must also be deferred. I shall conclude now by remarking that, as an immediate consequence of HAMILTON's principle, the world-line of a material point which is acted on only by a given gravitation field, will be a geodetic line, and that the equations which determine the gravitation field caused by material and electromagnetic systems will be found by the consideration of infinitely small variations of the indicatrices, by which the numerical values of all quantities that are measured by means of these surfaces will be changed.

**Physics.** — "*On* EINSTEIN'*s Theory of gravitation.*" II. By Prof. H. A. LORENTZ.

(Communicated in the meeting of March 25, 1916).

§ 15. In the first part of this communication the connexion between the electric and the magnetic force on one hand and the charge and the convection current on the other was expressed by the equation

$$\int \{[\mathsf{R}_e \,.\, \mathsf{N}] + [\mathsf{R}_h \,.\, \mathsf{N}]\}_x d\sigma = i \int \{\mathsf{q}\}_x \, d\Omega , \quad . \quad . \quad . \quad . \quad (10)$$

which has been discussed in § 13. It will now be shown that this formula is equivalent to the differential equations by which the connexion in question is expressed in the theory of EINSTEIN. For this purpose some further geometrical considerations must first be developed. They refer to the special case that the quantities $g_{ab}$ have the same values at every point of the field-figure.

If this condition is fulfilled, considerations which generally may be applied to infinitesimal extensions only are valid for finite extensions too.

§ 16. The factor required, in the measurement of four-dimensional domains, for the passage from $x$-units to natural units has now the same value at every point of the field-figure. Similarly, when any one-, two- or three-dimensional extension in the field-figure that is determined by linear equations ("linear extensions") is considered, the factor by means of which the said passage may be effected for parts of that extension, will be the same for all those parts. Moreover the factor in question will be the same for two "parallel" extensions of this kind, i.e. for two extensions the determining equations of which can be written in such a way that the coefficients of $x_1, \ldots x_4$ are the same in them.

It is obvious that linear one-dimensional extensions can be called "straight lines", also it will be clear what is to be understood by a "prism" (or "cylinder"). This latter is bounded by two mutually parallel linear three-dimensional extensions $\sigma_1$ and $\sigma_2$ and by a lateral surface which may be extended indefinitely to both sides and in which mutually parallel straight lines ("generating lines") can be drawn. We need not dwell upon the elementary properties of the prism.

§ 17. A vector may now be represented by a straight line of finite length; the quantities $X_1, \ldots X_4$, which have been introduced in § 10, are the changes of the coordinates caused by a displacement along that line. The magnitude of the vector, expressed in natural units, will be denoted by $S$. It is given by a formula similar to (1), viz. by

$$S^2 = \Sigma\,(ab)\, g_{ab}\, X_a\, X_b \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

A vector may be regarded as being the *same* everywhere in the field-figure, if $X_1, \ldots X_4$ have constant values. In the same way a rotation **R** (§ 11) may be said to be the same everywhere, if it can be represented by two vectors of this kind.

If from a point $P$ two vectors $PQ$ and $PR$ issue, denoted by $X_1', \ldots X_4'$, $S'$ and $X_1'', \ldots X_4''$, $S''$ resp., the angle between them (comp. (5)) is defined by

$$S'\,S''\cos(S', S'') = \Sigma\,(ab)\, g_{ab}\, X'_a\, X''_b. \quad . \quad . \quad . \quad . \quad (12)$$

We remark here that $X_a'$, $X_b''$ are real, positive or negative quantities and that $S'$ and $S''$ are expressed in the way indicated in § 5 ("absolute" values). It is to be understood that $S$ does not change when the signs of $X_1, \ldots X_4$ are reversed at the same time.

If $S'''$ is the value of the vector $RQ$ and if the angle between this vector and $RP$ is denoted by $(S'', S''')$, it follows further from (11) and (12) that

$$S'' = S'\cos(S', S'') + S'''\cos(S'', S''').$$

In the special case of a right angle $R$ we have

$$S'' = S'\cos(S', S''),$$

an equation expressing the connexion between a vector $PQ$ and its "projection" on a line $PR$. The angle $(S', S'')$ is the angle between the vector and its projection, both reckoned from the same point $P$.

§ 18. Let us now return to the prism $P$ mentioned in § 16. From a point $A_2$ of the boundary of the "upper face" $\sigma_2$ we can draw a line perpendicular to $\sigma_2$ and $\sigma_1$. Let $B_1$ be the point, where it cuts this last plane, the "base", and $A_1$ the point where this plane is encountered by the generating line through $A_2$. If then $\angle A_1 A_2 B_1 = \vartheta$, we have

$$\overline{A_2 B_1} = \overline{A_2 A_1} \cos \vartheta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The strokes over the letters indicate the absolute values of the distances $A_2 B_1$ and $A_2 A_1$.

It can be shown (§ 8) that, all quantities being expressed in natural units, the "volume" of the prism $P$ is found by taking the product of the numerical values of the base $\sigma_1$ and the "height" $A_2 B_1$.

Let now linear three-dimensional extensions perpendicular to $A_1 A_2$ be made to pass through $A_1$ and $A_2$. From these extensions the lateral boundary of the prism cuts the parts $\sigma_1'$ and $\sigma_2'$ and these parts, together with the lateral surface, enclose a new prism $P'$, the volume of which is equal to that of $P$. As now the volume of $P'$ is given by the product of $\overline{A_2 A_1}$ and $\sigma_1'$, we have with regard to (13)

$$\sigma'_1 = \sigma_1 \cos \vartheta .$$

If now we remember that, if a vector perpendicular to $\sigma_1$ is projected on the generating line, the ratio between the projection and the vector itself (viz. between their absolute values) is given by $\cos \vartheta$ and that a connexion similar to that which was found above between a normal section $\sigma'_1$ of the prism and $\sigma_1$ also exists between $\sigma'_1$ and any other oblique section, we easily find the following theorem:

Let $\sigma$ and $\overline{\sigma}$ be two arbitrarily chosen linear three-dimensional sections of the prism, $\mathsf{N}$ and $\overline{\mathsf{N}}$ two vectors, perpendicular to $\sigma$ and $\overline{\sigma}$ resp. and of the same length, $S$ and $\overline{S}$ the absolute values of the projections of $\mathsf{N}$ and $\overline{\mathsf{N}}$ on a generating line. Then we have

$$S\sigma = \overline{S}\overline{\sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

§ 19. After these preliminaries we can show that the left hand side of (10) is equal to 0, if the numbers $g_{ab}$ are constants and if moreover both the rotation $\mathsf{R}_e$ and the rotation $\mathsf{R}_h$ are everywhere the same. For the two parts of the integral the proof may be given in the same way, so that it suffices to consider the expression

$$\int [\mathsf{R}_e \, . \, \mathsf{N}]_r \, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Let $X_1, \ldots X_4$ be the components of the vector $\mathsf{N}$, expressed in $x$-units. From the distributive property of the vector product it then follows that each of the four components of

$$[\mathbf{R}_e \,.\, \mathbf{N}]_x$$

is a homogeneous linear function of $X_1, \dots X_4$. Under the special assumptions specified at the beginning of this § these are everywhere the same functions. Let us thus consider a definite component of (15) e.g. that which corresponds to the direction of the coordinate $x_a$. We can represent it by an expression of the form

$$\int (a_1 X_1 + \dots + a_4 X_4)\, d\sigma,$$

where $a_1, \dots . a_4$ are constants. It will therefore be sufficient to prove that the four integrals

$$\int X_1 \, d\sigma \dots \int X_4 \, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

vanish.

In order to calculate $\int X_1 \, d\sigma$ we consider an infinitely small prism, the edges of which have the direction $x_1$. This prism cuts from the boundary surface $\sigma$ two elements $d\sigma$ and $\overline{d\sigma}$. Proceeding along a generating line in the direction of the positive $x_1$ we shall enter the extension $\boldsymbol{\Omega}$ bounded by $\sigma$ through one of these elements and leave it through the other. Now the vectors perpendicular to $\sigma$, which occur in (15) and which we shall denote by $\mathbf{N}$ and $\overline{\mathbf{N}}$ for the two elements, have the same value. [1]) If, therefore, $S$ and $\overline{S}$ are the absolute values of the projections of $\mathbf{N}$ and $\overline{\mathbf{N}}$ on a line in the direction $x_1$, we have according to (14)

$$S\, d\sigma = \overline{S}\, \overline{d\sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Let first the four directions of coordinates be perpendicular to one another. Then the components of the vector obtained by projecting $\mathbf{N}$ on the above mentioned line are $X_1, 0, 0, 0$ and similarly those of the projection of $\overline{\mathbf{N}}$: $\overline{X}_1, 0, 0, 0$. But as, proceeding in the direction of $x_1$, we enter $\boldsymbol{\Omega}$ through one element and leave it through the other, while $\mathbf{N}$ and $\overline{\mathbf{N}}$ are both directed outward, $X_1$ and $\overline{X}_1$ must have opposite signs. So we have

$$S : \overline{S} = X_1 : -\overline{X}_1$$

and because of (17) we may now conclude that the elements $X_1\, d\sigma$

[1]) From § 10 it follows that if the length of a vector $\mathbf{A}$ that is represented by a line (§ 17) coincides with a radius-vector of the conjugate indicatrix, it is always represented by an imaginary number. We may however obtain a vector which in natural units is represented by a real number e.g. by **1** (§ 13) if we multiply the vector $\mathbf{A}$ by an imaginary factor, which means that its components and also those of a vector product in which it occurs are multiplied by that factor.

and $\overline{X}_1 \; \overline{d\sigma}$ in the first of the integrals (16) annul each other. It will be clear now that the whole integral vanishes and that similar considerations may be applied to the other three.

So we have proved that under the special assumptions made the left hand side of (10) will vanish in the special case that the directions of the coordinates are perpendicular to each other. This conclusion likewise holds for an other set of coordinates if only the assumption made at the beginning of this § is fulfilled. This is obvious, as we can pass from mutually perpendicular coordinates $x_1, \ldots x_4$ to arbitrarily chosen other ones $x'_1, \ldots x'_4$ which fulfil this latter condition by linear transformation formulae with constant coefficients. The $x$- and the $x'$-components of the vector

$$[\mathbf{R}_e \, . \, \mathbf{N}] + [\mathbf{R}_h \, . \, \mathbf{N}]$$

are then connected by homogeneous linear formulae with coefficients which have the same value at all points of the surface $\sigma$. Hence if, as has been shown above, the four $x$-components of the vector

$$\int \{ [\mathbf{R}_e \, . \, \mathbf{N}] + [\mathbf{R}_h \, . \, \mathbf{N}] \} \, d\sigma$$

vanish, the four $x'$-components are now seen to do so likewise. [1])

§ 20. The above considerations were intended to prepare a corollary which will be of use in the treatment of the integral on the left hand side of (10), if we now leave the special assumptions made above and suppose the quantities $g_{ab}$ to be functions of the coordinates while also the rotations $\mathbf{R}_e$ and $\mathbf{R}_h$ may change from point to point.

This corollary may be formulated as follows: If all dimensions of the limiting surface $\sigma$ are infinitely small of the first order, the integral

$$\int \{ [\mathbf{R}_e \, . \, \mathbf{N}] + [\mathbf{R}_h \, . \, \mathbf{N}] \}_x \, d\sigma$$

will be of the *fourth* order.

In order to make this clear let us suppose that in the calculation of the integral we confine ourselves to quantities of the third order. The surface $\sigma$ being already of that order we may then omit all infinitesimal values in the quantities by which $d\sigma$ is multiplied;

[1]) In the above considerations difficulties might arise if the vector $\mathbf{N}$ lay on the asymptotic cone of the indicatrix, our definition of a vector of the value 1 would then fail (comp. note 2, p. 1345). With a view to this we can choose the for mof the extension $\Omega$ (§ 13) in such a way that this case does not occur, a restriction leading to a boundary with sharp edges.

we may therefore neglect the infinitesimal changes of the quantities $g_{ab}$ over the extension considered, and also those of $\mathsf{R}_e$ and $\mathsf{R}_h$. By this we just come to the case considered in § 19. Thus it is evident, that as regards quantities of the third order the first part of (10) is 0. From this it follows that in reality it is at least of the fourth order.

§ 21. Let us now return to the general case that the extension $\Omega$ to which equation (10) refers, has finite dimensions. If by a surface $\bar{\sigma}$ this extension is divided into two extensions $\Omega_1$ and $\Omega_2$, the quantities on the two sides in (10) each consist of two parts referring to these extensions. For the right hand side this is immediately clear and as to the quantity on the left hand side, it follows from the consideration that the contributions of $\bar{\sigma}$ to the integrals over the boundaries of $\Omega_1$ and $\Omega_2$ are equal with opposite signs. In the two cases namely we must take for $\mathsf{N}$ equal but opposite vectors.

Also, if the extension $\Omega$ is divided into an arbitrary number of parts, each term in (10) will be the sum of a number of integrals, each relating to one of these parts.

By surfaces with the equations $x_1 = \text{const.}, \ldots x_4 = \text{const.}$ we can divide the extension $\Omega$ into elements which we shall denote by $(dx_1, \ldots dx_4)$. As a rule there will be left near the surface $\sigma$ certain infinitely small extensions of a different form. From the preceding § it is evident that, in the calculation of the integrals, these latter extensions may be neglected and that only the extensions $(dx_1, \ldots dx_4)$ have to be considered. From this we can conclude that equation (10) is valid for any finite extension, as soon at it holds for each of the elements $(dx_1, \ldots dx_4)$.

§ 22. We shall now show what equation (10) becomes for one element $(dx_1, \ldots dx_4)$. Besides the infinitesimal quantities $x_1, \ldots x_4$, occurring in the equation

$$F = \Sigma\,(ab)\, g_{ab}\, x_a\, x_b = \varepsilon^2$$

of the indicatrix we introduce four other quantities $\xi_1, \ldots \xi_4$, which we define by

$$\xi_a = \frac{1}{2}\frac{\partial F}{\partial x_a}, \quad \ldots \ldots \ldots \quad (18)$$

or

$$\left.\begin{array}{l} \xi_1 = g_{11}\, x_1 + g_{12}\, x_2 + \ldots + g_{14}\, x_4 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ \xi_4 = g_{41}\, x_1 + g_{42}\, x_2 + \ldots + g_{44}\, x_4 \end{array}\right\} \quad \ldots \quad (19)$$

with the equalities $g_{ba} = g_{ab}$.

To each of these quantities corresponds a definite direction, viz. that in which we have to proceed in order to make the considered quantity change in positive sense while the other three remain constant. If we denote these directions by 1*, 2*, 3*, 4* and in the same way the directions of the coordinates $x_1, x_2, x_3, x_4$ by 1, 2, 3, 4, it is evident that 1* is conjugate with 2, 3 and 4, 2* with 3, 1 and 4, and so on; inversely 1 with 2*, 3*, 4*; 2 with 3*, 1*, 4*, and so on. From what has been said above about the algebraic signs of $g_{11}$, $g_{22}$, $g_{33}$, $g_{44}$ it follows further that, if directions opposite to 1, 1* etc. are denoted by $-1$, $-1^*$ etc., the directions $-1$ and 1* will point to the same side of an extension $x_1 = \text{const}$. The same may be said of the directions $-2$ and 2* or $-3$ and 3* with respect to extensions $x_2 = \text{const.}$ or $x_3 = \text{const.}$, while with respect to an extension $x_4 = \text{const.}$ the directions 4 and 4* point to the same side.

Finally, we shall fix (§ 11) as far as is necessary, which direction corresponds to three others. For that purpose we shall imagine the directions of coordinates 1, . . . 4 to pass into mutually conjugate directions, which will also be called 1, . . . 4, by gradual changes, in such a way that never three of them come to lie in one plane. We shall agree that after this change $-4$ corresponds to 1, 2, 3.

Let $a, b, c, d$ be the numbers 1, 2, 3, 4 in an order obtained from the natural one by an *even* number of permutations. Then the rule of § 11 teaches us that the direction $-d$ corresponds to $a, b, c$. It is clear that this would be the case with $d$, if $a, b, c, d$ were obtained from 1, 2, 3, 4 by an odd number of permutations. If further it is kept in mind that, always in the new case, the directions 1*, 2*, 3*, 4* coincide with $-1$, $-2$, $-3$, 4, we come to the conclusion that the directions 1, 2, 3 and 4 correspond to the sets 2*, 3*, 4*; 3*, 1*, 4*; 1*, 2*, 4* and 1*, 2*, 3* respectively. The rule of gradual change (§ 11) involves that this holds also for the original case, in which 1, 2, 3, 4 were not yet mutually conjugate.

This is all that has to be said about the relations between the different directions. It must only be kept in mind, that whenever two of the first three directions are interchanged, the fourth must be reversed.

§ 23. In the neighbourhood of a point $P$ of the field-figure we may introduce as coordinates instead of $x_1, \ldots x_4$ the quantities $\xi_1, \ldots \xi_4$ defined by (19). Line-elements or finite vectors can be resolved in the directions of these coordinates, i.e. in the directions

1*, 2*, 3*, 4*. Their components and the magnitudes of different extensions can now be expressed in $\xi$-units in the same way as formerly in $x$-units. So the volume of a three-dimensional parallelepiped with the positive edges $d\xi_1$, $d\xi_2$, $d\xi_3$ is represented by the product $d\xi_1\, d\xi_2\, d\xi_3$.

Solving $x_1, \ldots x_4$ from (19) we obtain expressions of the form

$$\left.\begin{array}{l} x_1 = \gamma_{11}\,\xi_1 + \gamma_{21}\,\xi_2 + \ldots + \gamma_{41}\,\xi_4 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ x_4 = \gamma_{14}\,\xi_1 + \gamma_{24}\,\xi_2 + \ldots + \gamma_{44}\,\xi_4 \\ \qquad \gamma_{ba} = \gamma_{ab} \end{array}\right\} \quad \ldots \quad (20)$$

If we use the coordinates $\xi$ the coefficients $\gamma_{ab}$ play the same part as the coefficients $g_{ab}$ when the coordinates $x$ are used. According to (18) and (20) we have namely

$$F = \Sigma\,(a)\,\xi_a\,x_a = \Sigma\,(ab)\,\gamma_{ba}\,\xi_a\,\xi_b,$$

so that the equation of the indicatrix may be written

$$\Sigma\,(ab)\,\gamma_{ba}\,\xi_a\,\xi_b = \varepsilon^2.$$

§ 24. Let the rotations $\mathsf{R}_e$ and $\mathsf{R}_h$ of which we spoke in § 13 be defined by the vectors $\mathsf{A}^{\mathrm{I}}$, $\mathsf{A}^{\mathrm{II}}$ and $\mathsf{A}^{\mathrm{III}}$, $\mathsf{A}^{\mathrm{IV}}$ respectively, the resultants of the vectors $\mathsf{A}_{1^*}{}^{\mathrm{I}}, \ldots \mathsf{A}_{4^*}{}^{\mathrm{I}}$, etc. in the directions $1^*, \ldots 4^*$. Then, according to the properties of the vector product that were discussed in § 11,

$$[\mathsf{R}_e\,.\,\mathsf{N}] = [(\mathsf{A}_{1^*}{}^{\mathrm{I}} + \ldots + \mathsf{A}_{4^*}{}^{\mathrm{I}})\,.\,(\mathsf{A}_{1^*}{}^{\mathrm{II}} + \ldots + \mathsf{A}_{4^*}{}^{\mathrm{II}})\,.\,\mathsf{N}]$$
$$= \Sigma\,(\overline{ab})\,\{[\mathsf{A}_{a^*}{}^{\mathrm{I}}\,.\,\mathsf{A}_{b^*}{}^{\mathrm{II}}\,.\,\mathsf{N}] - [\mathsf{A}_{a^*}{}^{\mathrm{II}}\,,\,\mathsf{A}_{b^*}{}^{\mathrm{I}}\,.\,\mathsf{N}]\},$$

where the stroke over $ab$ indicates that each combination of two different numbers $a, b$ contributes one term to the sum. For the vector product $[\mathsf{R}_h\,.\,\mathsf{N}]$ we have a similar equation. Now two or more rotations in one and the same plane, e.g. in the plane $a^*b^*$, may be replaced by one rotation, which can be represented by means of two vectors with arbitrarily chosen directions in that plane, e.g. the directions $a^*$ and $b^*$. We may therefore introduce two vectors $\mathsf{B}_{a^*}$ and $\mathsf{B}_{b^*}$ directed along $a^*$ and $b^*$ resp., so that

$$[\mathsf{B}_{a^*}\,.\,\mathsf{B}_{b^*}] = [\mathsf{A}_{a^*}{}^{\mathrm{I}}\,.\,\mathsf{A}_{b^*}{}^{\mathrm{II}}] - [\mathsf{A}_{a^*}{}^{\mathrm{II}}\,.\,\mathsf{A}_{b^*}{}^{\mathrm{I}}] +$$
$$+ [\mathsf{A}_{a^*}{}^{\mathrm{III}}\,.\,\mathsf{A}_{b^*}{}^{\mathrm{IV}}] - [\mathsf{A}_{a^*}{}^{\mathrm{IV}}\,.\,\mathsf{A}_{b^*}{}^{\mathrm{III}}] \quad . \quad (21)$$

Then we must substitute in (10)

$$[\mathsf{R}_e\,.\,\mathsf{N}] + [\mathsf{R}_h\,.\,\mathsf{N}] = \Sigma\,(\overline{ab})\,[\mathsf{B}_{a^*}\,.\,\mathsf{B}_{b^*}\,.\,\mathsf{N}] \quad . \; . \; . \quad (22)$$

Here it must be remarked that the magnitude and the sense of one of the vectors $\mathsf{B}$ may be chosen arbitrarily; when this has been done, the other vector is perfectly determined.

In the following calculations the vector **N** has one of the directions $1^*, \ldots 4^*$. As this is also the case with the vectors $\mathbf{B}_{a^*}$ and $\mathbf{B}_{b^*}$, the vector product occurring in (22) can easily be expressed in $\xi$-units. After that we may pass to natural units and finally, as is necessary for the substitution in (10), to $x$-units.

In order to pass from $\xi$-units to natural units we have to multiply a vector in the direction $a^*$ by a certain coefficient $\lambda_a$, and a part of the extension $a^*$, $b^*$, $c^*$ by a coefficient $\lambda_{abc}$. These coefficients correspond to $l_a$ (§ 10) and $l_{abc}$ (§ 12). The factors $\lambda_{abc}$ e.g. can be expressed by means of the minors $\Gamma_{ab}$ of the determinant $\gamma$ of the quantities $\gamma_{ab}$. If this is worked out and if the equations

$$\gamma_{ab} = \frac{G_{ab}}{g} \quad , \quad g_{ab} = \frac{\Gamma_{ab}}{\gamma} \quad . \quad g\gamma = 1$$

are taken into consideration, we obtain the following corollary, which we shall soon use:

Let $a, b, c, d$ and also $a', b' c', d'$ be the numbers $1, 2, 3, 4$ in any order, $a'$ being not the same as $a$, then we have, if none of the two numbers $a$ and $a'$ is 4,

$$\frac{l_{bcd}\,\lambda_{b'c'd'}}{l_{a'}\,\lambda_a} = -1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and if one of the two is 4

$$\frac{l_{bcd}\,\lambda_{b'c'd'}}{l_{a'}\,\lambda_a} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

§ 25. We shall now suppose (comp. § 24) that in $\xi$-units the vector $\mathbf{B}_{a^*}$ has the value $+1$, and we shall write $\chi_{ab}$ for the value that must then be given to $\mathbf{B}_{b^*}$. If the $\xi$-components of the vectors $\mathbf{A}^{\mathrm{I}}$ etc. are denoted by $\Xi_1{}^{\mathrm{I}}, \ldots \Xi_4{}^{\mathrm{I}}$ etc., we find from (21)

$$\chi_{ab} = (\Xi_a{}^{\mathrm{I}}\, \Xi_b{}^{\mathrm{II}} - \Xi_a{}^{\mathrm{II}}\, \Xi_b{}^{\mathrm{I}}) + (\Xi_a{}^{\mathrm{III}}\, \Xi_b{}^{\mathrm{IV}} - \Xi_a{}^{\mathrm{IV}}\, \Xi_b{}^{\mathrm{III}}) \quad . \quad (25)$$

This formula involves that

$$\chi_{ba} = -\chi_{ab}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

It may be remarked that $\chi_{ba}$ is the value that must be given to the vector $B_{a^*}$ if $B_{b^*}$ is taken to be 1.

The quantities $\chi_{ab}$ may be said to represent the rotations $[B_{a^*} \,.\, B_{b^*}]$.

At the end of our calculations we shall introduce instead of $\chi_{ab}$ the quantities $\psi_{ab}$ defined by

$$\psi_{ab} = \chi_{a'b'}\,(a \mp b) \quad , \quad \psi_{aa} = 0 \quad . \quad . \quad . \quad . \quad . \quad (27)$$

In the first of these equations $a, b, a', b'$ are supposed to be the numbers $1, 2, 3, 4$, in an order obtained from $1, 2, 3, 4$ by an even number of permutations.

§ 26. We have now to calculate the left hand side of equation (10) for the case that $\sigma$ is the surface of an element $(dx_1, .. dx_4)$. For this purpose we shall each time take together two opposite sides, calculating for each pair the contributions due to the different terms on the right hand side of (22), or as we may say to the different rotations $\chi_{ab}$. It is convenient now to denote by $a, b, c$ the numbers 1, 2, 3 either in this order or in any other derived from it by a cyclic permutation, while the $x$-components of the vector we are calculating and which stands on the left hand side of (10) will be represented by $X_1, ... X_4$.

*a*. Let us first consider that one of the sides $(dx_a, dx_b, dx_c)$ which faces towards the side of the positive $x_4$. The vector **N** drawn outward has the direction 4* and in $\xi$-units the magnitude $\frac{1}{\lambda_4}$. As the direction $c$ corresponds to $a^*, b^*, 4^*$, the rotation $\chi_{ab}$ gives with **N** a vector product represented by a vector in the direction $c$. The magnitude of this vector is in $\xi$-units

$$\frac{1}{\lambda_4} \chi_{ab}$$

and in natural units

$$\frac{\lambda_{ab4}}{\lambda_4} \chi_{ab}.$$

This must be multiplied by $l_{abc}\, dx_a\, dx_b\, dx_c$, the magnitude of the side under consideration in natural units, and finally by $\frac{1}{l_c}$ to express the vector product in $x$-units. Because of (24) we may write for the result

$$\chi_{ab}\, dx_a\, dx_b\, dx_c = \psi_{c4}\, dx_a\, dx_b\, dx_c.$$

The opposite side gives a similar result with the opposite sign (**N** having for that side the direction $-4^*$), so that together the sides contribute the term

$$\frac{\partial \psi_{c4}}{\partial x_4} dW$$

to the component $X_c$. For shortness' sake we have put here

$$dx_1\, dx_2\, dx_3\, dx_4 = dW.$$

Finally we may take $c = 1, 2, 3$.

*b*. Secondly we consider a side $(dx_a, dx_b, dx_4)$ facing towards the positive $x_c$. The vector **N** has now the direction $-c^*$. We consider the vector products of this vector with the rotations $\chi_{b4}$, $\chi_{4a}$ and $\chi_{ba}$, which vector products have the directions $a$, $b$ and 4. A calculation exactly similar to the one we performed just now gives the contributions to $X_a$, $X_b$, $X_4$. For these we thus find the products of $dx_a\, dx_b\, dx_4$ by

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$$\frac{l_{ab4}\,\lambda_{bc4}}{l_a\,\lambda_c}\chi_{b4}=\chi_{4b}=\psi_{ac},$$

$$\frac{l_{ab4}\,\lambda_{ac4}}{l_b\,\lambda_c}\chi_{4a}=\chi_{a4}=\psi_{bc},$$

$$\frac{l_{ab4}\,\lambda_{abc}}{l_4\lambda_c}\chi_{ba}=\chi_{ba}=\psi_{4c}.$$

Taking also into consideration the opposite side $(dx_a, dx_b, dx_4)$ we find for $X_a$, $X_b$, $X_4$ the contributions

$$\frac{\partial\psi_{ac}}{\partial x_c}dW,\quad \frac{\partial\psi_{bc}}{\partial x_c}dW\quad \frac{\partial\psi_{4c}}{\partial x_c}dW.$$

This may be applied to each of the three pairs of sides not yet mentioned under $a$; we have only to take for $c$ successively 1, 2, 3.

Summing up what has been said in this § we may say: the components of the vector on the left hand side of (10) are

$$X_a=\Sigma\,(b)\frac{\partial\psi_{ab}}{\partial x_b}dW.$$

§ 27. For the components of the vector occurring on the right hand side of (10) we may write

$$i\,\mathsf{q}_a\,d\Omega,$$

if $\mathsf{q}_a$ is the component of the vector $\mathsf{q}$ in the direction $x_a$ expressed in $x$-units, while $d\Omega$ represents the magnitude of the element $(dx_1, \ldots dx_4)$ in natural units. This magnitude is

$$-i\sqrt{-g}\,dW,$$

so that by putting

$$\sqrt{-g}\,\mathsf{q}_a=w_a \quad \ldots\ldots\ldots \quad (28)$$

we find for equation (10)

$$\Sigma\,(b)\frac{\partial\psi_{ab}}{\partial x_b}=w_a \quad \ldots\ldots\ldots \quad (29)$$

The four relations contained in this equation have the same form as those expressed by formula (25) in my paper of last year[1]). We shall now show that the two sets of equations correspond in all respects. For this purpose it will be shown that the transformation formulae formerly deduced for $w_a$ and $\psi_{ac}$ follow from the way in which these quantities have been now defined. The notations from the former paper will again be used and we shall suppose the transformation determinant $p$ to be positive.

[1]) Zittingsverslag Akad. Amsterdam, **23** (1915), p. 1073; translated in Proceedings Amsterdam, **19** (1916), p. 751. Further on this last paper will be cited by l. c.

§ 28. Between the differentials of the original coordinates $x_a$ and the new coordinates $x'_a$ which we are going to introduce we have the relations

$$dx'_a = \Sigma (b)\, \pi_{ba}\, dx_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and formulae of the same form (comp. § 10) may be written down for the components of a vector expressed in $x$-measure. As the quantities $\mathfrak{q}_a$ constitute a vector and as

$$\sqrt{-g'} = p \sqrt{-g},$$

we have according to (28) [1])

$$\frac{1}{\sqrt{-g'}} w'_a = \frac{1}{\sqrt{-g}} \Sigma (b)\, \pi_{ba}\, w_b,$$

or

$$w'_a = p\, \Sigma (b)\, \pi_{ba}\, w_b\, .$$

Further we have for the infinitely small quantities $\xi_a$ [2]) defined by (19)

$$\xi'_a = \Sigma (b)\, p_{ba}\, \xi_b\, .$$

and in agreement with this for the components of a vector expressed in $\xi$-units

$$\Xi'_a = \Sigma (b)\, p_{ba}\, \Xi_b,$$

so that we find from (25) [3])

$$\chi'_{ab} = \Sigma (cd)\, p_{ca}\, p_{db}\, \chi_{cd}\, .$$

Interchanging here $c$ and $d$, we obtain

$$\chi'_{ab} = \Sigma (cd)\, p_{da}\, p_{cb}\, \chi_{dc} = - \Sigma (cd)\, p_{da}\, p_{cb}\, \chi_{cd}$$

and

$$\chi'_{ab} = \tfrac{1}{2} \Sigma (cd) (p_{ca}\, p_{db} - p_{da}\, p_{cb})\, \chi_{cd} \quad . \quad . \quad . \quad . \quad (31)$$

The quantity between brackets on the right hand side is a second order minor of the determinant $p$ and as is well known this minor

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1) Comp. § 7, l. c.

2) For the infinitesimal quantities $x_a$ occurring in (19) we have namely (comp. (30))

$$x'_a = \Sigma (b)\, \pi_{ba}\, x_b$$

and taking into consideration (19) and (20), i e.

$$\xi_a = \Sigma (b)\, g_{ab}\, x_b\, , \quad x_a = \Sigma (b)\, \gamma_{ba}\, \xi_b$$

and formula (7) l. c., we may write (comp. note 2, p. 758, l. c.)

$$\xi'_a = \Sigma (b)\, g'_{ab}\, x'_b = \Sigma (bcde)\, p_{ca}\, p_{db}\, \pi_{eb}\, g_{cd}\, x_e =$$

$$= \Sigma (cd)\, p_{ca}\, g_{cd}\, x_d = \Sigma (cdf)\, p_{ca}\, g_{cd}\, \gamma_{fd}\, \xi_f = \Sigma (c)\, p_{ca}\, \xi_c\, .$$

3) Put $\Xi_a^{I}\, \Xi_b^{II} = \vartheta_{ab}$. Then we have

$$\vartheta'_{ab} = \Xi_a^{I'}\, \Xi_b^{II'} = \Sigma (cd)\, p_{ca}\, p_{db}\, \Xi_c^{I}\, \Xi_d^{II} = \Sigma (cd)\, p_{ca}\, p_{db}\, \vartheta_{cd}$$

and similar formulae for the other three parts of (25).

is related to a similar minor of the determinant of the coefficients $\pi_{ab}$. If $a'b'$ corresponds to $ab$ in the way mentioned in § 25, and $c'd'$ in the same way to $cd$, we have

$$p_{ca}\, p_{db} - p_{da}\, p_{cb} = p\,(\pi_{c'a'}\, \pi_{d'b'} - \pi_{d'a'}\, \pi_{c'b'}),$$

so that (31) becomes

$$\chi'_{ab} = \tfrac{1}{2}\, p\, \Sigma\, (cd)\, (\pi_{c'a'}\, \pi_{d'b'} - \pi_{d'a'}\, \pi_{c'b'})\, \chi_{cd}.$$

According to (27) this becomes

$$\psi'_{a'b'} = \tfrac{1}{2}\, p\, \Sigma\, (cd)\, (\pi_{c'a'}\, \pi_{d'b'} - \pi_{d'a'}\, \pi_{c'b'})\, \psi_{c'd'},$$

for which we may write

$$\psi'_{ab} = \tfrac{1}{2}\, p\, \Sigma\, (cd)\, (\pi_{ca}\, \pi_{db} - \pi_{da}\, \pi_{cb})\, \psi_{cd}.$$

Interchanging $c$ and $d$ in the second of the two parts into which the sum on the right hand side can be decomposed, and taking into consideration that

$$\psi_{dc} = -\psi_{cd},$$

as is evident from (26) and (27), we find [1])

$$\psi'_{ab} = p\, \Sigma\, (cd)\, \pi_{ca}\, \pi_{db}\, \psi_{cd}.$$

§ 29. Finally it can be proved that if equation (10) holds for one system of coordinates $x_1, \ldots. x_4$, it will also be true for every other system $x'_1, \ldots. x'_4$, so that

$$\int \{ [\mathsf{R}_e \,.\, \mathsf{N}] + [\mathsf{R}_h \,.\, \mathsf{N}] \}_{x'}\, d\sigma = i \int \{ \mathsf{q} \}_{x'}\, d\Omega. \quad . \quad . \quad . \quad (32)$$

To show this we shall first assume that the extension $\Omega$, which is understood to be the same in the two cases, is the element $(dx_1, \ldots. dx_4)$.

For the four equations taken together in (10) we may then write

$$\int u_1\, d\sigma = v_1\, d\Omega, \ldots \int u_4\, d\sigma = v_4\, d\Omega \quad . \quad . \quad . \quad . \quad (33)$$

and in the same way for the four equations (32)

$$\int u'_1\, d\sigma = v'_1\, d\Omega, \ldots \int u'_4\, d\sigma = v'_4\, d\Omega \quad . \quad . \quad . \quad . \quad (34)$$

We have now to deduce these last equations from (33). In doing so we must keep in mind that $u_1, \ldots. u_4$ are the $x$-components and $u'_1 \ldots. u'_4$ the $x'$-components of one definite vector and that the same may be said of $v_1, \ldots. v_4$ and $v'_1 \ldots. v'_4$.

Hence, at a definite point (comp. (30))

$$v'_a = \Sigma\, (b)\, \pi_{ba}\, v_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

We shall particularly denote by $\pi_{ba}$ the values of these quantities belonging to the angle $P$ from which the edges $dx_1, \ldots. dx_4$ issue

[1]) Comp. (28) l. c.

in positive directions. To the right hand sides of the equations (34) we may apply transformation (35) with *these* values of $\pi_{ba}$, $d\Omega$ being infinitely small of the fourth order and it being allowed to confine ourselves to quantities of this order.

On the left hand sides of (34), however, we must take into consideration, the surface being of the third order, that the values of $\pi_{ba}$ change from point to point. Let $\mathfrak{x}_1, \ldots \mathfrak{x}_4$ be the changes which $x_1, \ldots x_4$ undergo when we pass from $P$ to any other point of the surface. Then we must write for the value of the coefficient at this last point

$$\pi_{ba} + \Sigma\,(c)\,\frac{d\pi_{ba}}{dx_c}\,\mathfrak{x}_c.$$

We thus have

$$\int u'_a\,d\sigma = \Sigma\,(b)\,\pi_{ba}\int u_b\,d\sigma + \Sigma\,(b)\int u_b\,\Sigma\,(c)\,\frac{\partial\pi_{ba}}{\partial x_c}\,\mathfrak{x}_c\,d\sigma.$$

It will be shown presently that the last term vanishes. This being proved, it is clear that the relations (34) follow from (33); indeed, multiplying equations (33) by $\pi_{1a}, \ldots \pi_{4a}$ respectively and adding them we find

$$\int u'_a\,d\sigma = v'_a\,d\Omega.$$

§ 30. The proof for

$$\Sigma\,(b)\int u_b\,\Sigma\,(c)\,\frac{\partial\pi_{ba}}{\partial x_c}\,\mathfrak{x}_c\,d\sigma = 0 \quad \ldots\ldots \quad (36)$$

rests on the relations

$$\frac{\partial\pi_{ba}}{\partial x_e} = \frac{\partial\pi_{ea}}{\partial x_b}, \quad \ldots\ldots\ldots\ldots \quad (37)$$

which follow from

$$\pi_{ba} = \frac{\partial x'_a}{\partial x_b}\quad,\quad \pi_{ea} = \frac{\partial x'_a}{\partial x_e}.$$

The integral which occurs in (36) differs from

$$\int u_b\,d\sigma \quad \ldots\ldots\ldots\ldots \quad (38)$$

by the infinitely small factor under the sign of integration

$$\Sigma\,(c)\,\frac{\partial\pi_{ba}}{\partial x_c}\,\mathfrak{x}_c.$$

Now we have calculated in § 26 integrals like (38) by taking together each time two opposite sides, one of which $\Sigma_1$ passes through $P$ while the second $\Sigma_2$ is obtained from the first by a shift in the direction of one of the coordinates e. g. of $x_e$ over the distance $dx_e$. We had then to keep in mind that for the two sides the values of $u_b$, which have opposite signs, are a little different; and it was precisely this difference that was of importance. In the calculation of the integral

$$\int u_b \, \Sigma \, (c) \frac{\partial \pi_{ba}}{\partial x_c} \mathbf{x}_c \, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

however it may be neglected. Hence, when we express the components $u_b$ in terms of the quantities $\psi_{ab}$, we may give to these latter the values which they have at the point $P$.

Let us consider two sides situated at the ends of the edges $dx_e$, and whose magnitude we may therefore express in $x$-units by $dx_j \, dx_k \, dx_l$ if $j$, $k$, $l$ are the numbers which are left of 1, 2, 3, 4 when the number $e$ is omitted. For the part contributed to (38) by the side $\Sigma_2$ we found in § 26

$$\psi_{be} \, dx_j \, dx_k \, dx_l \; .$$

We now find for the part of (39) due to the two sides

$$\psi_{bc} \, \Sigma \, (c) \frac{\partial \pi_{ba}}{\partial x_c} \left[ \int_2 \mathbf{x}_c \, d\sigma - \int_1 \mathbf{x}_c \, d\sigma \right]$$

where the first integral relates to $\Sigma_2$ and the second to $\Sigma_1$. It is clear that but one value of $c$, viz. $e$ has to be considered. As everywhere in $\Sigma_1 : \mathbf{x}_e = 0$ and everywhere in $\Sigma_2 : \mathbf{x}_e = dx_e$ it is further evident that the above expression becomes

$$\psi_{eb} \frac{\partial \pi_{ba}}{\partial x_e} \, dW.$$

This is one part contributed to the expression (36). A second part, the origin of which will be immediately understood, is found by interchanging $b$ and $e$. With a view to (37) and because of

$$\psi_{eb} = - \psi_{be}$$

we have for each term of (36) another by which it is cancelled. This is what had to be proved.

§ 31. Now that we have shown that equation (32) holds for each element $(dx_1, \ldots dx_4)$ we may conclude by the considerations of § 21 that this is equally true for any arbitrarily chosen magnitude and shape of the extension $\Omega$. In particular the equation may be applied to an element $(dx'_1 \ldots dx'_4)$ and by considerations exactly similar to those presented in § 26 we see that in the new coordinates as well as in the original ones we have equations of the form (29).

Whatever be our choice of the coordinates the part of the principal function indicated in § 14 can therefore be derived for a given current vector q.

In a sequel to this paper some conclusions that may be drawn from HAMILTON'S principle will be considered.