three straight lines F_1F_2, F_2F_3, F_3F_1, three trisecants, consecutively passing through F_1, F_2, F_3.

The three straight lines t, meeting in an arbitrary point P, are nodal lines on the surface Π^6, containing the points of support of the chords drawn through P of the curves of the $[\varrho^5]$. With the cone which projects the ϱ^5 passing through P, Π^6 has, besides this ϱ^5, only straight lines passing through P in common; they are the three trisecants out of P, which are nodal lines for both surfaces, and the seven singular bisecants PF_k. From the consideration of the points which Π^6 has in common with an arbitrary ϱ^5 follows that this surface has nodes in the seven fundamental points.

For a point S of the singular quadrisecant Π^6 passes into the monoid Σ^6.

Mathematics. — *"Bilinear congruences of elliptic and hyperelliptic twisted quintics."* By Prof. JAN DE VRIES.

(Communicated in the meeting of April 23, 1915).

1. We consider a net of cubic surfaces Φ^3 of which all figures have a *rational quartic*, σ^4, in common. Two arbitrary Φ^3 have moreover an elliptic quintic ϱ^5 in common, resting on σ^4 in *ten* points. A third surface of the net therefore intersects ϱ^5, outside σ^4, in *five* points F_k; they form with σ^4 the base of the net. As a Φ^3 passing through 13 points of σ^4 wholly contains this curve, only four of the points F_k may be taken arbitrarily for the determination of the net. The base-curves ϱ^5 of the pencils of the net form a *bilinear congruence, with singular curve σ^4 and five fundamental points F_k.*

The singular curve σ^4 may be replaced by the figure composed of a σ^3 with one of its secants, or by the figure composed of two conics, which have *one* point in common, or by the figure consisting of a conic and two straight lines intersecting it.

2. The curves ϱ^5, which intersect σ^4 in the *singular points* S, form a cubic surface Σ^3, with node S, which belongs to the net; S is therefore a *singular point of order three.* The monoids Σ^3 belonging to two points S have σ^4 and a curve ϱ^5 in common; through two points of σ^4 passes therefore in general *one* curve ϱ^5. The groups of 10 points which σ^4 has in common with the curves of the congruence form therefore an involution of the second rank.

On σ^4 lie consequently 36 pairs of points, each bearing ∞^1 curves ϱ^5; in other words, the net contains 36 dimonoids, of which the two nodes are lying on σ^4. The congruence further contains 24 curves ϱ^5, which osculate the singular curve σ^4.

The curves ϱ^5 lying on the monoid Σ^3, are, by central projection out of S, represented by a pencil of plane curves φ^4, with two double base-points and eight single base-points; to it belong the images of the five fundamental points. The remaining three are the intersections of *three singular bisecants* b; through each point of such a straight line passes a ϱ^5 of Σ^3. The two nodes are the intersections of *two singular trisecants* t; each straight line t is moreover intersected in two points by each ϱ^5 of the monoid; for two ϱ^5 the line t is a tangent. The three straight lines b, and the two straight lines t lie of course on Σ^3; the sixth straight line passing through S is a *trisecant* d of σ^4. It is component part of a *degenerate* ϱ^5; for all Φ^3 passing through an arbitrary point of d contain this straight line and have moreover another *elliptic curve* ϱ^4 in common.

3. The locus of the straight lines d is the *hyperboloid* Δ^2, which may be laid through σ^4. The latter has with a monoid Σ^3 the singular curve σ^4 and two trisecants d in common. Consequently Σ^3 contains a straight line d *not* passing through S; the curve ϱ^4 coupled to this straight line must contain the point S. It is represented by a curve φ^3, containing the intersections of the straight lines t, b and the images of the points F, while the line connecting the intersections of the two singular trisecants is the image of the straight line d belonging to this ϱ^4.

The locus of the curves ϱ^4 has in common with Σ^3 the curves σ^4 and two curves ϱ^4; so it is a *surface of order four*, Δ^4. With Δ^2 the surface Δ^4 has in common the curve σ^4; the remaining section is a rational curve δ^4, being the locus of the point $D \equiv (d, \varrho^4)$. As the trisecants of δ^4 form the second system of straight lines of Δ^2, δ^4 and σ^4 have ten points in common. This is confirmed by the observation that the pairs d, ϱ^4 determine on σ^4 a correspondence (7, 3), which has the said ten points as coincidences.

4. The locus of the pairs of points which the curves ϱ^5 have in common with their chords drawn through a point P is a surface Π^6, with a quadruple point P. The tangents in P form the cone $\mathfrak{K}^4$, which projects the curve ϱ^5 laid through P; the two trisecants t of this curve are nodal edges of that cone and at the same time nodal lines of Π^6. The cone, which projects σ^4 out of P has in common with $\mathfrak{K}^4$ the 10 edges containing the points of intersection of σ^4 and ϱ^5; the remaining 6 common edges q are singular bisecants. For q is chord of the curve ϱ^5 passing through P, and moreover of a ϱ^5 intersecting it on σ^4, but in that case it must be chord of ∞^1 curves ϱ^5. The surface Φ^3, which may be laid through q, σ^4 and ϱ^5 does belong to the net; the other surfaces of this net consequently intersect this net in the pairs of a quadratic involution; in other words, q is a singular bisecant.

The six straight lines q lie apparently on Π^6; this surface also contains the five straight lines $f_k \equiv PF_k$, which, as the above mentioned straight lines b, are *particular* (parabolic) *singular bisecants*; through each point f passes a ϱ^5, which has its second point of support in F, so that the involution of the points of support is parabolic. The section of Π^6 and $\mathfrak{K}^4$ apparently consists of a ϱ^5, two straight lines t (which are nodal lines for both surfaces) five straight lines f and six straight lines q.

For a point S of the singular curve σ^4 the surface Π^6 consists of two parts: the *monoid* Σ^3 and a *cubic cone* formed by the singular bisecants q, which intersect σ^4 in S. As a plane contains four points S, consequently 4×3 straight lines q, the singular bisecants form a congruence of rays (6, 12), belonging to the complex of secants of σ^4, which congruence of rays possesses in σ^4 a singular curve of order three.

5. The singular trisecants t form, as has been proved, a congruence of rays of *order two*. The latter has the five fundamental points F as *singular points*, for each of those points bears ∞^1 straight lines t, which form a cone $\mathfrak{T}$. With the cone $\mathfrak{H}^4$, which projects an arbitrary ϱ^5 out of F, $\mathfrak{T}$ has the four straight lines to the remaining points in common and further the two straight lines t, passing through F. As these straight lines are nodal edges of $\mathfrak{H}^4$, $\mathfrak{T}$ must be a quadric cone. The congruence $[t]$ has therefore *five singular points of order two.*

The trisecants t of an elliptic ϱ^5 form [1] a ruled surface $\mathfrak{R}^5$, with nodal curve ϱ^5. The axial ruled surface $\mathfrak{A}$ formed by the straight lines t which intersect a given straight line a, has in common with an arbitrary ϱ^5 in the first place 5×3 points, in which ϱ^5 is intersected by the five straight lines t resting on a. Moreover they have in common the five points F, which, however, are nodes of $\mathfrak{A}$. Consequently $\mathfrak{A}$ is a ruled surface of order five. As a is nodal line

[1]) Vid. e.g. my paper in volume II (p. 374) of these Proceedings.

of $\mathfrak{A}^5$, a plane passing through a contains three straight lines more hence the *singular trisecants* form *a congruence* (2, 3).

6. A straight line l intersects three curves ϱ^5 of a monoid Σ^3; consequently σ^4 is a *triple curve* on the surface A formed by the ϱ^5, intersecting l. As two surfaces A^x, outside σ^4, have but x curves ϱ^5 in common, we have $x^2 = 5x + 36$, hence $x = 9$. An arbitrary curve ϱ^5 intersects A^9 on σ^4 in 10×3 points, consequently fifteen times in F_k; so A^9 has *five triple points* F_k. On A^9 lie (§ 3) *six* straight lines and *six* elliptic curves ϱ^4; the ϱ^5, for which l is a chord, is a *nodal curve*.

In a plane λ passing through l, the congruence $[\varrho^5]$ determines a quintuple-involution possessing four singular points S of order three. It transforms a straight line l into a curve λ^8 with four triple points, and has a *curve of coincidence* of order six, γ^6, with four nodes S. With an arbitrary surface A^9 the curve γ^6, has outside S_k, $9 \times 6 - 4 \times 3 \times 2 = 30$ points in common. The curves ϱ^5, touching a plane φ, consequently form a surface Φ^{30}; on it σ^4 is a *decuple curve* (Σ^3 intersects γ^6, outside S_k, in $3 \times 6 - 4 \times 2$ points) while F_k are *decuple points* (an arbitrary ϱ^5 intersects Φ^{30}, outside σ^4, in $5 \times 30 - 10 \times 10$ points).

Φ^{30} has in common with φ another curve φ^{18}, possessing four sextuple points S; it touches φ^6 in 30 points; φ is therefore *osculated* by *thirty curves* ϱ^5.

Two surfaces Φ^{30} have, outside σ^4, 100 curves ϱ^5 in common, *two planes* are therefore *touched* by 100 curves ϱ^5.

7. When all the surfaces Φ^3 of a net have an *elliptic twisted curve* σ^4 in common, the variable base-curves ϱ^5 of the pencils comprised in the net form a *bilinear congruence of hyperelliptic curves*. Each ϱ^5 rests in *eight* points on σ^4 and has with an arbitrary surface Φ^3 moreover *seven fundamental points* F_k in common. As the net is completely determined by σ^4 and five points F, the points F cannot be taken arbitrarily.

The *singular curve* σ^4 may be replaced by the figure composed of a curve σ^3 and one of its chords, or by two conics having two points in common.[1])

8. The *monoid* Σ^3, which has the *singular point* S as node

[1]) In both cases a Φ^3, containing 12 points of the base-figure, will contain it entirely. This elucidates the fact that Φ^3 needs only to be laid through 12 points of the elliptic σ^4 in order to contain it entirely.

and belongs to the net $[\Phi^3]$, again contains all the ϱ^5 intersecting the singular curve σ^4 in S. In representing Σ^3 on a plane φ the system of those curves passes into a pencil of hyperelliptic curves φ^4, with a double base-point and 12 simple base-points. The first is the intersection of a singular trisecant t, consequently of a straight line passing through S, which is moreover twice intersected by all the ϱ^5 lying on Σ^3.

To the simple base-points belong the central projections of the 7 fundamental points. The remaining five are *singular bisecants* b, consequently straight lines, which have a second point in common with any ϱ^5 passing through S. With the trisecant already mentioned they form the six straight lines of Σ^3 passing through S. The straight lines b, are, as well as the straight lines f passing through the fundamental points, *parabolic bisecants.*

9. In the same way as above (§ 4) it is proved that an arbitrary point bears *eight singular bisecants* q, i.e. straight lines, which are intersected by $[\Phi^3]$ in the pairs of an involution; they belong to the complex of secants of σ^4. The straight lines q passing through a point S of σ^4 again form a *cubic cone,* so that $[q]$ is a congruence of rays (8, 12).

The singular trisecants t form a congruence of *order one,* which has the points F as singular points. The singular cone $\mathfrak{T}$ belonging to F is a quadric cone as it has in common with the cone $\mathfrak{F}^4$, which projects an arbitrary ϱ^5 out of F, six straight lines FF' and a trisecant t, which is nodal edge of $\mathfrak{F}^4$. As the trisecants of ϱ^5 form a ruled surface $\mathfrak{R}^2$, the axial ruled surface $\mathfrak{A}$, belonging to a straight line a, has in common with a ϱ^5 the six points of support of two trisecants and the seven nodes F, consequently is of order four. But in that case $[t]$ is of *class three,* consequently the congruence of the bisecants of a *cubic* τ^3, passing through the seven points F.

As in § 6 we find that two arbitrary straight lines are intersected by *nine* curves ϱ^5, that two arbitrary planes are touched by *a hundred* curves, that there are *thirty* curves osculating a given plane.

Here too, the fundamental points are triple on Λ^9, decuple on Φ^3.