Mathematics. — *"Bilinear congruences of twisted curves, which are determined by nets of cubic surfaces."* By Prof. JAN DE VRIES.

(Communicated in the meeting of May 29, 1915)

1. The base-curves ϱ^9 of the pencils belonging to a general net $[\Phi^3]$ of cubic surfaces, form a bilinear congruence. For through an arbitrary point passes *one* curve ϱ^9, and the involution of the second rank, which the net determines on an arbitrary straight line, has *one* neutral pair, so that there is *one* ϱ^9 for which that straight line is bisecant.

The 27 base-points of the net are fundamental points of the congruence. Any straight line f passing through one of those points F is *singular bisecant*; for through any point of f passes a ϱ^9, at the same time containing F. As the points of support of the curves resting on f form a parabolic involution, f may be called a parabolic bisecant.

Let t be a trisecant of a ϱ^9; through an arbitrary point of t passes *one* Φ^3, and this surface contains *all* the points of t. By the remaining surfaces of the net, t is intersected in the triplets of an involution; consequently t is a *singular trisecant.* The singular trisecants therefore form a congruence of rays; it is at the same time the congruence of the straight lines lying on the surfaces of the net.

A curve ϱ^9 has 18 apparent nodes, is therefore of genus 10. The cone of order eight $\mathfrak{K}^8$, projecting it out of one of the points F has therefore 11 double edges t [1]).

Any point F is a singular point for the congruence $[t]$, consequently vertex of a cone $\mathfrak{T}$ formed by trisecants t. With $\mathfrak{K}^8$ this cone has, besides the 26 straight lines FF' to the remaining fundamental points, the 11 double edges of $\mathfrak{K}^8$ in common. Consequently $\mathfrak{T}$ is a cone of order six; the congruence $[t]$ has therefore 27 *singular points of order six.*

The trisecants of a ϱ^9 form a ruled surface, on which ϱ^9 is an elevenfold curve. With an arbitrary surface Φ^3 this ruled surface has moreover the 27 straight lines of Φ^3 in common; the complete section is consequently a figure of order 126, and the ruled surface in question has the order 42.

Let us now consider the axial ruled surface $\mathfrak{A}$, formed by the rays of the congruence $[t]$ resting on a straight line a. With an arbitrary ϱ^9 it has first in common the 27 sextuple points F; the

1) A curve ϱ^n with h apparent nodes is intersected in each of its points by $h-(n-2)$ trisecants.

remaining intersections lie three by three on the 42 trisecants of ϱ^9, resting on a. From this it follows that $\mathfrak{A}$ is a ruled surface of order 32. As an arbitrary point bears *eleven* straight lines t, a is elevenfold straight line of $\mathfrak{A}^{32}$, and a plane passing through a contains moreover 21 straight lines t. *The singular trisecants form therefore a congruence* (11, 21).

In order to investigate whether the congruence $[\varrho^9]$ possesses other singular bisecants besides, we consider the surface Π, which contains the points of support of the chords, which the curves ϱ^9 send through a given point P. A straight line r passing through P, is, in general, chord of *one* ϱ^9, therefore contains two points of Π lying outside P. One of those points of support comes in P, as soon as r becomes chord of the ϱ^9, passing through P. The cone $\mathfrak{K}^8$ projecting this ϱ^9 out of P, is therefore the cone of contact of the octuple point P and Π is of order 10. The 11 straight lines t passing through P are nodal edges of $\mathfrak{K}^8$ and at the same time nodal lines of Π^{10}. The complete section of these two surfaces consists of the 11 double lines mentioned, the curve ϱ^9_P and the 27 straight lines PF. From this it ensues that the straight lines f are the only singular bisecants.

With an arbitrary ϱ^9 Π^{10} has the points of support of the 18 chords in common, which the curve sends through P; the remaining 54 intersections lie in the points F; consequently Π^{10} has nodes in the 27 fundamental points.

2. If two surfaces Φ^3 touch each other, the point of contact D is node of their section δ^9 and at the same time node of a surface belonging to the net. The locus of D is a curve δ^{24}. In order to find the locus of the nodal curves δ^9, we consider two pencils of the net. Each surface of the first pencil has 72 points D in common with δ^{24}, is therefore touched by 72 surfaces of the second pencil; by this a correspondence (72,72) is determined between those pencils. The intersections of homologous surfaces with a straight line l are homologous points in a correspondence (216,216); and both pencils produce therefore a figure of order 432. But the surface that the pencils have in common has been assigned 72 times to itself; the real product is therefore of order 216 only. From this it appears that *the nodal curves δ^9 form a surface of order* 216, Δ^{216}.

An arbitrary ϱ^9 can intersect this surface in the points F only; consequently Δ^{216} has the fundamental points as 72-fold points.

3. The pencils mentioned above are brought in a correspondence

(3,3), when each two surfaces, intersecting on a straight line l, are considered as homologous. It is found then that the locus of the curves ϱ^9, resting on l, is a surface Λ of order *nine*, which has the fundamental points as *triple* points.[1])

Two straight lines are therefore intersected by *nine* curves ϱ^9.

The curves ϱ^9, intersecting a straight line f, form therefore a surface of order six, with nodes F.

A plane passing through l, intersects Λ^9 moreover along a curve λ^8, the latter has in common with l the points of support of the curve ϱ^9, which has l as chord (*nodal curve* of Λ^9). In each of the remaining points of intersection of l with λ^8 the plane is touched by curves ϱ^9. The points, in which a plane φ is touched by curves of the congruence lie therefore in a curve φ^6. The latter is the *curve of coincidence of the nonuple involution* which $[\varrho^9]$ determines in φ; this involution possesses no exceptional points; each point belongs to *one* group.

As each point of intersection of φ^6 with a surface Λ^9 belonging to an arbitrary straight line l indicates a curve ϱ^9, which touches φ and rests on l, the curves ϱ^9 touching φ form a surface Φ^{54}. This surface has moreover in common with φ a curve φ^{42}; as the latter can only touch the curve φ^6, there are 126 *curves* ϱ^9, *osculating a given plane.*

If the curve φ^6 is brought in connection with the surface Ψ^{54}, belonging to a plane ψ, then it appears that *two arbitrary planes are touched by* 324 *curves* ϱ^9.

4. If the surfaces of a net $[\Phi^3]$ have the straight line q in common, the *base-curves* ϱ^8 of the pencils form a bilinear congruence with *singular quadrisecant* q. As a ϱ^8 is cut by a surface Φ, outside q, in 20 points, the congruence has 20 fundamental points F.

Each point S of q is singular; the ∞^1 curves ϱ^8 passing through S form a monoid Σ^3 belonging to the net, with nodal point in S. In order to confirm this more specially we consider two pencils of the net, and make them projective by associating any two surfaces, which touch in S. The figure which they produce then consists of the common figure of the pencils and the monoid Σ^3.

If Σ^3 is represented by central projection out of S on a plane φ, the images of the curves ϱ^8 form a pencil of curves φ^7. The image of the quadrisecant is triple base-point, the images of the five trisecants t, which a ϱ^8 sends moreover through S, are double base-points. The remaining 20 base-points are the images of the points

[1]) A ϱ^9 which does not intersect l will cut Λ^9 only in the 27 points F.

F. The five straight lines t lie, like q, entirely on Σ^3; they are apparently *singular trisecants*. Each straight line t is intersected by the curves ϱ^8 in S, and in a pair of an involution.

Two monoids have the straight line q and a ϱ^8 in common. Consequently in general a curve of the congruence is determined by two of its intersections with q. The sets of four points of support form therefore an involution of the second rank. So there are on q three pairs of points, which each bear ∞^1 curves ϱ^8; in other words, the net contains *three binodal surfaces*, of which the two nodes lie on q. We may further observe that q is stationary tangent of *six* ϱ^8 and bitangent of *four* ϱ^8.

Each trisecant t of a ϱ^8 is singular (§ 1); the straight lines t form a congruence of order 8, with 20 singular points F. The cone $\mathfrak{F}^7$, with vertex F, which projects a ϱ^8, has 8 double edges and contains 19 straight lines FF'; from this it ensues that the straight lines passing through F form a cone $\mathfrak{F}^5$, so that F appears to be a *singular point of order five.*

In any plane passing through q lie 6 chords of a ϱ^8, through any point of q pass 8 chords. The straight lines resting on q and twice on ϱ^8, form therefore a ruled surface of order 14. As they belong to the trisecants of the figure (q, ϱ^8), the trisecants of ϱ^8 must form a ruled surface of order 28.

Let us now again consider the axial ruled surface $\mathfrak{A}$, formed by the trisecants resting on the straight line a. With a definite ϱ^8 $\mathfrak{A}$ has the 20 quintuple points F and 28 triplets of points of support in common; from this it ensues that $\mathfrak{A}$ is of order 23. *The singular trisecants consequently form a congruence* (8, 15).

5. The surface Π is here of order 9; it contains q and has 20 nodes F (§ 1). Its section with the cone, $\mathfrak{K}^7$, which projects the ϱ^8 laid through P, consists of that curve, 8 singular trisecants (which are nodal lines for both surfaces) the 20 singular bisecants PF (each with a parabolic involution of points of support) and moreover three straight lines b, which apparently must also be *singular bisecants.* These straight lines we find moreover by paying attention to the intersections of $\mathfrak{K}^7$ with q; to them belong the four points, which q has in common with the ϱ^8 projected by that cone. If S is one of the remaining three intersections the straight line PS belongs to a Φ^3 of the net, is consequently cut by that net in the pairs of an involution and is therefore bisecant of ∞^1 curves ϱ^8.

For a point S of q Π consists of the monoid Σ^3 and a *cone of order six.* For, a bisecant of a ϱ^8 not laid through S is at the same time bisecant of a ϱ^8 belonging to Σ^3, consequently a singular bisecant b. The locus of the straight lines b drawn through S forms therefore with Σ^3 the surface Π. An arbitrary plane consequently contains six straight lines b, and *the singular bisecants form a congruence* (3,6). The three rays b out of a point P lie in the plane (Pq); the six rays in a plane π meet in the point (πq).

The curves ϱ^8 meeting a straight line l form again a surface Λ^9. On it q is *triple straight line,* for each monoid Σ^3 contains three curves resting on l and meeting in S. Two surfaces Λ have besides q the 9 curves ϱ^8 in common, resting in the two straight lines l. The points F appear this time again to be *triple.*

In a plane φ the congruence $[\varrho^8]$ determines an *octuple involution,* which possesses a *singular point of order three* (intersection S of q). The *curve of coincidence* φ^6 (§ 3) has now a *node* S.

As Λ^9 and φ^6 have now, outside S, 48 points in common, the curves ϱ^8, which touch the plane φ, form a surface Φ^{48}. On it q is a 16-fold straight line; for the monoid that has an arbitrary point of q as vertex, cuts φ^6, outside q, in 16 points. The plane φ cuts Φ^{48} moreover along a curve φ^{36} with 12-fold point S. The curves φ^{36} and φ^6 have 24 intersections in S; as their remaining common points must coincide in pairs, there are 96 *curves* ϱ^8 *osculating* φ.

The curve φ^6 has with the surface Ψ^{48} (belonging to a plane ψ) $6 \times 48 - 2 \times 16 = 256$ points in common outside q; there are consequently 256 *curves* ϱ^8,*which touch two given planes.*

6. If the surfaces Φ^3 of a net have two non-intersecting straight lines q and q' in common, they determine a bilinear congruence of *twisted curves* ϱ^7, of genus *four,* for which q and q' *are singular quadrisecants*; it has 13 *fundamental points* F. The curves ϱ^7 have 11 apparent nodes.

If the monoid Σ^3 containing the curves ϱ^7, which intersect q in S, is represented in the usual way, the system of those curves passes into a pencil of curves φ^6, which has a triple base-point on q and double base-points in the intersections of three other straight lines t of the monoid; the remaining base-points are the images of the points F, and the intersections of the two straight lines b^*, which may moreover be drawn on Σ^3 through S (and which apparently rest on q'). The straight lines b^* are *singular bisecants* (parabolic bisecants), the straight lines t are *singular trisecants.* The locus of the singular bisecants b^* is a ruled surface of order four with nodal lines q and q'.

Through an arbitrary point P pass six singular trisecants; they are nodal lines of the surface Π^8 determined by P and nodal edges of the cone $\mathfrak{K}^6$, which projects the curve ϱ^7 laid through P. These two surfaces have besides that ϱ^7 and the six straight lines t, moreover the 13 parabolic bisecants PF and *four singular bisecants* b in common. The straight lines b are found back if $\mathfrak{K}^6$ is brought to intersection with q and q'; on each of the singular quadrisecants rest therefore two straight lines b.

Each point of q or q' bears a cone of order 5 (completing Σ^3 into a surface Π^8) formed by singular bisecants. *The singular bisecants* consequently *form two congruences* (2,5).

The locus of the trisecants of the figure (q, q', ϱ^7) consists of four ruled surfaces, together forming a figure of order 42. The straight lines intersecting q, q' and ϱ^7 apparently form a ruled surface $\mathfrak{R}^6$. The bisecants of ϱ^7 resting on q or on q' lie on a $\mathfrak{R}^8$ with quintuple straight line. Consequently the trisecants t of ϱ^7 will form a $\mathfrak{R}^{20}$ (with sextuple curve ϱ^7).

According to the method followed above (§§ 1, 4) we find now that *the singular trisecants* t form *a congruence* (6, 10), possessing in the 13 fundamental points F singular points of order six.

On two arbitrary straight lines nine curves of the congruence rest now too. The surface Λ^9 has two triple straight lines, q and q'. In a plane φ arises a *septuple involution* with a curve of coincidence φ^6 possessing *two nodes*, where the involution has *singular points of order three.* The curves ϱ^7 touching φ, form a Φ^{42} with 14-fold straight lines q and q'.

There are 70 curves ϱ^7 *osculating* a plane φ, and 196 curves *touching two given planes.*

7. If the surfaces Φ^3 of a net have a conic σ^2 in common they determine a bilinear congruence of *twisted curves* ϱ^7, of genus *five.* Every ϱ^7 rests in six points on the singular conic σ^2. The congruence possesses consequently 15 *fundamental points* F.

In representing the monoid Σ^3, containing the curves ϱ^7, which intersect σ^2 in a point S, the system of those curves passes into a pencil of curves φ^6. They have five nodes in the intersections of the singular trisecants t, meeting in S, the remaining base-points are the images of the 15 points F, and the intersection of the straight line b^* of Σ^3, which forms with the 5 straight lines t the set of six straight lines passing through S. Apparently b^* is here also a *singular bisecant* (parabolic bisecant).

The surface Π^8 belonging to a point P and the corresponding cone $\mathfrak{K}^6$ have in common a ϱ^7, five singular trisecants t (nodal lines for both surfaces), the 15 parabolic bisecants PF and *six singular bisecants* b.

For a point S of σ^2 Π^8 consists of the monoid Σ^3 and a cone of order five formed by straight lines b. Hence *the singular bisecants* b *form a congruence* (6, 10).

Let us now consider the straight lines which intersect the figure (ϱ^7, σ^2) thrice, consequently form together a figure of order 42. Any point of σ^2 bears 10 chords of ϱ^7; in the plane σ of that conic there are 6 of them, viz. the straight lines connecting the 6 intersections of σ^2 and ϱ^7 with the point R, which ϱ^7 has moreover in common with σ. The chords of ϱ^7 meeting σ^2 consequently form a $\mathfrak{R}^{26}$. The chords of σ^2 meeting ϱ^7, form the plane pencil (R, σ). Consequently the trisecants of ϱ^7 form a $\mathfrak{R}^{15}$.

In connection with this we easily find now that the *singular trisecants* form *a congruence* (5,10) possessing 15 singular points F of order four.

The surface Λ^9 has now a triple conic, σ^2, and 15 triple points F. In a plane φ $[\varrho^7]$ determines again a septuple involution with two singular points of order three. In connection with this we find for this congruence $[\varrho^7]$ the same characteristic number as for the $[\varrho^7]$ treated in § 6.

8. Passing on to congruences of twisted curves ϱ^6, we suppose in the first place, that the surfaces of $[\Phi^3]$ have three non-intersecting straight lines q, q', q'' in common They are then *singular quadrisecants* of the congruence $[\varrho^6]$; consequently the curves ϱ^6 (genus *one*) pass through *six fundamental* points F.

The curves ϱ^6 intersecting q in S form again a monoid Σ^3. They are represented by a pencil (φ^5), having a triple base-point on q and double base-points in the intersections of two straight lines t. To the base belong further the images of the points F and the intersections of two straight lines b^* (singular bisecants).

The sixth straight line of Σ^3, passing through S, is component part of a *degenerate curve* ϱ^6. It is the transversal d of q' and q'' passing through S; for through an arbitrary point of that transversal pass ∞^1 surfaces Φ^3 having d in common and therefore intersecting moreover along a curve σ^5 (of genus *one*), which has q, q', q'' as trisecants. The planes (dq') and (dq'') each intersect Σ^3 along one of the straight lines b^*.

The ruled surface $\mathfrak{D}^2$ with directrices q, q', q'' contains all the straight lines d forming the second system of straight lines. With a monoid Σ^3 $\mathfrak{D}^2$ has three straight lines d in common of which one passes through S. Consequently there lie on Σ^3 two curves σ^5, which pass through S. The locus Δ of the curves σ^5 has consequently three nodal lines q, q', q''; its section with a Σ^3 consists further of three curves σ^5, is therefore of order 21. Hence Δ is *a surface of order seven.*

The figures (d, σ^5) determine on q a correspondence (3,2); so there lie on q five points $D \equiv (d, \sigma^5)$. The locus of the points D is therefore a *twisted curve* $(D)^6$, intersecting each of the straight lines q, q', q'' five times.

Now $\mathfrak{D}^2$ and Δ^7 have the three straight lines q and the curve $(D)^6$ in common, consequently another figure of order two. This figure must consist of two straight lines d, hence there is a figure of $[\varrho^6]$ consisting of two straight lines d and a curve σ^4. This curve has q, q', q'' as bisecants and intersects $\mathfrak{D}^2$ moreover in two points D.

Through an arbitrary point P pass five singular trisecants; they are nodal lines of Π^7 and $\mathfrak{K}^5$. These surfaces have moreover in common the curve ϱ^6_P laid through P, the six parabolic bisecants PF and three straight lines b. The straight lines b are determined by the points which the straight lines q outside the curve ϱ^6_P have in common with the cone $\mathfrak{N}^5$; hence they are *singular bisecants.*

If P is supposed to be on q, Π^7 is replaced by the figure composed of Σ^3 and a cone $(b)^4$. *The singular bisecants form consequently three congruences* (1, 4).

The locus of the straight lines which intersect a figure (ϱ^6, q, q', q'') thrice, consists of the hyperboloid $(q\, q'\, q'')$, three ruled surfaces $\mathfrak{R}^4$ with nodal lines q, q' and the ruled surface of the trisecants of ϱ^6; this is therefore of order 16.

From this it is now deduced, in the way followed before, that *the singular trisecants* form *a congruence* (5, 6) possessing six singular points F of order three.

The surface Λ^9 has three triple straight lines q, q', q''. In a plane φ the congruence $[\varrho^6]$ determines a *sextuple involution with three singular points of order three,* which are at the same time nodes of the curve of coincidence φ^6. The curves ϱ^6, touching φ, form a Φ^{36} with 12-fold straight lines q, q', q''. There are 48 curves ϱ^6, *osculating* one plane, and 144 curves *touching two planes.*

9. Let us now consider the case that all the surfaces of the net $[\Phi^3]$ have in common a conic σ^2 and a straight line q not intersecting it. Any two surfaces then determine a *twisted curve* ϱ^6, which rests in six points on σ^2, in four points on q. A third surface intersects ϱ^6 moreover in eight points. The congruence $[\varrho^6]$ possesses therefore *eight fundamental points F*. The curves ϱ^6 have eight apparent nodes, they are consequently of genus *two*.

The monoid Σ^3 belonging to a point S of the *singular quadrisecant* q contains *a singular trisecant* passing through S. From the image of Σ^3 it appears that the remaining four straight lines of Σ^3 passing through S are *singular bisecants* b^*.

The curves ϱ^6 intersecting the singular conic σ^2 in a point S^* also form a monoid Σ^3. These curves are represented by a pencil (φ^5), which has double base-points in the intersections of the *four singular trisecants* t meeting in S^*. The simple base-points are the images of the 8 points F and the intersection of a *singular bisecant* b^*.

The sixth straight line passing through S^* must be component part of a *compound* ϱ^6. It must cut σ^2 and q, belongs therefore to the plane pencil in the plane σ of σ^2, which has the point Q of q as vertex.

Any ray d of that plane pencil is component part of a degenerate σ^6, for an arbitrary point of d determines a pencil (Φ^3) of which all figures pass through d, consequently have a curve ϱ^5 in common besides, which intersects σ^2 four times, q three times, consequently possesses four apparent nodes. To the surfaces Φ^3 passing through the figure $(\sigma^2, q, d, \delta^5)$ belongs the figure composed of the plane σ and the *hyperboloid* $\mathfrak{D}^2$ passing through q and the points F; this degenerate figure apparently replaces the monoid belonging to Q. The hyperboloid $\mathfrak{D}^2$ is *the locus of the curves* δ^5; its intersection δ^2 on σ contains the points $D \equiv (d, \delta^5)$; all curves δ^5 pass through the four intersections of δ^2 with σ^2.

From the consideration of the surfaces Π^7 and $\mathfrak{R}^5$, which are determined by a point P it follows readily that P bears five singular bisecants b. Four of these straight lines rest on σ^2, the fifth on q. Any point of σ^2 or of q is the vertex of a cone of order four, formed by straight lines b. The *singular bisecants* consequently form two congruences; a *congruence* (1,4) with directrix q, a *congruence* (4,8) with singular curve σ^2.

The singular trisecants t form a congruence possessing eight singular points, F, of order three. The trisecants of a ϱ^6 form a ruled surface $\mathfrak{R}^{12}$. In connection with this we find that the straight lines t determine a *congruence* (4,6).

As $[\varrho^6]$ again intersects a plane φ along a *sextuple involution* with three singular points of order three, we find for the characteristic numbers connected with it the same values as in § 8.

10. A net $[\Phi^3]$, of which the figures have a cubic σ^3 (or a degeneration of it) in common, determines a congruence of twisted curves ϱ^6, of genus *three*, intersecting the singular curve σ^3 eight times [1]. The congruence possesses accordingly *ten fundamental points F.*

As ϱ^6 has seven apparent nodes, σ^3 is intersected in each of its points S by *three singular trisecants t*. Using the image of the monoid Σ^3 belonging to S, we find that the remaining three straight lines of Σ^3 meeting in S are *singular bisecants* b^*.

Through an arbitrary point P pass seven singular bisecants b. Each point of σ^3 is vertex of a cone of order four formed by straight lines b. From this it ensues that *the singular bisecants form a congruence* (7,12).

The *singular trisecants* form *a congruence* (3,6) with ten singular points F, of order three.

The characteristic numbers, connected with the surface Δ^9, have the same values as with the congruence $[\iota^6]$ already dealt with.

11. The surfaces of a net $[\Phi^3]$, which have a plane curve σ^3 in common, determine a congruence of *twisted curves* ϱ^6 of genus *four*, which possesses *twelve fundamental points F.*

As ϱ^6 has now six apparent nodes, each point S of the singular curve σ^3 bears *two singular trisecants.*

To the surfaces Φ^3 passing through a figure (σ^3, ϱ^6) belongs a figure consisting of the plane σ of σ^3 and a hyperboloid; ϱ^6 is therefore the complete section of a hyperboloid with a cubic surface. In connection with this the curves ϱ^6 intersecting σ^3 in a point S, form a *hyperboloid* Σ^2, passing through the points F. The surfaces Σ^2 form a pencil [2] with *base-curve* β^4, which determines in σ a pencil of conics ϱ^2. Any point of the plane σ bears therefore a *figure consisting of a* ϱ^2 *and the curve* β^4.

The section of σ with the surface Δ belonging to the straight line l consists of the nodal curve σ^3 and the conics ϱ^2 intersected by l; hence Δ is of *order eight.*

Two surfaces Δ^8 have the singular curve σ^3, the curve β^4, and eight curves ϱ^6 in common.

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1) If σ^3 is replaced by a conic σ^2 and a straight line s intersecting it, we understand easily that any ϱ^6 has five points in common with σ^2, and three points with s.

2) The net $[\Phi^3]$ may be represented by the equation

$$a_x^3 + \lambda\,(a_x^3 + b_x^2\,x_4) + \mu\,(a_x^3 + c_x^2\,x_4) = 0.$$

Through a point of $x_4 = 0$ passes the pencil for which $1 + \lambda + \mu = 0$. It consists therefore of the plane $x_4 = 0$, and the pencil $\lambda\,(b_x^2 - c_x^2) - c_x^2 = 0$, with base-curve $b_x^2 = 0$, c_x^2 0.

The sextuple involution, which $[\varrho^6]$ determines in a plane φ, has *three singular points S of order two* lying in a straight line s and (in the intersections of β^4) *four singular points of order one,* which are completed into sets of six by the pairs of an involution lying on s.

Any trisecant t of a ϱ^6 is trisecant of ∞^1 curves of the congruence and in particular of a figure (ϱ^2, β^4). The *congruence of the singular trisecants* is therefore identical with the congruence of the chords of β^4, is consequently a (2, 6).

The cone projecting a ϱ^6 out of one of its points has in common with σ^3 the 6 intersections of the two curves; the remaining 9 points determine each a singular bisecant b.

The surface Π^7 belonging to a point S of σ^3 consists of Σ^2, the plane σ (of which any straight line is singular bisecant) and a cone $(b)^4$. Consequently the *singular bisecants b* form *a congruence* (9, 12).

A plane φ contains a curve φ^5 being the locus of the points of contact of curves ϱ^6. As φ^5 has 34 points in common with Δ^8, outside σ^3, the curves ϱ^6 touching φ form a Φ^{34}, which is moreover intersected by φ in a curve φ^{24}. As φ^5 is intersected by an arbitrary Σ^2 in 10 points, σ^3 is decuple curve of Φ^{34}; so φ^{24} has three octuple points S. From this it ensues further that φ^5 and φ^{24}, apart from the points S, have 96 points in common, so that φ is *osculated* by 48 curves ϱ^6.

As φ^5 has outside σ^3 140 points in common with Ψ^{34} there are 140 curves ϱ^6 *touching two planes.*

The bilinear congruences of twisted curves ϱ^5 and ϱ^4, which are determined by nets of cubic surfaces I have considered in communications published in volume XVII, p. 1250, in volume XVIII, p. 43 and in vol. XVI, p. 733 and 1186 of these *Proceedings.* The congruence of twisted cubics determined by a $[\Phi^3]$ was extensively treated by STUYVAERT (Bull. Acad. de Belgique, 1907, p. 470—514).

Mathematics. — *"Associated points with respect to a complex of quadrics."* By CHS. H. VAN OS. (Communicated by Professor JAN DE VRIES).

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Let a triply infinite linear system (*complex*) be given of quadrics Φ^2. The surfaces passing through a point P form a net and have moreover *seven* points Q in common. If we associate those points to P we get a correspondence, which will be considered here.