

If these values are substituted in the expression for ψ, the coefficient of x_z^2 will give us the value of $\frac{1}{2}\mu'$. Hence

$$\mu' = 2g(1 + 2\mathfrak{z}) + 2l(\mathfrak{x} + \mathfrak{z}),$$

or, if we replace $2l$ by a and observe that, for $\mathfrak{x} = 0$, $\mathfrak{z} = 0$, μ' must be equal to μ,

$$\mu' = \mu(1 + 2\mathfrak{z}) + a(\mathfrak{x} + \mathfrak{z}).$$

$$G = 2\pi l\left[4(\lambda+\mu)rs+2(\lambda+\mu)r^2\frac{ds}{dr}+2\lambda rq\right]+\frac{\pi\Theta^2}{l}(4\mu+a+b)r^3. \quad (24)$$

$$F = 2\pi l\left[2(\lambda+\mu)r^2s+(\lambda+2\mu)r^3\frac{ds}{dr}+\lambda r^2q\right]+\frac{\pi\Theta^2}{l}(\mu+b)r^4 \quad . \quad (25)$$

By partial integration of the second term (23) becomes

$$\delta\psi = \Big| F\delta s \Big|_0^R + \int_0^R \left(G-\frac{dF}{dr}\right)\delta s\, dr.$$

As for $r=0$ the dilatations s and $r\frac{ds}{dr}$ must have finite values, the function F vanishes for $r=0$, so that we obtain

$$\delta\psi = F_{r=R}\,\delta\mathbf{s} + \int_0^R \left(G-\frac{dF}{dr}\right)\delta s\, dr. \quad . \quad . \quad . \quad . \quad (26)$$

If now we put $\delta\mathbf{s}=0$, only the last term remains, so that we are led to the condition

$$G = \frac{dF}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

or after some transformation

$$\frac{d}{dr}\left(r^3\frac{ds}{dr}\right) = \frac{\Theta^2}{2l^2}\,\frac{a-3b}{\lambda+2\mu}\,r^3.$$

But for $r=0$ we may put $r^3\frac{ds}{dr}=0$. We find therefore

$$s = \frac{\Theta^2}{16l^2}\cdot\frac{a-3b}{\lambda+2\mu}(r^2-R^2)+\mathbf{s}. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

If this value is substituted in (22) we obtain ψ as a function of the external parameters Θ, $\mathbf{s}$ and q. By differentiation with respect to these variables we may calculate the external forces corresponding to them. We need only the two last ones, S and Q, of which S is immediately determined by (26). For according to this formula we have

$$\delta\psi = F_{r=R}\,\delta\mathbf{s},$$

so that

$$S = \frac{\partial\psi}{\partial\mathbf{s}} = F_{r=R},$$

which can be calculated by means of (25) and (28). As to

$$Q = \frac{\partial\psi}{\partial q}$$

this quantity is found if, after differentiating in (22) under the signs of integration we substitute the value (28) and then perform the integrations.

The result is

$$S = 2\pi l\,[\lambda R^2 q + 2(\lambda+\mu) R^2 \mathbf{s}] + \frac{\pi\Theta^2}{4l}(4\mu + a + b) R^4$$

$$Q = \pi l\,[(\lambda + 2\mu) R^2 q + 2\lambda R^2 \mathbf{s}] + \frac{\pi\Theta^2}{4l}(\mu + a) R^4 .$$

If no stretching forces act on the ends of the wire, nor any forces on the surface, we have $Q = 0$, $S = 0$, so that

$$2\lambda q + 4(\lambda+\mu)\mathbf{s} = -\frac{\Theta^2 R^2}{4l^2}(4\mu + a + b), \quad . \quad . \quad . \quad (29)$$

$$(\lambda + 2\mu) q + 2\lambda \mathbf{s} = -\frac{\Theta^2 R^2}{4l^2}(\mu + a) . \quad . \quad . \quad . \quad . \quad (30)$$

When the coefficients of elasticity λ, μ and the coefficients a and b are given, we can derive from these equations the changes in length and diameter (q and $\mathbf{s}$) caused by a torsion.

§ 14. We shall use formulae (29) and (30) to calculate the coefficients a and b from POYNTING's measurements.

POYNTING has worked with two steel wires and one copper wire, for which he has determined in the first place YOUNG's modulus $\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}$ and POISSON's ratio $\frac{\lambda}{2(\lambda+\mu)}$. From these quantities we can calculate λ and μ. Further he has measured q and $\mathbf{s}$, so that a and b can be found. The results are given in the following table, in which everything has been expressed in C.G.S. units. The length of the wire was in all cases

$$l = 160{,}5 \text{ cm}$$

and the numbers given for q and $\mathbf{s}$ refer to the value $\Theta = 2\pi$; so they indicate by what part of the original value the length and the diameter change, if one end of the wire is once twisted round.

	R	YOUNG's modulus	POISSON's ratio	λ	μ	q	$\mathbf{s}$	a	b
:l 1	0,0493	$2{,}12.10^{12}$	0,270	$9{,}77.10^{11}$	$8{,}35.10^{11}$	$1{,}71.10^{-6}$	$-3{,}19.10^{-7}$	$-5{,}03.10^{12}$	$0{,}58.10^{12}$
:l 2	0,0605	$2{,}12.10^{12}$	0,287	$11{,}09.10^{11}$	$8{,}24.10^{11}$	$2{,}90.10^{-6}$	$-5{,}24.10^{-7}$	$-5{,}70.10^{12}$	$0{,}70.10^{12}$
per	0,06095	$1{,}31.10^{12}$	0,331	$9{,}64.10^{11}$	$4{,}92.10^{11}$	$4{,}25.10^{-6}$	$-1{,}75.10^{-6}$	$-3{,}94.10^{12}$	$3{,}37.10^{12}$

§ 15. We can further calculate $\frac{d\log\mu}{d\log v}$ by means of the values found for a and b. Let us suppose the metal to be stretched equally in all directions, so that there is an infinitely small cubical dilatation $d\log v$. Then we have according to (21) in which expression we must put $\mathbf{x}=\mathbf{y}=\mathbf{z}=\frac{1}{3}\,d\log v$,

$$d\mu = \tfrac{1}{3}(2\mu + 2a + b)\,d\log v,$$

$$\frac{d\log\mu}{d\log v} = \frac{2\mu + 2a + b}{3\mu}.$$

To calculate from this the coefficient of dilatation, we shall suppose that, when the volume is increased, λ and μ change proportionally to each other.

The differential coefficient in (17) then becomes

$$-\tfrac{3}{2}\frac{d\log\mu}{d\log v},$$

and the formula itself

$$v - v_0 = \varkappa\left[-\tfrac{1}{2}\frac{d\log\mu}{d\log v} - \tfrac{1}{6}\right]E.$$

Treating the coefficient of E as a constant (comp. § 9) we find from this for the coefficient of cubical expansion

$$\alpha = \varkappa\left[-\tfrac{1}{2}\frac{d\log\mu}{d\log v} - \tfrac{1}{6}\right]A\,c\,\varrho.$$

If the coefficient of compressibility $\varkappa$ is derived from λ and μ, this equation gives the following results:

	$\frac{d\log\mu}{d\log v}$	$\varkappa$	c	ϱ	α	
					calc.	obs.
Steel 1	— 3,1	$6{,}5.10^{-13}$	0,11	7,8	$3{,}2.10^{-5}$	$3{,}3.10^{-5}$
„ 2	— 3,7	$6{,}0.10^{-13}$	0,11	7,8	$3{,}6.10^{-5}$	$3{,}3.10^{-5}$
Copper	— 2,4	$7{,}7.10^{-13}$	0,093	8,9	$2{,}8.10^{-5}$	$5{,}1.10^{-5}$

The only inaccuracy in the above calculation of the terms in (11) corresponding to transverse vibrations is the application of the ordinary formulae of the theory of elasticity to very short waves. For the determination of the terms referring to the longitudinal vibrations, however, we had to make the assumption that λ changes proportionally to μ. As however the transverse vibrations have a greater part in the heat motion than the longitudinal ones we may

perhaps hope that the error introduced by *this* assumption will not be considerable [1]).

We mentioned already the analogy between the problem treated in §§ 11—13 and that of the thermal expansion. In the one case the torsion plays the same part as the heat motion in the other and the quantities that have been indicated by q in the two problems are comparable with each other; the similarity of the mathematical treatment in the two cases is likewise evident. POYNTING remarks that a dilatation of the wire will also take place when it executes torsional vibrations or when vibrations of this kind are propagated in it. With similar phenomena we are generally concerned, when an elastic body is traversed by waves, and when we consider the very short waves especially, this leads us directly to an insight into the nature of thermal dilatation.

Finally it deserves our attention that, though the phenomena discussed in this paper are chiefly determined by the change of the elastic constants caused by a previous deformation, yet there are as well in equation (17) as in (29) and (30) terms that are independent of this change.

Physics. – "*On* EINSTEIN's *Theory of gravitation.*" I. By Prof. H. A. LORENTZ.

(Communicated in the meeting of February 26, 1916).

§ 1. In pursuance of his important researches on gravitation EINSTEIN has recently attained the aim which he had constantly kept in view; he has succeeded in establishing equations whose form is not changed by an arbitrarily chosen change of the system of coordinates [2]). Shortly afterwards, working out an idea that had been expressed already in one of EINSTEIN's papers, HILBERT [3]) has shown the use that may be made of a variation law that may be regarded as HAMILTON's principle in a suitably generalized form. By these results the "general theory of relativity" may be said to have taken a definitive form, though much remains still to be done in further

[1]) This paper had already gone to press, when an article of FÖRSTERLING came under my notice (Ann. d. Phys. **47** (1915) p 1127) in which considerations similar to those here developed are put forward.

[2]) A. EINSTEIN, Zur allgemeinen Relativitätstheorie, Berliner Sitzungsberichte 1915, pp. 778 799; Die Feldgleichungen der Gravitation, ibid. 1915, p. 844.

[3]) D. HILBERT, Die Grundlagen der Physik I, Göttinger Nachrichten, Math.-phys. Klasse, Nov. 1915.