Table III (p. 326) shows the values of the wind velocity, the direction of the wind and the angle of deviation as calculated from equation (2) for 16 different directions of the gradient and a uniform field of 1 mm. difference of pressure per degree of latitude.

A comparison of these results with those of table II shows that the use of an average wind for the whole country has induced a more regular course in the numbers, but also that considerable differences are due to this method. The wind velocity and the angles of deviation have become smaller as also the azimuths and wind directions. From this result we may conclude that the northerly stations behave differently in many respects from Flushing and that a combination as made in this inquiry is not desirable.

Physics. — *"On a General Electromagnetic Thesis and its Application to the Magnetic State of a Twisted Iron Bar".* By Dr. G. J. ELIAS. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of May 29, 1915).

WIEDEMANN has already observed that in a longitudinally resp. circularly magnetized iron bar a circular resp. longitudinal magnetisation arises in consequence of torsion. Moreover he discovered that a bar which is at the same time longitudinally and circularly magnetized, is twisted. These observations formed the starting point of the following considerations.

In a magnetic field, in which the magnetic induction can be an arbitrary vector function of the magnetic force variable from point to point, whereas the media in the field can be anisotropic also with respect to the conductivity, but in which no phenomena of hysteresis occur, the equation

$$T = \frac{1}{c}\int \Sigma\, i dM \quad [1] \quad \ldots \ldots \ldots \quad (1)$$

holds for the magnetic field energy.

In this i means the current in a circuit M, the induction flux passing through this circuit, c representing the ratio of the electromagnetic to the electrostatic unity of electricity. The summation extends over all the circuits, the integration covering a range from M for $i = 0$ to the final value which M assumes.

[1]) In this and following formulae LORENTZ's system of unities is used.

1. Let us now consider two linear conductors (circuits), in which currents i_1 and i_2 run. Let M_1 be the induction flux passing through the first, M_2 that through the second wire. If M_1 and M_2 change infinitely little, then follows from (1)

$$dT = \frac{1}{c}(i_1 dM_1 + i_2\, dM_2),$$

for which we may put:

$$dT = \frac{1}{c}\, d(i_1 M_1 + i_2 M_2) - \frac{1}{c} M_1\, di_1 - \frac{1}{c} M_2\, di_2$$

The first member of this equation is a total differential, as T is perfectly determined by i_1 and i_2, hence

$$M_1\, di_1 + M_2\, di_2$$

must also be a total differential, from which follows:

$$\frac{\partial M_1}{\partial i_2} = \frac{\partial M_2}{\partial i_1}\ ^{1)}, \quad \ldots\ldots\ldots \quad (2)$$

i.e. the increase of the induction flux passing through the first circuit, caused by an infinitely small current variation in the second, is equal to the increase of induction flux passing through the second circuit, caused by an equal change of current in the first.

An increase of the induction flux dM will give rise to an electrical impulse, in which through every section of the circuit the quantity of electricity

$$de = -\frac{1}{c}\frac{dM}{w}$$

passes, if w represents the resistance of the circuit. The negative sign means that the direction of the current, is in lefthand cyclical order with the increase of the induction flux.

If now the current i_1 increases by the infinitely small amount of di_1, the induction flux through the second circuit will increase by:

$$dM_2 = \frac{\partial M_2}{\partial i_1} di_1.$$

Hence for a short time an induction current will pass through the second conductor. If after the lapse of this time the current in this conductor has again the same value as before, then the "integral current", i.e. the total quantity of electricity set in motion by the induction current amounts to:

1) For so far as I have been able to ascertain, this relation, as well as those following later (3), (8), (15) and (17) is new.

$$de_2 = -\frac{1}{c} \cdot \frac{dM_2}{w_2} = -\frac{1}{c \cdot w_2} \frac{\partial M_2}{\partial i_1} di_1 .$$

In the same way for an infinitesimal change of i_2 the integral current in the first conductor will amount to

$$de_1 = -\frac{1}{c \cdot w_1} \frac{\partial M_1}{\partial i_2} di_2 .$$

If $di_1 = di_2$, it follows from this by the aid of (2)

$$w_1 de_1 = w_2 de_2 \quad \ldots \ldots \ldots \quad (3')$$

If by $\frac{\partial e_1}{\partial i_2}$ resp. $\frac{\partial e_2}{\partial i_1}$ we denote the quotient of the integral current in the first resp. second conductor and the change of current in the second resp. first conductor, we may also write:

$$w_1 \frac{\partial e_1}{\partial i_2} = w_2 \frac{\partial e_2}{\partial i_1} \quad \ldots \ldots \ldots \quad (3)$$

In case the permeability is independent of the intensity of the field, so that $\mathfrak{B}$ in general is a linear vector function of $\mathfrak{H}$, both $\mathfrak{H}$ and $\mathfrak{B}$ are linear functions of i_1 and i_2, hence M_1 and M_2 too. Then we may write:

$$\left. \begin{aligned} M_1 &= L_{11} i_1 + L_{12} i_2 \\ M_2 &= L_{21} i_1 + L_{22} i_2 \end{aligned} \right\} \quad \ldots \ldots \ldots \quad (4)$$

From (2) then follows the known thesis:

$$L_{12} = L_{21} \quad \ldots \ldots \ldots \quad (5)$$

i.e. with equal currents in the two circuits the first sends as many induction lines through the second as the second through the first. For this case the magnetic field energy becomes according to (1):

$$T = \frac{1}{2c} L_{11} i_1^2 + \frac{1}{c} L_{12} i_1 i_2 + \frac{1}{2c} L_{22} i_2^2 \quad \ldots \quad (6)$$

If the current in the first circuit increases by di_1, then the integral current in the second amounts to:

$$de_2 = -\frac{dM_2}{c \cdot w_2} = -\frac{L_{21} \cdot di_1}{c \cdot w_2} .$$

On increase of the current in the second conductor by di_2 the integral current:

$$de_1 = -\frac{L_{12} \cdot di_2}{c \cdot w_1} .$$

flows through the first.

Both expressions can be integrated. If e_1 resp. e_2 represent the integral currents, which pass through the first resp. second circuit on increase of the current in the second resp. first circuit to the same amount i, the relation

$$e_1 w_1 = e_2 w_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

exists between these quantities.

2. We shall now consider the case that the function which represents the relation between $\mathfrak{B}$ and $\mathfrak{H}$ is variable in some parts of the field. This variability is meant in very general sense: we may e. g. imagine it as a dependence of volume, pressure, temperature etc. or as variations in consequence of elastic deformations, while also motions of the particles of the medium may be understood by it. We except, however, such changes which are attended with motions of the current conductors or parts of them. Let the variability be expressed by means of the general coordinate α. Then the induction flux through the circuits will in general depend on α. With a variation of α the relation (2) holds both before and after the change, so that we get:

$$\frac{\partial}{\partial \alpha} \frac{\partial M_1}{\partial i_2} = \frac{\partial}{\partial \alpha} \frac{\partial M_2}{\partial i_1},$$

for which we may write, seeing that

$$\frac{1}{c} dM_1 = - w_1 de_1 \qquad \frac{1}{c} dM_2 = - w_2 de_2$$

$$\frac{\partial}{\partial \alpha} \left(w_1 \frac{\partial e_1}{\partial i_2} \right) = \frac{\partial}{\partial \alpha} \left(w_2 \frac{\partial e_2}{\partial i_1} \right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

when we attach analogous signification to the partial differential quotients $\frac{\partial e_1}{\partial i_2}$ and $\frac{\partial e_2}{\partial i_1}$ as above for (3). If the resistances are not dependent on α, we get:

$$w_1 \frac{\partial}{\partial i_2} \frac{\partial e_1}{\partial \alpha} = w_2 \frac{\partial}{\partial i_1} \frac{\partial e_2}{\partial \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8')$$

We may express this relation in the following words: Successively we measure four quantities of electricity: 1. the integral current $(de_1)_{i_1, i_2}$ in the first conductor, which is the result of the change $d\alpha$, whilst the currents i_1 and i_2 run through the two conductors; 2. the integral current $(de_2)_{i_1, i_2}$, which flows in the second conductor under the same circumstances; 3. the integral current $(de_1)_{i_1, i_2 + di}$ in the first conductor, which is the result of the same change as under 1, with this difference however, that the current in the second conductor is $i_2 + di$; 4. the integral current $(de_2)_{i_1 + di, i_2}$ in the second conductor, which is the consequence of the same change as under

2, with this difference, however, that the current in the first conductor is $i_i + di$.

Now according to (8′) the difference of $(de_1)_{i_1, i_2}$ and $(de_1)_{i_1, i_2 + di}$, multiplied by the resistance of the first conductor must be equal to the difference of $(de_2)_{i_1, i_2}$ and $(de_2)_{i_1 + di, i_2}$, multiplied by the resistance of the second conductor.

If the relation between $\mathfrak{B}$ and $\mathfrak{H}$ is linear, then on change of α the relation (7) will hold, both before and after the change, so that we have quite generally

$$\frac{\partial}{\partial \alpha}(w_1 e_1) = \frac{\partial}{\partial \alpha}(w_2 e_2) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

If the resistances are not dependent on α we have

$$w_1 \frac{\partial e_1}{\partial \alpha} = w_2 \frac{\partial e_2}{\partial \alpha} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9')$$

i.e. when in the first circuit there runs a current i, the second being without current, and the change $d\alpha$ is accompanied with an integral current de_2 in the second conductor, then the product of de_2 with the resistance of the second circuit will be equal to the product of the resistance of the first circuit with the integral current de_1, which flows through the first circuit in consequence of the change $d\alpha$, when the current i now exists in the second conductor, the first being currentless.

3. Up to now we only considered linear conductors. In order to be able to apply the above derived relations to three-dimensional conductors, we shall first prove a general thesis.

We imagine an arbitrary conductor in which certain electrical forces are active. Let the conductor be an anisotropic body, of such a symmetry, however, that there are three main directions which are vertical with respect to each other, in which the current coincides with the electrical force. In this case:

$$\left.\begin{aligned} \mathfrak{J}_x &= \sigma_{11}\,\mathfrak{E}_x + \sigma_{12}\,\mathfrak{E}_y + \sigma_{13}\,\mathfrak{E}_z \\ \mathfrak{J}_y &= \sigma_{21}\,\mathfrak{E}_x + \sigma_{22}\,\mathfrak{E}_y + \sigma_{23}\,\mathfrak{E}_z \\ \mathfrak{J}_z &= \sigma_{31}\,\mathfrak{E}_x + \sigma_{32}\,\mathfrak{E}_y + \sigma_{33}\,\mathfrak{E}_z \end{aligned}\right\} , \quad . \; . \; . \; . \; . \quad (10)$$

in which

$$\sigma_{12} = \sigma_{21} \qquad \sigma_{23} = \sigma_{32} \qquad \sigma_{31} = \sigma_{13}$$

Now let a system of electrical forces $\mathfrak{E}^{(1)}$ give rise to a current $\mathfrak{J}^{(1)}$, the system $\mathfrak{E}^{(2)}$ giving rise to a current $\mathfrak{J}^{(2)}$. Then the following equation will hold for every volume element, as is easy to see by the aid of (10):

$$(\mathfrak{E}^{(1)} . \mathfrak{J}^{(2)}) = (\mathfrak{E}^{(2)} . \mathfrak{J}^{(1)}).$$

Integrated with respect to an arbitrary volume of the conductor this yields:

$$\int (\mathfrak{E}^{(1)} . \mathfrak{J}^{(2)}) . dS = \int (\mathfrak{E}^{(2)} . \mathfrak{J}^{(1)}) . dS \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This we apply to a conductor consisting of two parts, one of which, A, is a three-dimensional body, whereas the other, B, which is to be considered as linear, is in contact with the three-dimensional part in its initial point P and its final point Q. Let us suppose in the linear part a galvanometer G, which we use to measure the current I in the linear part. The case that arbitrary electrical forces are active in this system, e.g. originating from induction actions which can vary from moment to moment, we shall denote by (1). In case (2) on the other hand we imagine a constant electromotive force to act in the linear part. Then there will exist a potential difference $\varphi_Q - \varphi_P$ between the points Q and P.

In both cases we divide the three-dimensional part A into the circuits that compose the current. Let us call the current in each circuit i and let us denote an element of the circuit by ds, then the relation (11) gives:

$$\Sigma \int \mathfrak{E}^{(1)}_{(2)} . i^{(2)} ds^{(2)} = \Sigma \int \mathfrak{E}^{(2)}_{s(1)} . i^{(1)} . ds^{(1)}.$$

In this the integration takes place along the circuits, the summation extending over all the circuits. In the lefthand member we may write $i^{(2)} = \frac{1}{w^{(2)}} (\varphi_Q - \varphi_P)$, when $w^{(2)}$ denotes the resistance of a circuit in case (2). For every circuit this current is multiplied by the linear integral of the electrical force in case (1) along the circuit. In the righthand member we shall have to distinguish between circuits which are closed in themselves inside the part A, and circuits which start in Q and terminate in P. For the first kind:

$$\int \mathfrak{E}^{(2)}_{s(1)} \, ds_{(1)} = 0,$$

seeing that

$$\mathfrak{E}^{(2)} = - \nabla \varphi.$$

For the second kind:

$$\int \mathfrak{E}^{(2)}_{s(1)} ds^{(1)} = \varphi_Q - \varphi_P,$$

further holding for this:

$$\Sigma i^{(1)} = I,$$

when I is the current measured by the aid of the galvanometer G. If we divide both parts by $\varphi_Q - \varphi_P$, we get finally:

$$I = \Sigma \frac{1}{w^{(2)}} \int^{(1)} \mathfrak{E}_{s^{(2)}}\, ds^{(2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

or expressed in words: the total current flowing through the linear part B is obtained by division of the part A into those circuits which are the consequence of the presence of a constant electromotive force in the linear part B, by integration of the electric force $\mathfrak{E}$ along every circuit, by division every time of these line integrals by the resistance of the circuit, and by taking finally the sum of all these quotients.

If we now call an element of a circuit in case (2) briefly ds, we can, with the omission of the indices, also write:

$$I = \Sigma \frac{1}{w} \int \mathfrak{E}_s\, ds.$$

Hence we may assign an imaginary current to every circuit

$$i = \frac{1}{w} \cdot \int_Q^P \mathfrak{E}_s\, ds,$$

from which follows:

$$i \,.\, w = \int_Q^P \mathfrak{E}_s\, ds.$$

On the other hand:

$$I \,.\, W_0 = \int_P^Q \mathfrak{E}_s\, ds$$

holds according to the law of Ohm for the linear part, when W_0 represents the resistance of this.

By adding the two last relations we get:

$$iw + I\, W_0 = \int \mathfrak{E}_s\, ds,$$

in which the integration is extended all along the circuit. If we put:

$$\int \mathfrak{E}_s\, ds = E,$$

we get:

$$i = \frac{E}{w} - \frac{I\, W_0}{w}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and further by summation over all the circuits and introduction of:

$$\Sigma i = I \qquad \Sigma \frac{1}{w} = \frac{1}{W_1},$$

if W_1 is the resistance of the part A,

$$I = \frac{W_1}{W_1 + W_0} \Sigma \frac{E}{w} = \frac{W_1}{W} \Sigma \frac{E}{w},$$

if W is the resistance of the whole system.

If now by w we represent a resistance which is $\frac{W}{W_1}$ times as great as that of the circuit between Q and P, we get:

$$I = \Sigma \frac{E}{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The resistance w introduced here is practically the resistance of a circuit closed in itself, to which the circuits of case (2) discussed above can be supplemented by continuation into the linear part of the conductor. The summation is extended here over all the circuits of the case indicated above by (2).

5. We shall now consider the case of two current conductors of the kind considered just now, so each consisting of a three-dimensional and a linear part. When currents pass through these conductors, either in one of them or in both, and we want to examine the induction action which is the consequence of a change, either of the current in these conductors or of the properties of the surrounding field, then we may, therefore, according to what was derived just now, divide these conductors into the circuits which are the consequence of the presence of a constant electromotive force in the linear part of these conductors, examine the induction action in each of these circuits and take the sum of these.

Let the resistances of the conductors be W_1 and W_2, the currents, measured in the linear part, I_1 and I_2. We shall examine the influence of a change of these currents. We can now divide the first conductor into m circuits, each with a current i_1, the second into n circuits, each with a current i_2, so that we shall have:

$$I_1 = mi_1 \qquad I_2 = ni_2.$$

The resistance of each circuit of the first conductor amounts to $m \cdot W_1$, of the second conductor to $n \cdot W_2$, as the electromotive force must be taken the same for all of them on division into circuits. If we increase the current in every circuit of the second conductor by di_2, then the total induction flux through the p^{th} circuit of the first conductor will be increased by:

$$dM_{1,p} = \sum_n \frac{\partial M_{1,p}}{\partial i_{2,q}} \cdot di_2.$$

As the resistance of every circuit amounts to $m.W_1$ we get for the integral current, which flows through the linear part of the first conductor:

$$de_1 = -\frac{di_2}{c\, m.\, W_1} \sum_m \sum_n \frac{\partial M_{1,p}}{\partial i_{2,q}}$$

For this may also be written:

$$de_1 = -\frac{dI_2}{c.m.n.W_1} \sum_m \sum_n \frac{\partial M_{1,p}}{\partial i_{2,q}}.$$

In the same way the integral current

$$de_2 = -\frac{dI_1}{c\, m.n.\, W_2} \sum_m \sum_n \frac{\partial M_{2\,q}}{\partial i_{1\,p}}$$

flows through the linear part of the second conductor on a change dI_1 of the current in the first conductor.

If

$$dI_1 = dI_2,$$

then follows, when (2) is used:

$$W_1 \cdot de_1 = W_2 \cdot de_2 \quad \ldots \ldots \ldots \quad (15')$$

In general:

$$W_1 \frac{\partial e_1}{\partial I_2} = W_2 \frac{\partial e_2}{\partial I_1} \quad \ldots \ldots \ldots \quad (15)$$

in which the meaning of the differential quotients is analagous to that which was attached to them above in (3).

This relation is analagous to (3). It holds quite generally, so long as $\mathfrak{B}$ is a univalent function of $\mathfrak{H}$, which, however, can be quite arbitrary for the rest.

If the permeability is independent of the strength of the field, so that there exists a linear relation between $\mathfrak{B}$ and $\mathfrak{H}$, we shall be able to integrate equation (15). So we get:

$$W_1 \cdot e_1 = W_2 \cdot e_2 \quad \ldots \ldots \ldots \quad (16)$$

analogous to relation (7). Here just as there e_1 resp. e_2 will mean the integral currents which flow through the linear part of the first resp. second conductor, when the current in the second resp. first conductor increases from zero to the same value I, the other conductor being without current.

5. Just as we did before in the case of two circuits we can also

now consider the case of an infinitely small change of the function which indicates the relation between $\mathfrak{B}$ and $\mathfrak{H}$, in some parts of the field, as result of an infinitely small change of a general coordinate α.

In general (15) will be valid both before and after the change of α, so that analogous to (8) we get from this:

$$\frac{\partial}{\partial \alpha}\left(W_1 \frac{\partial e_1}{\partial I_2}\right) = \frac{\partial}{\partial \alpha}\left(W_2 \frac{\partial e_2}{\partial I_1}\right) \quad . \quad . \quad . \quad . \quad (17)$$

If the resistances remain unchanged we get analogous to (8').

$$W_1 \frac{\partial}{\partial I_2} \frac{\partial e_1}{\partial \alpha} = W_2 \frac{\partial}{\partial I_1} \frac{\partial e_2}{\partial \alpha} \quad , \quad . \quad . \quad . \quad . \quad . \quad (17')$$

which relation is also open to analogous interpretation.

In the special case of a linear relation between $\mathfrak{B}$ and $\mathfrak{H}$ we shall get in the same way analogous to (9):

$$\frac{\partial}{\partial \alpha}(W_1 e_1) = \frac{\partial}{\partial \alpha}(W_2 e_2) \, , \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which becomes for invariable resistances:

$$W_1 \frac{\partial e_1}{\partial \alpha} = W_2 \frac{\partial e_2}{\partial \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18')$$

Here e_1 and e_2 have the same signification as above in (16).

6. We now inquire into the work of the ponderomotive forces, being accompanied with a modification in the magnetic field, which is the consequence of the infinitely small change $d\alpha$. We assume that at the change $d\alpha$ the external electromotive forces remain unchanged, and likewise the coefficients σ, which in the most general case determine the relation between the electrical force and the current.

If $\mathfrak{E}$ represents the electric force, and $\mathfrak{E}^e$ the external electromotive force, then the quantity of energy

$$\{(\mathfrak{E} + \mathfrak{E}^e) \, . \, \mathfrak{J}\} \, . \, dS \, . \, dt.$$

will be consumed as JOULE heat in the volume element dS in the time dt.

On the other hand the energy supplied by the current generators in the time dt is:

$$(\mathfrak{E}^e \, . \, \mathfrak{J}) \, . \, dS \, . \, dt.$$

The difference of these two expressions:

$$-(\mathfrak{E} \, . \, \mathfrak{J}) \, . \, dS \, . \, dt$$

passes into other modes of energy. Integrated with respect to all the conductors this becomes:

$$\int (\mathfrak{E} \cdot \mathfrak{J}) \cdot dS \cdot dt.$$

If we introduce

$$\mathfrak{J} = c \; curl \; \mathfrak{H},$$

and if we make use of the known thesis of the vector calculus that the following equation holds generally:

$$div \, [\mathfrak{A} \, \mathfrak{B}] = \mathfrak{B} \; curl \; \mathfrak{A} - \mathfrak{A} \; curl \; \mathfrak{B}$$

then we get for the above expression:

$$- c \int (curl \, \mathfrak{E}, \mathfrak{H}) \, dS \, . \, dt + c \int div \, [\mathfrak{E}, \mathfrak{H}] \, dS \, . \, dt.$$

Introducing further:

$$curl \, \mathfrak{E} = - \frac{1}{c} \frac{d\mathfrak{B}}{dt}$$

and making use of GAUSS's theorem, we get:

$$\int \left(\frac{d\mathfrak{B}}{dt}, \mathfrak{H} \right) dS \, . \, dt + c \int [\mathfrak{E}, \mathfrak{H}]_\nu \, d\sigma \, . \, dt.$$

The second term vanishes, as on the surfaces of the current conductors the normal component of $[\mathfrak{E}, \mathfrak{H}]$ is continuous, and the integral amounts to zero over the plane in infinity. Accordingly the first term only remains. This will have to be equal to the increase of the energy of the magnetic field and the work of the ponderomotive forces. Hence we get:

$$dT + dA = \int \left(\frac{d\mathfrak{B}}{dt}, \mathfrak{H} \right) dS \, . \, dt.$$

Per volume and time unity:

$$dT + dA = \left(\frac{d\mathfrak{B}}{dt}, \mathfrak{H} \right) dt.$$

For the energy of the magnetic field per volume unity the expression:

$$T = \int_0^{\mathfrak{H}} (\mathfrak{H}, d\mathfrak{B}).$$

holds generally.

With the change $d\alpha$ we shall get:

$$dT = \int_0^{\mathfrak{H} + d\mathfrak{H}} (\mathfrak{H}, d\mathfrak{B}') - \int_0^{\mathfrak{H}} (\mathfrak{H}, d\mathfrak{B}),$$

in which $d\mathfrak{H}$ represents the change of the final value of $\mathfrak{H}$, and $\mathfrak{B}'$ the value of $\mathfrak{B}$ corresponding to $\mathfrak{H}$ in the changed state. Now we get:

$$-dA = \int_0^{\mathfrak{H}+d\mathfrak{H}} (\mathfrak{H}, d\mathfrak{B}') - \int_0^{\mathfrak{H}} (\mathfrak{H}, d\mathfrak{B}) - (d\mathfrak{B}, \mathfrak{H}),$$

from which easily follows:

$$dA = \frac{\partial}{\partial a} \int_0^{\mathfrak{H}} (\mathfrak{B}, d\mathfrak{H}) \, . \, da$$

Integrated with respect to the whole field this becomes:

$$dA = \frac{\partial}{\partial a} \int dS \int \mathfrak{B} \, . \, d\,\mathfrak{H} . \, da \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

We can always split up the vector $\mathfrak{H}$ into two parts, $\mathfrak{H}^0$, for which holds $div\ \mathfrak{H}^0 = 0$, and $\mathfrak{H}^1$, for which holds $curl\ \mathfrak{H}^1 = 0$ [1]). Taking into consideration that generally

$$\int dS\, (\mathfrak{A} \, . \, \mathfrak{B}) = 0$$

on integration over the whole space, when

$$div\ \mathfrak{A} = 0, \quad curl\ \mathfrak{B} = 0,$$

we get:

$$dA = \frac{\partial}{\partial a} \int dS \int \mathfrak{B} \, . \, d\mathfrak{H}^0$$

Making use of the equation:

$$\mathfrak{B} = \mathfrak{H} + \mathfrak{M},$$

we get:

$$dA = \frac{\partial}{\partial a} \left\{ \int dS \int \mathfrak{H} d\mathfrak{H}^0 + \int dS \int \mathfrak{M} d\,\mathfrak{H}^0 \right\} . \, da.$$

As in the first term we can again split up $\mathfrak{H}$ into $\mathfrak{H}^0$ and $\mathfrak{H}^1$, in which $\mathfrak{H}^0$ is independent of a — $\mathfrak{H}^0$ being determined by the current $\mathfrak{J}$ — and as the product $\mathfrak{H}^1\, d\, \mathfrak{H}^0$ integrated over the whole field yields zero, this term will vanish, so that there remains:

$$dA = \int dS \int \frac{\partial \mathfrak{M}}{\partial a} d\mathfrak{H}^0 \, . \, da \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In this $\frac{\partial \mathfrak{M}}{\partial a}$ denotes the change of the magnetisation in consequence of a change da, in which the external electromotive forces and also the coefficients determining the conductivity, remain unchanged.

1) In general we shall understand by $\mathfrak{H}^0$ the intensity of the field as it would be without the presence of the iron, $\mathfrak{H}$ representing the real strength of the field. The difference is $\mathfrak{H}^1$.

7. We shall now consider a special case. Let us imagine a system of two currents, one passing through a vertical cylindrical iron bar, the other through a vertical solenoid which is concentric with the iron bar. We suppose the iron bar, whose length is assumed to be large with respect to the diameter, to be in the middle part of the solenoid, and that the latter on both sides projects far beyond the bar. For the present we assume for simplicity's sake that the permeability of the iron has a constant value.

The first current I_1 gives rise to a circular magnetisation in the iron, the second I_2 to a longitudinal magnetisation. If I_1 and I_2 are in righthand cyclical order the corresponding strengths of the field $\mathfrak{H}_1{}^0$ and $\mathfrak{H}_2{}^0$ are so too.

The resistances of the conductors are called W_1 and W_2.

We can now twist the iron bar, I_1 being $= I$ and $I_2 = 0$; in consequence of this three main directions will arise in the iron with different permeability, which will also cause a longitudinal magnetisation in the bar, which is accompanied with an impulse of current in the second conductor. Likewise we may twist the bar when $I_1 = 0$ and $I_2 = I$, which gives rise to a circular magnetisation of the bar, and accompanying this an impulse of current in the first conductor. We shall compute for both cases the quantities of electricity which pass through every section in consequence of the impulses of current.

If the radius of the iron bar is R, then

$$\mathfrak{H}^0{}_1 = \frac{rI_1}{2\pi . R^2 . c} .$$

holds for the intensity of te field $\mathfrak{H}^0{}_1$ inside the iron at the distance r from the axis of the cylinder.

If the solenoid has m windings per unity of length, the intensity of the field in the middle part in which the iron bar is found, is:

$$\mathfrak{H}^0{}_2 = \frac{1}{c} . m . I_2 .$$

We shall assume the bar, which has a length l, to be twisted over an angle $\varphi = l.\alpha$, and this in such a way that while one extremity, where the current I_1 enters, is held fast, the other extremity is twisted over an angle φ in the sense of the current I_2. In consequence of this an originally square surface element with sides of a length one of a cylinder surface concentric with the axis of the bar, with radius r, will assume a rhombic shape.

In this the angle which the sides of the rhomb, which were originally parallel to the axis, form with the direction of the axis, becomes equal to $r\,\alpha$, so long as the second and higher powers of α are neglected. The diagonals of the rhomb become resp.:

$$\sqrt{2}\,(1 + \tfrac{1}{2}\,r\,\alpha) \qquad \text{and} \qquad \sqrt{2}\,(1 - \tfrac{1}{2}\,r\,\alpha),$$

hence the ratio between this and the original length resp.

$$1 + \tfrac{1}{2}\,r\,\alpha \qquad \text{and} \qquad 1 - \tfrac{1}{2}\,r\,\alpha.$$

We call the direction of the strength of the field $\mathfrak{H}^0{}_1\; x$, that of the strength of the field $\mathfrak{H}^0{}_2\; y$.

In consequence of the twisting the considered surface element has obtained two main directions, which coincide with the diagonals of the rhomb [1]). We call the direction of the diagonal which falls between the positive x-direction and the positive y-direction, u, that of the other diagonal v. In the direction u the iron is elongated, in the direction v compressed. The elongation resp. compression amounts to $\frac{1}{2}\,r\,\alpha$ per unity of length. Let $k\,.\,\lambda$ be the increase of the permeability in a certain direction, when the elongation per unity of length amounts to λ in that direction, the compression per unity of length normal to that direction being of the same value. Then

$$\mu_u = \mu + \tfrac{1}{2}\,k\,r\,\alpha \qquad\qquad \mu_v = \mu - \tfrac{1}{2}\,k\,r\,\alpha.$$

We assume k to be independent of the strength of the field.

If we further assume the angles which the directions u and v form with x and y to amount to 45°, which is permissible so long as we confine ourselves to quantities of the first order in α, we get:

$$\mathfrak{H}_u = \tfrac{1}{2}\sqrt{2}\,(\mathfrak{H}_x + \mathfrak{H}_y).$$
$$\mathfrak{H}_v = \tfrac{1}{2}\sqrt{2}\,(-\,\mathfrak{H}_x + \mathfrak{H}_y),$$

and further, as:

$$\mathfrak{B}_u = \mu_u \,.\, \mathfrak{H}_u \qquad\qquad \mathfrak{B}_v = \mu_v \,.\, \mathfrak{H}_v,$$

$$\mathfrak{B}_u = \frac{\mu}{2}\sqrt{2}\,(\mathfrak{H}_x + \mathfrak{H}_y) + \tfrac{1}{4}\,k\,r\,\alpha\sqrt{2}\,(\mathfrak{H}_x + \mathfrak{H}_y)$$

$$\mathfrak{B}_v = \frac{\mu}{2}\sqrt{2}\,(-\,\mathfrak{H}_x + \mathfrak{H}_y) + \tfrac{1}{4}\,k\,r\,\alpha\sqrt{2}\,(-\,\mathfrak{H}_x + \mathfrak{H}_y),$$

from which follows:

$$\mathfrak{B}_x = \mu\,\mathfrak{H}_x + \tfrac{1}{2}\,k\,r\,\alpha\,\mathfrak{H}_y$$
$$\mathfrak{B}_y = \mu\,\mathfrak{H}_y + \tfrac{1}{2}\,k\,r\,\alpha\,\mathfrak{H}_x.$$

We see that here the relation $\mu_{12} = \mu_{21}$ holding universally for anisotropic media with three mutually normal main directions is satisfied.

In the twisted bar $\mathfrak{H}_x$ has everywhere the same value at a certain distance from the axis, when we move along a circle normal to the axis, as there is radial symmetry with respect to this axis. The

1) The third radially directed main direction may be left out of consideration, as no change takes place in that direction.

line integral of $\mathfrak{H}_x$ along this circle amounts therefore to $2\pi r \,.\, \mathfrak{H}_x$; this line integral also amounts to $2\pi r \,.\, \mathfrak{H}_1^0$, so that we get:

$$\mathfrak{H}_x = {}_1\mathfrak{H}^0 .$$

We shall further assume the length of the bar to be large with respect to its diameter, in which case the influence of the magnetisation at the extremities in the determination of the field intensity inside the bar in case of longitudinal magnetisation will be small, so that we may assume

$$\mathfrak{H}_y = \mathfrak{H}_2^0 .$$

Inside the bar the following equations hold

$$\mathfrak{B}_x = \mu \mathfrak{H}_1^0 + \tfrac{1}{2} kra\, \mathfrak{H}_2^0$$

$$\mathfrak{B}_y = \mu \mathfrak{H}_2^0 + \tfrac{1}{2} kra\, \mathfrak{H}_1^0$$

The change of the magnetic induction within the bar in consequence of the twisting amounts to

$$\Delta \mathfrak{B}_x = \tfrac{1}{2} kra\, \mathfrak{H}_2^0$$

$$\Delta \mathfrak{B}_y = \tfrac{1}{2} kra\, \mathfrak{H}_1^0$$

In the same way we have for the magnetisation·

$$\mathfrak{M}_x = \varkappa \,.\, \mathfrak{H}_1^0 + \tfrac{1}{2} kra\, \mathfrak{H}_2^0$$

$$\mathfrak{M}_y = \varkappa \,.\, \mathfrak{H}_2^0 + \tfrac{1}{2} kra\, \mathfrak{H}_1^0$$

Also outside the magnetic induction changes in consequence of the twisting. On account of the change of $\mathfrak{B}_y$ the quantity of magnetism will namely change at the extremities of the bar which will give rise to a change of strength of the field outside the bar. If there was no iron inside the solenoid, and if this was infinitely long, the change of the magnetism at the extremities would not give rise to an induction current at all, because every quantity of magnetism sends its induction lines through the windings lying on either side, and the sense of rotation of the induced electric force is directed for the windings on one side opposite to that on the other side. We commit an error on account of the presence of the iron inside the solenoid in as much as the magnetic induction inside the iron does not change in the same way as that outside it. As we have, however, assumed that as far as the magnetic induction inside the iron is concerned, we may disregard the magnetism at the extremities, we may also leave this error out of account.

In order to calculate the induction impulse, we must therefore integrate the just mentioned amounts of $\Delta \mathfrak{B}_x$ and $\Delta \mathfrak{B}_y$ inside the bar over the surface which is surrounded by every circuit, and then sum up over all the circuits.

We explicitly excepted (§ 2 above) movements of the current conductors. Here, however, such movements occur in consequence of the twisting. Now in case of longitudinal magnetisation of the bar the movement of matter, which is the consequence of the torsion, will give rise to an induction impulse in radial direction, which has no influence on the induction impulse in longitudinal direction. In the case of circular magnetisation on the other hand no induction lines will be cut by the matter on twisting, so that no induction impulse takes place. The movement of the substance will, therefore, have no influence in these cases on the induction impulses, which are accordingly exclusively the consequence of the change of the properties of the substance.

A If we now first suppose $I_1 = I, I_2 = 0$, hence the case of circular magnetisation, then:

$$\mathfrak{H}_1^0 = \frac{rI}{2\pi R^2 . c} \qquad \mathfrak{H}_2^0 = 0$$

$$\Delta \mathfrak{B}_y = \frac{kIar^2}{4\pi R^2 . c} \qquad \Delta \mathfrak{B}_x = 0.$$

Now $\Delta \mathfrak{B}_y$ must be integrated over all the surface elements which are normal to the direction y, so over all the windings of the solenoid. The increase of the flux of induction through one winding amounts to:

$$\Delta M_y = 2\pi \int_0^R \Delta \mathfrak{B}_y . r dr = \frac{kIaR^2}{8c}$$

As there are $m.l.$ windings to the length l of the bar, the total increase of the induction flux will be $m.l. \; \Delta M_y$ and the electricity set in motion:

$$e_2 = - \frac{ml . kIaR^2}{8 W_2 . c^2}.$$

If we introduce the angle of twisting $\varphi = l. a$, we get:

$$e_2 = - \frac{m\varphi kIR^2}{8 W_2 . c^2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (21)$$

With a positive value of k we come to the conclusion that for the considered twisting the sense in which the impulse takes place, is in lefthand cyclical order with the current I.

In the other circuit the impulse is zero, as $\Delta \mathfrak{B}_y = 0$.

B. Let us now suppose $I_1 = 0$, $I_2 = I$, hence the case of longitudinal magnetisation; then:

$$\mathfrak{H}_1^0 = 0 \qquad \mathfrak{H}_2^0 = \frac{mI}{c}$$

$$\Delta \mathfrak{B}_x = \frac{1}{2c} mkraI \qquad \Delta \mathfrak{B}_y = 0.$$

In order to calculate the impulse in the first circuit, we shall divide the first conductor into conducting tubes, which each of them again consists of circuits. Let the conducting tubes, which are concentric and cylindrical in the iron, have a radius r there and a thickness dr. When we then give them dimensions proportional to this in the other parts of the conductor, the resistance of such a tube will be:

$$w = \frac{R^2}{2r \; dr} \cdot W_1$$

The increase of the induction flux through the surface surrounded by every circuit belonging to the conducting tube, amounts to:

$$\Delta M_r = l \int_r^R dr \, . \, \Delta \mathfrak{B}_x = \frac{1}{4c} mlkIa \, (R^2 - r^2).$$

The quantity of electricity set in motion in the conductor, now becomes, when we make use of the mode of calculation explained in § 3, which finds expression in (14):

$$e_1 = - \Sigma \frac{\Delta M_r}{c.w} = - \frac{mlkIa}{2R^2 W_1 c^2} \int_0^R (R^2 - r^2) \, r \, dr = - \frac{mlkIaR^2}{8 W_1 c^2} .$$

With introduction of the angle of twisting φ this becomes:

$$e_1 = - \frac{m\varphi kIR^2}{8 W_1 c^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (22)$$

Hence from (21) and (22) we find really

$$e_1 W_1 = e_2 W_2$$

in agreement with (18′).

If k is positive, then the sense in which the impulse takes place, is in lefthand cyclical order with the current I.

As $\Delta \mathfrak{B}_y = 0$, the impulse in the second conductor is zero.

We may assume that the circuits run parallel to the axis over the greater part of the length. The direction of the current can, however, be different for different circuits. In this case we shall be allowed to use the formula (13) for the real current. It follows from this that the circuits where the motion of electricity is zero, will lie on a cylinder surface, the radius r of which is given by the equation:

$$\frac{1}{c} \Delta M_r = - e_1 \, . \, W_{01},$$

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in which W_{01} is the resistance of the linear part of the circuit. From this we get:

$$r = R\sqrt{\frac{2W_1 - W_{01}}{2W_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

When W_{01} is small compared with W_1, r will differ little from R. As a rule, however, the reverse will be the case, from which ensues that r approaches the value $\frac{1}{2}R\sqrt{2}$. We can calculate the current through the central part of the bar by means of the relation (13). For this we get:

$$e_1^{(c)} = -\frac{m\varphi k I . R^2}{32c^2(W_1 - W_{01})}\left(\frac{2W_1 - W_{01}}{W_1}\right)^2 . \quad . \quad . \quad . \quad (24)$$

When $W_1 - W_{01}$ is small with respect to W_1 this quantity of electricity will become much larger than e_1; it can become arbitrarily large with respect to e_1 when $W_1 - W_{01}$ is made small enough with respect to W_1. On the other hand when W_{01} was small with respect to W_1, $e_1^{(c)}$ would differ only little from e_1.

Let us now suppose that a current I_1 runs in the first conductor, a current I_2 in the second. We assume that then the state of equilibrium is characterized by this that the bar is twisted over an angle α per unity of length. The torsion couple amounting to $KR^4 . \alpha$, the elastic energy of the bar is $\frac{1}{2}KR^4 . \alpha^2 . l$ in the twisted state. We make this state undergo an infinitesimal change so that α increases by the amount $d\alpha$. Then the elastic energy increases by the amount $KR^4\, \alpha\, l\, d\alpha$, the work of the ponderomotive forces being found from (20) for the considered change, which formula, after introduction of $\mathfrak{M}_x$ and $\mathfrak{M}_y$, produces

$$dA = \frac{1}{2}\, k . d\alpha \int r . \mathfrak{H}_1^0\, \mathfrak{H}_2^0 . dS = \frac{1}{8c^2}\, mlk\, I_1 I_2 R^2 . d\alpha.$$

In case of equilibrium this work must be equal to the increase of the elastic energy, from which we find for the angle α:

$$\alpha = \frac{kmI_1I_2}{8c^2KR^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The whole torsion becomes:

$$\varphi = \frac{kmlI_1I_2}{8c^2KR^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25')$$

If k is positive, then with the given current directions of I_1 and I_2 the bar will be twisted so that when the extremity where I_1 enters, is kept in fixed position, the other extremity is rotated in the sense of the current I_2, hence counter clockwise, when we look towards this extremity. Of course the sense of the rotation changes on reversal of one of the currents.

Hence the bar assumes the shape of a righthand screw, when the currents I_1 and I_2 are in righthand cyclical order. Further the angle over which the bar is twisted, is proportional to the total number of windings of the solenoid, which falls on the length of the bar, to the intensities of the currents, and in inverse ratio to the square of the radius.

Above we found an expression for the work of the ponderomotive forces dA on the increase of the torsion $d\alpha$. If the torsion amounts to α, we can integrate this expression, through which we get:

$$A = \frac{1}{8c^2} m\,k\,\varphi\,I_1\,I_2\,R^2$$

We find this work back in the first place in the elastic energy U of the bar. If into the expression for this $\frac{1}{2} KR^4 \alpha\varphi$, we introduce the above found expression for α, we get for this:

$$U = \frac{1}{16c^2} m\,k\,\varphi\,I_1\,I_2\,R^2.$$

The rest, which is of the same amount as U, is converted into kinetic energy, or when we make the motion take place infinitely slowly by means of external couples, into external work.

Let us now inquire into the increase of magnetic field energy. For this purpose we make use of the expression:

$$T = \int d\,S \int \mathfrak{H}\,d\,\mathfrak{B},$$

which can be easily derived from (1).

Here we introduce:

$$\mathfrak{H}_x = \frac{rI_1}{2\pi cR^2} \qquad \mathfrak{H}_y = \frac{m}{c} I_2$$

$$\Delta\,\mathfrak{B}_x = \frac{1}{2c} m\,k\,r\,\alpha\,I_2 \qquad \Delta\,\mathfrak{B}_y = -\frac{kr^2\alpha I_1}{R^2}$$

We get then:

$$\Delta T = \int d\,S \int (\mathfrak{H}_x\,d\,\Delta\,\mathfrak{B}_x + \mathfrak{H}_y\,d\,\Delta\,\mathfrak{B}_y) = \frac{1}{8c^2} m\,\varphi\,k\,I_1\,I_2\,R^2$$

Hence

$$A + \Delta T = \frac{1}{4c^2} m\varphi k\,I_1 I_2 R^2.$$

On the other hand on account of the torsion the quantity of electricity

$$e_1 = - \frac{m\varphi k I_2 R^2}{8 W_1 c^2}$$

is circulated in the first conductor. The electromotive force in that conductor amounts to $E_1 = I_1 . W_1$. In consequence of the circulation of the quantity of electricity e_1, the generator of the current yields, besides the JOULE heat, the quantity of energy $-E_1 e_1$, which amounts to:

$$-E_1 e_1 = \frac{m\varphi k I_1 I_2 R^2}{8c^2}.$$

We find in the same way that after subtraction of the JOULE heat, an equal amount of energy is yielded by the second generator of current. Together the total quantity of energy yielded by the generator of current, amounts therefore to:

$$\frac{1}{4c^2} m\varphi k \, I_1 I_2 R^2,$$

which corresponds with the value $A + \Delta T$, required for the work of the ponderomotive forces and the increase of the magnetic energy.

Chemistry. — *"Molecular-Allotropy and Phase-Allotropy in Organic Chemistry."* By Prof. A. SMITS. (Communicated by Prof. J. D. VAN DER WAALS).

1. *Survey of organic pseudo-systems.*

I have indicated the appearance of a substance in two or more similar phases by the name *phase-allotropy,* and the occurrence of different kinds of molecules of the same substance by the name of *molecular-allotropy.* It may be assumed as known that one of the conclusions to which the theory of allotropy leads, is this that phase-allotropy is based on molecular-allotropy.

The region in which the existence of molecular allotropy is easiest to demonstrate is the region of organic chemistry, and I think that I have to attribute this fact to this that the velocity of conversion between the different kinds of molecules which present the phenomenon of isomery or polymery, is on the whole much smaller in organic chemistry than in anorganic chemistry ; in organic substances it seems even not perceptible in many cases. The substances, for which this is, however, the case, and which were formerly called