perhaps hope that the error introduced by *this* assumption will not be considerable [1]).

We mentioned already the analogy between the problem treated in §§ 11—13 and that of the thermal expansion. In the one case the torsion plays the same part as the heat motion in the other and the quantities that have been indicated by q in the two problems are comparable with each other; the similarity of the mathematical treatment in the two cases is likewise evident. POYNTING remarks that a dilatation of the wire will also take place when it executes torsional vibrations or when vibrations of this kind are propagated in it. With similar phenomena we are generally concerned, when an elastic body is traversed by waves, and when we consider the very short waves especially, this leads us directly to an insight into the nature of thermal dilatation.

Finally it deserves our attention that, though the phenomena discussed in this paper are chiefly determined by the change of the elastic constants caused by a previous deformation, yet there are as well in equation (17) as in (29) and (30) terms that are independent of this change.

Physics. - "*On* EINSTEIN'S *Theory of gravitation.*" I. By Prof. H. A. LORENTZ.

(Communicated in the meeting of February 26, 1916).

§ 1. In pursuance of his important researches on gravitation EINSTEIN has recently attained the aim which he had constantly kept in view; he has succeeded in establishing equations whose form is not changed by an arbitrarily chosen change of the system of coordinates [2]). Shortly afterwards, working out an idea that had been expressed already in one of EINSTEIN'S papers, HILBERT [3]) has shown the use that may be made of a variation law that may be regarded as HAMILTON'S principle in a suitably generalized form. By these results the "general theory of relativity" may be said to have taken a definitive form, though much remains still to be done in further

1) This paper had already gone to press, when an article of FÖRSTERLING came under my notice (Ann. d. Phys. **47** (1915) p 1127) in which considerations similar to those here developed are put forward.

2) A. EINSTEIN, Zur allgemeinen Relativitätstheorie, Berliner Sitzungsberichte 1915, pp. 778 799; Die Feldgleichungen der Gravitation, ibid. 1915, p. 844.

3) D. HILBERT, Die Grundlagen der Physik I, Göttinger Nachrichten, Math.-phys. Klasse, Nov. 1915.

developing it and in applying it to special problems. It will also be desirable to present the fundamental ideas in a form as simple as possible.

In this communication it will be shown that a four-dimensional geometric representation may be of much use for this latter purpose; by means of it we shall be able to indicate for a system containing a number of material points and an electromagnetic field (or eventually only one of these) the quantity H, which occurs in the variation theorem, and which we may call the *principal function*. This quantity consists of three parts, of which the first relates to the material points, the second to the electromagnetic field and the third to the gravitation field itself.

As to the material points, it will be assumed that the only connexion between them is that which results from their mutual gravitational attraction.

§ 2. We shall be concerned with a four-dimensional extension R_4, in which "space" and "time" are combined, so that each point P in it indicates a definite place A and at the same time a definite moment of time t. If we say that P refers to a material point we mean that at the time t this point is found at the place A. In the course of time the material point is represented every moment by a new point P; all these points lie on the "world-line", which represents the state of motion (or eventually the state of rest) of the material point [1]). In the same sense we may speak of the world-line of a propagated light-vibration. An intersection of two world-lines means that the two objects to which they belong meet at a certain moment, that a "coincidence" takes place [2]). Now EINSTEIN has made the striking remark [3]) that the only thing we can learn from our observations and with which our theories are essentially concerned, is the existence of these coincidences. Let us suppose e.g. that we have observed an occultation of a star by the moon or rather the reappearance of a star at the moon's border. Then the world-line of a certain light-vibration starting from a point on the world-line of the star has in its further course intersected the world-line of a

[1]) It will be known that in the theory of relativity MINKOWSKI was the first who used this geometric representation in an extension of four dimensions. The name "world-line" has been borrowed from him.

[2]) For the sake of simplicity we shall imagine the two motions not to be disturbed by this coincidence, so that e.g. two material points penetrate each other or pass each other at an extremely small distance without any mutual influence.

[3]) In a correspondence I had with him.

point of the border of the moon and finally that of the observer's eye. A similar remark may be made when the moment of reappearance is read on a clock. Let us suppose that the light-vibration itself lights the dial-plate, reaching it when the hand is at the point a; then we may say that three world-lines, viz. that of the light-vibration, that of the hand and that of the point a intersect.

§ 3. We may imagine that, in order to investigate a gravitation field as e.g. that of the sun, a great number of material points, moving in all directions and with different velocities, are thrown into it, that light-beams are also made to traverse the field and that all coincidences are noted [1]). It would be possible to represent the results of these observations by world-lines in a four-dimensional figure — let us say in a "field-figure" — the lines being drawn in such a way that each observed coincidence is represented by an intersection of two lines and that the points of intersection of one line with a number of the others succeed each other in the right order.

Now, as we have to attend only to the intersections, we have a great degree of liberty in the construction of the "field-figure". If, independently of each other, two persons were to describe the same observations, their figures would probably look quite different and if these figures were deformed in an arbitrary way, without break of continuity, they would not cease to serve the purpose.

After having constructed a field-figure F we may introduce "coordinates", by which we mean that to each point P we ascribe four numbers x_1, x_2, x_3, x_4, in such a way that along any line in the field-figure these numbers change continuously and that never two different points get the same four numbers. Having done this we may for each point P seek a point P' in a four-dimensional extension R'_4, in which the numbers $x_1, \ldots x_4$ ascribed to P are the Cartesian coordinates of the point P'. In this way we obtain in R'_4 a figure F', which just as well as F can serve as field-figure and which of course may be quite different according to the choice of the numbers $x_1 \ldots x_4$ that have been ascribed to the points of F.

If now it is true that the coincidences only are of importance it must be possible to express the fundamental laws of the phenomena by geometric considerations referring to the field-figure, in such a way that this mode of expression is the same for all possible field-figures; from our point of view all these figures can be considered as being the *same*. In such a geometric treatment the introduction of

[1]) In other terms, that the data procured by astronomical observations can be extended arbitrarily and unboundedly.

coordinates will be of secondary importance; with a single exception (§ 13) it only serves for short calculations which we have to intercalate (for the proof of certain geometric propositions) and for establishing the final equations, which have to be used for the solution of special problems. In the discussion of the general principles coordinates play no part; and it is thus seen that the formulation of these principles can take place in the same way whatever be our choice of coordinates. So we are sure beforehand of the general covariancy of the equations that was postulated by EINSTEIN.

§ 4. EINSTEIN ascribes to a line-element PQ in the field-figure a length ds defined by the equation

$$ds^2 = \Sigma\,(ab)\, g_{ab}\, dx_a\, dx_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$(g_{ab} = g_{ba})$$

Here $dx_1 \ldots dx_4$ are the changes of the coordinates when we pass from P to Q, while the coefficients g_{ab} depend in one way or another on the coordinates. The gravitation field is known when these 10 quantities are given as functions of $x_1 \ldots x_4$. Here it must be remarked that in all real cases the coordinates can be chosen in such a way that for one point arbitrarily chosen (1) becomes

$$ds^2 = -\,dx_1{}^2 - dx_2{}^2 - dx_3{}^2 + dx_4{}^2.$$

This requires that the determinant g of the coefficients of (1) be always negative. The minor of this determinant corresponding to the coefficient g_{ab} will be denoted by G_{ab}.

Around each point P of the field-figure as a centre we may now construct an infinitesimal surface [1]), which, when P is chosen as origin of coordinates, is determined by the equation

$$\Sigma\,(ab)\, g_{ab}\, x_a x_b = \varepsilon^2\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ε is an infinitely small positive constant which we shall fix once for all. This surface, which we shall call the *indicatrix*, is a hyperboloid with one real axis and three imaginary ones. We shall also introduce the surface determined by the equation

$$\Sigma\,(ab)\, g_{ab}\, x_a x_b = -\,\varepsilon^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which differs from (2) only by the sign of ε^2. We shall call this the *conjugate indicatrix*. It is to be understood that the indicatrices and conjugate indicatrices take part in the changes to which the field-figure may be subjected. As these surfaces are infinitely small,

1) A "surface" determined by one equation between the coordinates is a three-dimensional extension. It will cause no confusion if sometimes we apply the name of "plane" to certain two-dimensional extensions, if we speak e.g. of the "plane" determined by two line-elements.

they always remain hyperboloids of the said kind. The gravitation field will now be determined by these indicatrices, which we can imagine to have been constructed in the field-figure without the introduction of coordinates. When we have occasion to use these latter, we shall so choose them that the "axes" x_1, x_2, x_3 intersect the conjugate indicatrix constructed around their starting point, while the indicatrix itself is intersected by the axis x_4. This involves that the coefficients g_{11}, g_{22}, g_{33} are negative and that g_{44} is positive.

§ 5. The indicatrices will give us the units in which we shall express the length of lines in the field-figure and the magnitude of two-, three or four-dimensional extensions. When we use these units we shall say that the quantities in question are expressed in *natural measure*.

In the case of a line-element PQ the unit might simply be the radius-vector in the direction PQ of the indicatrix or the conjugate indicatrix described about P. It is however desirable to distinguish the two cases that PQ intersects the indicatrix itself or the conjugate indicatrix. In the latter case we shall ascribe an imaginary length to the line-element [1]). Besides, by taking as unit not the radius-vector itself but a length proportional to it, the numerical value of a line-element may be made to be independent of the choice of the quantity ε.

These considerations lead us to define the length that will be ascribed to line-elements by the assumption that each radius-vector of the indicatrix has in natural measure the length ε, while each radius-vector of the conjugate indicatrix has the length $i\varepsilon$. [2])

It will now be clear that the length of an arbitrary line in the field-figure can be found by integration, each of its elements being measured by means of the indicatrix or the conjugate indicatrix belonging to the position of the element. In virtue of our definitions a deformation of the field-figure will not change the length of lines expressed in natural measure and a geodetic line will remain a geodetic line.

§ 6. We are now in a position to indicate the first part H_1 of the principal function (§ 1). Let σ be a closed surface in the field-figure and let us confine ourselves to the principal func-

1) This corresponds to the negative value which (1) gives for ds^2.

2) For a radius-vector on the asymptotic cone we may take either of these values: this makes no difference, as the numerical value of a line-element in the direction of such a radius-vector becomes 0 in both cases.

tion so far as it belongs to the space Ω enclosed by that surface. Then the quantity H_1 is the sum, taken with the negative sign, of the lengths of all world-lines of material points so far as they lie within Ω, each length multiplied by a constant m, characteristic of the point in question and to be called its *mass.*[1])

It must be remarked that the elements of the world-lines of material points intersect the corresponding indicatrices themselves. The lengths of these lines are therefore real positive quantities.

A deformation of the field-figure leaves H_1 unchanged.

§ 7. We shall now pass on to the part of the principal function belonging to the gravitation field. The mathematical expression for this part was communicated to me by EINSTEIN in our correspondence. It is also to be found in HILBERT's paper in which it is remarked that the quantity in question may be regarded as the measure of the *curvature* of the four-dimensional extension to which (1) relates. Here we have to speak only of the interpretation of this quantity. To find this the following geometrical considerations may be used.

Let PQ and PR be two line-elements starting from a point P of the field-figure, QR the line-element joining the extremities Q and R. If then the lengths of these elements in natural measure are

$$PQ = ds', \quad PR = ds'', \quad QR = ds,$$

we define the angle (s', s'') between PQ and PR by the well known trigonometric formula

$$ds^2 = ds'^2 + ds''^2 - 2ds'ds'' \cos(s', s''),$$

$$\cos(s', s'') = \frac{ds'^2 + ds''^2 - ds^2}{2ds'ds''} \quad \ldots \ldots \quad (4)$$

from which one can derive

$$\cos(s', s'') = \Sigma(ab)\, g_{ab} \frac{dx'_a}{ds'} \frac{dx''_b}{ds''}. \quad \ldots \ldots \quad (5)$$

By means of this formula we are able to determine the angle between any two intersecting lines. Of course the two other angles of the triangle PQR can be calculated in the same way.

Now two cases must be distinguished.

a. The plane of the triangle PQR cuts the conjugate indicatrix, but not the indicatrix itself. Then the three sides have positive imaginary values. Moreover each of them proves to be smaller than

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[1]) This agrees with the value of the LAGRANGIAN function, which is to be found e.g. in my paper on "HAMILTON's principle in EINSTEIN's theory of gravitation." These Proceedings **19** (1916), p. 751.

the sum of the others, from which one finds that the angles have real values and that their sum is π.

b. The plane PQR cuts both the indicatrix and the conjugate indicatrix. In this case different positions of the triangle are still possible. We can however confine ourselves to triangles the three sides of which are real. These are really possible, for in the plane of a hyperbola we can draw triangles the sides of which are parallel to radius-vectors drawn from the centre to points of the curve (and not of the conjugate hyperbola).

By a closer consideration of the triangles now in question it is found however that by the choice of our "natural" units one side is necessarily longer than the sum of the other two. Formula (4) then shows that the cosines of the angles are real quantities, greater than 1 in absolute value, two of them being positive, and the third negative. We must therefore ascribe to the angles imaginary or complex values. If for $p > +1$ we put

$$arc\ cos\ p = i\ log\ (p + \sqrt{p^2 - 1})$$

and

$$arc\ cos\ (-p) = \pi - arc\ cos\ p\ ,$$

we find for the three angles expressions of the form

$$i\,\alpha,\ i\,\beta \text{ and } \pi - i\,(\alpha + \beta)\ ,$$

so that the sum is again π.

From the cosine calculated by (4) or (5) the sine can be derived by means of the formula

$$sin\ \varphi = \sqrt{1 - cos^2\,\varphi}\ ,$$

where for the case $cos^2\,\varphi > 1$ we can confine ourselves to the value

$$sin\ \varphi = i \sqrt{cos^2\,\varphi - 1}$$

with the positive sign.

It deserves special notice that two conjugate radius-vectors of the indicatrix and the conjugate indicatrix are perpendicular to each other and that a deformation of the field-figure does not change the angle between two intersecting lines determined according to our definitions.

§ 8. Before proceeding further we must now indicate the natural units (§ 5) for two-, three-, or four-dimensional extensions in the field-figure. Like the unit of length, these are defined for each point separately, so that the numerical value of a finite extension is found by dividing it into infinitely small parts.

A two-dimensional extension cuts the conjugate indicatrix in an ellipse, or the indicatrix itself and the conjugate indicatrix in two conjugate hyperbolae. In both cases we derive our unit from the area of a parallelogram described on conjugate radius-vectors.

A three-dimensional extension cuts the conjugate indicatrix in an ellipsoid, or the indicatrix and its conjugate in two conjugate hyperboloids. Now our unit will be derived from the volume of a parallelepiped described on three conjugate radius-vectors.

In a similar way the magnitude of four-dimensional extensions will be determined by comparison with a parallelepiped the edges of which are four conjugate radius-vectors of the indicatrix and the conjugate indicatrix.

It must here be kept in mind that, according to well known theorems, the area of the parallelogram and the volume of the parallelepipeds in question are independent of the special choice of the conjugate radius-vectors.

We shall further specify the units in such a way (comp. § 5) that the numerical magnitude of a parallelogram or a parallelepiped described on conjugate radius-vectors is found by multiplying the numbers by which the edges are expressed in natural measure.

From what has been said it follows that the area of the parallelogram described on two line-elements is given by the product of the lengths of these elements and the sine of the enclosed angle. Similarly the area of an infinitely small triangle is determined by half the product of two sides and the sine of the angle between them.

We need hardly add that the numerical value of any two-, three- or four-dimensional domain expressed in natural measure is not changed by a deformation of the field-figure.

§ 9. Let, at any point P of the field-figure, 1, 2, 3, 4 be four arbitrarily chosen conjugate radius-vectors of the indicatrix. Two of these determine an infinitely small part V of a two-dimensional extension. We may prolong this part to finite distances from P by drawing from this point geodetic lines whose initial directions lie in the plane V. In this way we obtain six two-dimensional extensions (1,2), (2,3), (3,1), (1,4), (2,4) and (3,4). Let us now consider in one of these e. g. (a, b) an infinitesimal triangle near the point P, the sides of which are geodetic lines (viz. geodetic lines *in* (a, b)). If in calculating the angles of this triangle we go to quantities of the second order with respect to the sides and to the distances from P, the sum s of the angles proves to have no longer the value π (comp. § 7). The "excess" $e = s - \pi$ is proportional to the area Δ of the triangle, independently of the length of the sides, of their ratios and of the position of the triangle in the extension (a, b). For the three extensions (1,2) (2,3), (3,1), which do not intersect the indicatrix itself but the conjugate indicatrix, this proposition follows from a well-known theorem of GAUSS in the theory of curvature of surfaces; for the other three (1,4), (2,4), (3,4), which cut the indicatrix itself, the proof can be given by direct calculation. The considerations necessary for this, and some other calculations with which we shall be concerned further on will be communicated in a later paper.

In considering the three last-mentioned extensions I have confined myself to triangles with real sides (§ 7, *b*).

The quotient

$$\frac{e}{\Delta} = K_{ab}$$

is now for each extension a definite number, which we may consider as a measure of the *curvature* of the two-dimensional extension (a, b); the sum K of the six numbers K_{ab} may be called the *curvature of the field-figure* at the point P in question. This quantity is the same that has been introduced by HILBERT; this results from the calculation of its value, which at the same time shows K to be independent of the special choice of the directions 1, 2, 3, 4 introduced in the beginning of this §.

The numbers K_{ab} are all real and have a meaning that can be indicated without the introduction of coordinates; moreover their sum K is not changed by a deformation of the field-figure.

If now $d\Omega$ is an element of the four-dimensional extension of the field-figure, expressed in natural measure, the part of the principal function belonging to the gravitation field is

$$H_1 = \frac{i}{\varkappa}\int K d\Omega\,, \quad \ldots \ldots \ldots \quad (6)$$

where the integration is extended to the domain considered (§ 6) while $\varkappa$ is the gravitation constant. H_1 too is not changed by a deformation of the field-figure.

The factor i has been introduced in order to obtain a real value for H_1, the element $d\Omega$ being represented in natural measure by a negative imaginary number (§ 8).

§ 10. What we have to say of the electromagnetic field must be preceded by some considerations belonging to what may be called the "vector theory" of the field-figure.

A line-element PQ, taken in a definite direction (indicated by the order of the letters), may be called a *vector*. Such vectors can be compounded or decomposed by means of parallelograms or parallelepipeds. Especially, when coordinates $x_1, \ldots x_4$ have been chosen,

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a vector may be resolved into four components which have the directions of the coordinates, viz. such directions that a shift along the first e.g. changes only x_1, while x_2, x_3, x_4 remain constant. The four components in question are determined by the differentials $dx_1, \dots dx_4$ corresponding to PQ. We shall say that by these they are expressed in "x-measure". Their values in natural measure are found by multiplying $dx_1, \dots dx_4$ by certain factors. If we keep in mind that the radius-vectors of the conjugate indicatrix and the indicatrix in the directions of the axes are expressed in "x measure" by

$$\frac{\varepsilon}{\sqrt{-g_{11}}},\quad \frac{\varepsilon}{\sqrt{-g_{22}}},\quad \frac{\varepsilon}{\sqrt{-g_{33}}},\quad \frac{\varepsilon}{\sqrt{g_{44}}},$$

and in natural units by

$$i\varepsilon,\quad i\varepsilon,\quad i\varepsilon,\quad \varepsilon$$

we find for the reducing factors

$$l_1 = i\sqrt{-g_{11}},\quad l_2 = i\sqrt{-g_{22}},\quad l_3 = i\sqrt{-g_{33}},\quad l_4 = \sqrt{g_{44}} \quad . \quad (7)$$

In the language of vector-analysis the vector obtained by the composition of two or more vectors is also called the *sum* of these vectors.

We shall also speak of *finite* vectors, i.e. of directed quantities which can be represented on an infinitely reduced scale by line-elements in the field-figure. If ω is the constant "reduction factor" chosen for this purpose, a vector **A** will be represented by a line-element ω**A**, the direction of which is also ascribed to **A**. It will now be evident that two finite vectors, as well as two infinitely small ones, determine an infinitesimal two-dimensional extension and that finite vectors can be compounded and resolved by means of parallelograms and parallelepipeds. Also that we may speak of the "magnitude" of such figures, that e.g. the rule given in § 8 applies to the parallelogram described on two vectors.

The components of a vector in the directions of the coordinates expressed in x-measure will be called X_1, X_2, X_3, X_4. This means that $\omega X_1, \dots \omega X_4$ are equal to the differentials $dx_1, \dots dx_4$ corresponding to the infinitely small vector ω**A**.

If we want to know the components of **A** in natural units we must multiply $X_1, \dots X_4$ by the factors (7).

§ 11. Two vectors **A** and **B** starting from a point P of the field-figure and lying in a plane V, determine what we shall call a *rotation* **R** in that plane. We ascribe to it the direction indicated by the order **AB** and a value given by the parallelogram described on

A and B and expressed in natural measure [1]). This involves that the same rotation may be represented in many different ways by two vectors in the plane V.

For the rotation R we shall also use the symbol [A . B].

By the *vector product* [A . B . C] of three vectors A, B, C at a point of the field-figure and not lying in one plane we shall understand a vector D the direction of which is conjugate with each of the three vectors (and therefore with the three-dimensional extension A, B, C), the direction of D corresponding to those of A, B and C in a way presently to be indicated, while the magnitude of D, expressed in natural measure, is equal to that of the parallelepiped described on A, B and C and expressed in the same measure. This definition involves that the value 0 is ascribed to the vector product of three vectors lying in one and the same plane.

A further statement about the direction of D is necessary because *two* opposite directions are conjugate with A, B, C. For one set of three directions A_0, B_0, C_0 we shall choose arbitrarily which of its two conjugate directions will be said to correspond to it. If this is the direction D_0, then the direction D corresponding to A, B, C will be determined by the rule that D_0 passes into D by a gradual passage of the first three vectors from A_0, B_0, C_0 into A, B, C, this latter passage being effected in such a way that during the change the vectors never come to lie in one plane.

The vector product [A . B . C] takes the opposite direction when one of the vectors is reversed as well as when two of them are interchanged. We must therefore always attend to the order of the symbols in [A . B . C].

The vector product possesses the distributive property with respect to each of the three vectors, so that e.g. if A_1 and A_2 are vectors,

$$[(A_1 + A_2) . B . C] = [A_1 . B . C] + [A_2 . B . C].$$

From this we can infer that [A . B . C] depends only on C and the rotation R determined by A and B. For this reason we write for the vector product also [R . C]; in calculating it we are free to replace the rotation R by any two vectors by means of which it can be represented.

If R, R_1 and R_2 are rotations in the same plane, such that the value and direction of R are found by adding R_1 and R_2 algebraically, we have, in virtue of the distributive property

$$[R_1 . C] + [R_2 . C] = [R . C]$$

[1]) If, according to circumstances, different signs are given to R, the angle whose sine occurs in the formula for the area of a parallelogram must be understood to be positive in one case and negative in the other

§ 12. In what precedes we were concerned with the volumes of parallelepipeds expressed in natural units. When we have introduced coordinates $x_1, \ldots x_4$ we may also express these volumes in the "x-units" corresponding to the coordinates chosen.

Let us consider e.g. the three-dimensional extension $x_4 = \text{const.}$, which cuts the conjugate indicatrix in the ellipsoid

$$g_{11}x_1^2 + g_{22}x_2^2 + g_{33}x_3^2 + 2g_{12}x_1x_2 + 2g_{23}x_2x_3 + 2g_{31}x_3x_1 = -\varepsilon^2.$$

If we agree that in x-measure spaces in this extension will be represented by positive numbers and that a parallelepiped with the positive edges dx_1, dx_2, dx_3 will have the volume $dx_1\, dx_2\, dx_3$, we find for that of the parallelepiped on three conjugate radius-vectors

$$\frac{\varepsilon^3}{\sqrt{-G_{44}}},$$

where it has been taken into consideration that G_{44} is negative.

The volume of the same parallelepiped being expressed in natural measure by $-i\varepsilon^3$ (§ 8), we have to multiply by

$$l_{123} = -i\sqrt{-G_{44}} \quad \ldots \ldots \ldots \quad (8)$$

if we want to pass from the expression in x-measure to that in natural measure.

For the extension (x_2, x_3, x_4), i.e. $x_1 = 0$ the corresponding factor is

$$l_{234} = -\sqrt{G_{11}} \quad \ldots \ldots \ldots \quad (9)$$

§ 13. In the theory of electromagnetic phenomena we are concerned in the first place with the electric charge and the convection current. So far as these quantities belong to a definite element $d\Omega$ of the field-figure they may be combined into

$$\mathsf{q}\, d\Omega$$

where q is a vector which we may call the *current vector*. When it is resolved into four components having the directions of the axes, the first three components determine the convection current, while the fourth component gives the density of the electric charge.

As to the electric and the magnetic force, these two taken together can be represented at each point of the field-figure by two rotations

$$\mathsf{R}_e \quad \text{and} \quad \mathsf{R}_h$$

in definite, mutually conjugate two-dimensional extensions. These quantities are closely connected with the current vector, for after having introduced coordinates $x_1, \ldots x_4$ we have for each closed surface σ the vector equation

$$\int \{[R_e \cdot N] + [R_h \cdot N]\}_x \, d\sigma = i \int \{q\}_x \, d\Omega, \quad . \quad . \quad . \quad (10)$$

where the second integral has to be taken over the domain Ω enclosed by σ. On the left hand side $d\sigma$ represents a three-dimensional surface-element expressed in natural units and N a vector of the magnitude 1 in natural measure conjugate with or perpendicular to that element (§ 7) and directed towards the outside of the domain Ω. The index x shows that the vector $[R_e.N]+[R_h.N]$ must be expressed in x-measure. At each point of the surface we must resolve the vector along the four directions of the coordinates, express each component in x-measure (§ 10) and finally, after multiplication by $d\sigma$, we must add algebraically all x_1-components; similarly all x_2-components and so on.

It must be expressly remarked that if an equation like (10) in which we are concerned with the composition of vectors at *different* points of the field-figure, shall have a definite meaning we must know which components are to be considered as having the same direction, so that they can be added. This has been determined by the introduction of coordinates.

On the right hand side of the equation the index x means that the vector q must be expressed in x-measure and the factor i had to be introduced because $d\Omega$ is imaginary.

One can prove that equation (10) is equivalent to the differential equations which in EINSTEIN's theory serve for the same purpose and further that when the equation holds for one choice of coordinates it will also be true for any other choice.

§ 14. The proof for these assertions must be deferred to the second part of this communication. For the present we shall only add that the part of the principal function referring to the electromagnetic field is given by

$$H_2 = i \int \tfrac{1}{2} (R_e^2 + R_h^2) \, d\Omega,$$

where R_e and R_h are, expressed in natural units, the two rotations that are characteristic of the field. Like the two other parts of the principal function, H_2 is not changed by a deformation of the field-figure. In this statement it is to be understood that the parallelograms by which R_e and R_h are represented take part in the deformation.

Some remarks on the way in which, starting from the principal function, we may obtain the fundamental equations of the theory must also be deferred. I shall conclude now by remarking that, as an immediate consequence of HAMILTON's principle, the world-line of a material point which is acted on only by a given *gravitation field*, will be a geodetic line, and that the equations which determine the gravitation field caused by material and electromagnetic systems will be found by the consideration of infinitely small variations of the indicatrices, by which the numerical values of all quantities that are measured by means of these surfaces will be changed.

Physics. — *"On* EINSTEIN's *Theory of gravitation."* II. By Prof. H. A. LORENTZ.

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§ 15. In the first part of this communication the connexion between the electric and the magnetic force on one hand and the charge and the convection current on the other was expressed by the equation

$$\int \{[\mathsf{R}_e \,.\, \mathsf{N}] + [\mathsf{R}_h \,.\, \mathsf{N}]\}_\lambda d\sigma = i \int \{\mathsf{q}\}_x \, d\Omega \,, \quad . \quad . \quad . \quad . \qquad (10)$$

which has been discussed in § 13. It will now be shown that this formula is equivalent to the differential equations by which the connexion in question is expressed in the theory of EINSTEIN. For this purpose some further geometrical considerations must first be developed. They refer to the special case that the quantities g_{ab} have the same values at every point of the field-figure.

If this condition is fulfilled, considerations which generally may be applied to infinitesimal extensions only are valid for finite extensions too.

§ 16. The factor required, in the measurement of four-dimensional domains, for the passage from x-units to natural units has now the same value at every point of the field-figure. Similarly, when any one-, two- or three-dimensional extension in the field-figure that is determined by linear equations ("linear extensions") is considered, the factor by means of which the said passage may be effected for parts of that extension, will be the same for all those parts. Moreover the factor in question will be the same for two "parallel" extensions of this kind, i.e. for two extensions the determining equations of which can be written in such a way that the coefficients of $x_1, \ldots x_4$ are the same in them.