Physics. — *"The viscosity of liquefied gases. I. The rotational oscillations of a sphere in a viscous liquid."* By Prof. J. E. VERSCHAFFELT. Comm. N°. 148*b* from the Physical Laboratory at Leiden. (Communicated by Prof. H. KAMERLINGH ONNES).

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1. With a view to an investigation of the viscosity of liquefied gases at low temperatures, especially in the case of hydrogen, which on the invitation of Professor KAMERLINGH ONNES I hope to undertake, in conjunction with Mr. Ch. NICAISE, by the method of damped rotational oscillations of a sphere suspended in the liquids in question, I shall here give the theory of the method. The problem has been dealt with before by a number of writers [1]) and the formulae which embody the results of their calculations have also found application in the discussion of different experiments; still I do not consider it superfluous to publish my method of dealing with the problem, because in my opinion it is simpler and less involved than the one followed by previous writers, while the formulae which I have arrived at are much better adopted to numerical calculations.

The sphere will be supposed to swing freely about a diameter under the action of a couple of forces (the torsional moment of the suspension) the moment of which $M\alpha$ is proportional to the angle of deflection α. In the absence of friction the sphere would perform a harmonic oscillation with a time of swing given by:

$$T_0 = 2\pi \sqrt{\frac{K}{M}}, \quad \ldots \ldots \ldots \quad (1)$$

K being the moment of inertia of the sphere about a diameter (or more correctly the moment of inertia of the vibrating system of which the sphere forms parts), M the angular moment per unit of angle. If the sphere swings in a viscous liquid, the motion is damped and it appears (although properly speaking an experimental confirmation is lacking), that when the friction is not too strong the sphere executes a damped harmonic vibration, according to the formula:

[1]) C. J. H. LAMPE, Programm des städt. Gymn. zu Danzig, 1866.
G. KIRCHHOFF, Vorlesungen über mathematische Physik, No. 26, 1877.
IG. KLEMENCIC, Wien Ber. II. 84, 146, 1882.
G. G. STOKES, Math. and Phys. Papers. Vol. V, p 207.
W. KÖNIG, Wied. Ann. 32, 193, 1887.
H. LAMB. Hydrodynamics, 1906, p. 571, 599, 581.
G. ZEMPLÉN, Ann. d. Phys. 19, 783, 1906; 29, 899, 1909.
M. BRILLOUIN. Leçons sur la viscosité des liquides et des gaz, 1907; 1ère partie p. 96.

$$a = a\, e^{-\delta \frac{t}{T}} \cos 2\pi \frac{t}{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where T is the new time of vibration and δ the logarithmic decrement of the elongations for one vibration.[1] The problem before us is, how δ and T depend upon the specific properties of the liquid, in particular on the viscosity η, and how η may be calculated from observations on the two quantities in question.

2. We shall confine our investigation to the two cases in which the liquid is either externally unlimited (i e. practically speaking, fills a space the dimensions of which are very large compared with the radius of the sphere) or is limited by a stationary spherical surface which is concentric with the oscillating sphere; in these cases we may naturally assume, that the motion in the liquid is such, that it divides itself into spherical, concentric layers, which each separately oscillates as a solid shell about the same axis as the sphere, with the same periodic time and the same logarithmic decrement; it will be shown further down that this assumed state of motion is actually a possible one, at least when the motion is very slow [2]). In that case it is only the amplitude and the phase of the motion which differ from one shell to another, and for a shell of radius r we may therefore put:

$$a_r = a_r\, e^{-\delta \frac{t}{T}} \cos 2\pi \left(\frac{t}{T} - \varphi_r\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

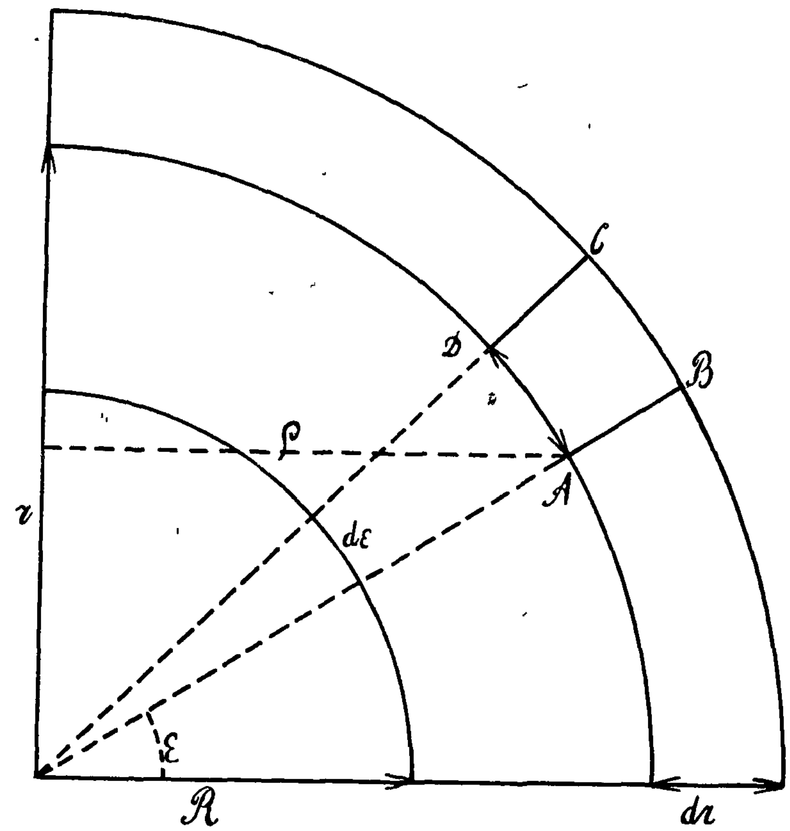
where δ_r and φ_r are functions of r. If we further assume that the liquid layer which is contiguous to the sphere, adheres to it, as is well known to be generally the case, expression (3) must become identical with (2) for $r = R$; thus $a_R = a$ and $\varphi_R = 0$.

3. In order to find the functions a_R and φ_R we proceed to establish the equation of motion for a spherical liquid shell. For this purpose we shall consider the ring whose section is $ABCD = r.d\varepsilon.dr$ (comp. adjoining figure) and whose radius is $\varrho = r \cos \varepsilon$. On its side-faces AB and CD this ring according to our assumption does not experience any friction; on the inner surface AB, owing to friction against a shell closer to the centre, it experiences a tangential force F per unit area in the direction of its motion, and on the outer surface BC similarly a force $-\left(F + \frac{\partial F}{\partial r} dr\right)$; writing down the

[1]) If the motion of the sphere without friction were a compound harmonic motion, as would be the case, if the sphere were coupled to other oscillating systems, the motion with friction would be compounded of damped harmonic vibrations.

[2]) For the necessary condition of slowness of the motion see note in Comm N⁰. 148*d*.

54*



condition, that the work of these forces during a small angular displacement equals the increase of the kinetic energy $\frac{1}{2} m v_r^2$ of the ring, we find, when the density of the liquid is μ,

$$F \,.\, 2\pi r \cos\varepsilon \,.\, r d\varepsilon \,.\, r \cos\varepsilon \frac{\partial \alpha_r}{\partial t} - \left(F + \frac{\partial F}{\partial r} dr\right) 2\pi (r + dr)^3 \cos^2\varepsilon \,.\, d\varepsilon \,.\, \frac{\partial \alpha r}{\partial t} =$$

$$= \frac{\partial}{\partial t}\left(\tfrac{1}{2} m v_r^2\right) = 2\pi r \cos\varepsilon \,.\, r d\varepsilon \,.\, dr \,.\, \mu \,.\, v_r \frac{\partial v_r}{\partial t}$$

or

$$-\frac{\partial F}{\partial r} - \frac{3F}{r} = \mu \frac{\partial v_r}{\partial t} = \mu\, r \cos\varepsilon \frac{\partial^2 \alpha_r}{\partial t^2}.$$

According to the elementary laws of internal friction the force F is proportional to the velocity-gradient in the direction of the radius; in determining this slope we must only take into account the gradient which is due to the change of the angular velocity with r[1]). The velocity-gradient thus becomes equal to $r \cos\varepsilon \frac{\partial}{\partial r}\left(\frac{\partial \alpha_r}{\partial t}\right)$, and therefore

[1]) The gradient of velocity which is the consequence of a uniform rotation of the liquid does not produce any friction. In the classical hydrodynamical theory this results from the circumstance that in a uniform rotation there is no deformation and consequently no stress. (Note added in the translation).

$$F = -\eta\, r \cos \varepsilon \frac{\partial \omega}{\partial r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

when $\omega = \frac{\partial \alpha,}{\partial t}$ represents the angular velocity of the shell under consideration and η the viscosity of the liquid. The equation of motion of the sphericall shell may now be written in the form

$$\frac{\partial^2 \omega}{\partial r^2} + \frac{4}{r}\frac{\partial \omega}{\partial r} = \frac{\mu}{\eta}\frac{\partial \omega}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

4. This equation determines how ω depends on r; as it does not contain the angle ε, it is in accordance with our assumption, that the individual shells oscillate to and fro as solid bodies [1]). As regards the law of dependence of ω on t, which we have already presupposed in equation (3), it appears that it also is compatible with (5); substituting (3) in (5) and expressing the condition, that equation (5) must be fulfilled at all times (by putting the coefficients of *cos* and *sin* equal to zero), two differential equations are obtained, which do not contain the time and which determine the functions a_r and φ_r.

This method is, however, very cumbrous. It is much simpler first to reduce (3) to the form

$$a_r = e^{-\delta \frac{t}{T}} \left(x \cos 2\pi \frac{t}{T} + y \sin 2\pi \frac{t}{T} \right), \quad . \quad . \quad . \quad . \quad (6)$$

where x and y are new functions which for $r = R$ become equal to a and 0 respectively and are determined by the two differential equations:

$$\left.\begin{aligned} \frac{d^2 x}{dr^2} + \frac{4}{r}\frac{dx}{dr} + \frac{\mu}{\eta T}(\delta x - 2\pi y) = 0 \\ \frac{d^2 y}{dr^2} + \frac{4}{r}\frac{dy}{dr} + \frac{\mu}{\eta T}(\delta y + 2\pi x) = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The simplest method of all is to consider (6) as the real part of an exponential function

$$a_r = u\, e^{kt} \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where u and k are in general complex quantities; in that case (2) is the real part of

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[1]) It should not be overlooked that in this manner the possibility of the aforesaid assumption has been proved, not its necessity (for this proof, see LAMB, loc. cit.). It is moreover easily seen, that with a different law of friction, e g. in which $\varkappa$ would also depend on the velocity itself, the assumption would become unallowable.

$$a = a e^{kt} \quad , \qquad \ldots \ldots \ldots \quad (8')$$

and u is a function of r only, which for $r = R$ obtains the value a. Putting

$$k = k' + k'' i \quad (i = \sqrt{-1}) \, , \qquad \ldots \ldots \quad (9)$$

it follows by equating (6) to the real part of (8) that

$$k' = -\frac{\delta}{T} \quad \text{and} \quad k'' = \frac{2\pi}{T} \qquad \ldots \ldots \quad (9')$$

The real angular velocity ω is the real part of the complex quantity

$$\omega = k u e^{kt} , \qquad \ldots \ldots \quad (10)$$

the function u satisfying the equation

$$\frac{d^2 u}{dr^2} + \frac{4}{r} \frac{du}{dr} = \frac{\mu}{\eta} k u \, , \qquad \ldots \ldots \quad (11)$$

which is obtained by substituting (10) in (5). [1]

5. The general solution of (11) is well known to be

$$u = \frac{1}{r^3} \left[A e^{-br} (br + 1) + B e^{br} (br - 1) \right] ,$$

or

$$u = \frac{1}{r^3} \left[P e^{-b(r-R)} (br + 1) + Q e^{b(r-R)} (br - 1) \right] , \quad . \quad . \quad (12)$$

where

[1] Equation (10) is a particular solution of equation (5). The mode of motion which it represents is, therefore, a possible one but not necessarily the actually existing one. The reason why we only consider this solution is that we suppose the sphere not to perform forced vibrations In the case of a compound harmonic motion ω would consist of a number of terms, each with its own k, the u's of which would satisfy as many equations (11).

It is also obvious, that the condition of motion considered cannot exist from the beginning, but can only be reached after a theoretically infinite period, so that the motion of the sphere cannot correspond either to equation (2) from the moment at which the motion begins. The experiments show, however, that the final condition is practically reached after a comparatively short time (a few minutes), i. e. very soon T and δ have become constant; this may be expressed mathematically by saying, that the assumed condition of motion is the limiting condition to which the real motion approaches asymptotically and this approach is in general so rapid, that even after a comparatively short time the deviations of the actual motion from the final limit are within the limits of the errors of observation. The question as to the real motion during the said period of approach is one which would have to be settled by a separate theoretical and experimental investigation, but is of no importance for our present purpose.

$$b = \sqrt{\frac{\mu}{\eta} k}\,; \qquad \ldots \ldots \quad (13)$$

A, B, P and Q are complex constants which are determined by the conditions at the boundaries.

In the first place we have $u = a$ for $r = R$, so that

$$P(bR + 1) + Q(bR - 1) = aR^3 \quad \ldots \ldots \quad (14)$$

If the liquid is unlimited or at any rate may practically be considered as unlimited, $u = 0$ for $r = \infty$; this leads to the condition $Q = 0$ (unless b were a pure imaginary quantity, i. e. k were real, in which case the motion would be aperiodic, a case which we do not consider here), and therefore

$$u = a \frac{R^3}{r^3} \frac{br+1}{bR+1} e^{-b(r-R)} \quad \ldots \ldots \quad (15)$$

On the other hand, if the liquid is bounded by a stationary spherical surface of radius R', the condition is that $u = 0$ for $r = R'$ at all times (again in the supposition that the liquid adheres to the surface of the sphere) so that

$$Pe^{-b(R'-R)}(bR' + 1) + Qe^{b(R'-R)}(bR' - 1) = 0\,; \quad \ldots \quad (16)$$

in that case

$$P = \frac{aR^3(bR'-1)e^{b(R'-R)}}{D}\,, \quad Q = -\frac{aR^3(bR'+1)e^{-b(R'-R)}}{D}\,, \quad (17)$$

where

$$D = (bR + 1)(bR' - 1)\,e^{b(R'-R)} - (bR - 1)(bR' + 1)\,e^{-b(R'-R)}, \quad (17')$$

so that

$$u = \frac{aR^3}{Dr^3}\left[(br + 1)(bR' - 1)\,e^{b(R'-r)} - (br - 1)(bR' + 1)\,e^{-b(R'-r)}\right] \quad (17'')$$

6. If we put

$$\sqrt{k} = \pm(\gamma' + \gamma'' i)$$

it follows that

$$\gamma'^2 - \gamma''^2 = k' \text{ and } 2\gamma'\gamma'' = k'',$$

and therefore, seeing that γ' and γ'' from their nature represent real quantities:

$$\left.\begin{aligned} \gamma' &= \sqrt{\tfrac{1}{2}k' + \tfrac{1}{2}\sqrt{k'^2 + k''^2}} = \sqrt{\frac{-\delta + \sqrt{\delta^2 + 4\pi^2}}{2T}} \\ \gamma'' &= \sqrt{-\tfrac{1}{2}k' + \tfrac{1}{2}\sqrt{k'^2 + k''^2}} = \sqrt{\frac{\delta + \sqrt{\delta^2 + 4\pi^2}}{2T}} \end{aligned}\right\} \quad (18)$$

As a rule the circumstances under which the experiments are conducted are such, that σ is a small number, of the order of magnitude 0,1; in that case the expressions (18) can be developed into series progressing according to the ascending powers of $\chi = \frac{\sigma}{2\pi}$, which leads to:

$$\left.\begin{aligned} \gamma' &= \sqrt{\frac{\pi}{T}}\left(1 - \tfrac{1}{2}\chi + \tfrac{1}{8}\chi^2 + \dots\right) \\ \gamma'' &= \sqrt{\frac{\pi}{T}}\left(1 + \tfrac{1}{2}\chi + \tfrac{1}{8}\chi^2 + \dots\right) \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

so that

$$b = \sqrt{\frac{\pi\mu}{\eta T}}\left[(1+i) - (1-i)\frac{\chi}{2} + (1+i)\frac{\chi^2}{8} + \dots\right]. \qquad (20)$$

7. As mentioned above in section 1, the real part of (8) may in general be written in the form

$$\begin{aligned} a_{,} &= e^{k't-b'r}\,[X_1 \cos(k''t - b''r) + Y_1 \sin(k't - b''r)] \\ &+ e^{k't+b'r}\,[X_2 \cos(k''t + b''r) + Y_2 \sin(k''t + b''r)], \quad . \quad . \quad (21) \end{aligned}$$

where X_1, X_2, Y_1 and Y_2 are again functions of r, but now real quantities. This form shows, that the motion of the liquid is the result of the propagation of two waves, the one moving away from the oscillating sphere, the other moving towards the sphere; writing $k''t \pm b''r$ in the form $\frac{2\pi}{T}\left(t \pm \frac{r}{V}\right)$, the speed of propagation appears to be

$$V = \frac{k''}{b''} = \frac{2\pi}{b''T}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

this velocity therefore depends not on the specific properties of the liquid only, but in addition on the time of swing of the sphere. The wave-length is $\lambda = \frac{2\pi}{b''}$.

For σ very small we have by (19'),

$$V = 2\sqrt{\frac{\pi\eta}{\mu T}} \text{ and } \lambda = 2\sqrt{\frac{\pi\eta T}{\mu}}. \quad . \quad . \quad . \quad (22')$$

When the liquid extends to infinity (practically), we have only to deal with the former of the two waves: but when the liquid is bounded, the wave which is emitted by the oscillating sphere is reflected on the fixed wall, in such a manner that the phase is reversed, and thereby the amplitude u becomes zero at the wall.

In addition the waves undergo a damping effect during propagation, in such a manner that, independently of the algebraic dependence on r, the amplitude is reduced in the ratio $\Delta:1$ over a distance 1, where $\Delta = e^{-b'}$.

With a small value of σ according to (20) the damping increases as T becomes smaller and with a sufficiently small value of T it may happen, that even a comparatively narrowly bounded liquid is practically unbounded, because the motion which starts from the sphere is practically completely damped, before it reaches the external boundary; to this point we shall return later on (§ 12).

8. We can now proceed to calculate the time of swing and the logarithmic decrement of the damped oscillations of the sphere from the specific constants of the liquid (viz. the viscosity η and the density μ). The equation of motion of the oscillating sphere is

$$K\frac{d^2\alpha}{dt^2} - C + M\alpha = 0\,,[1] \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where C, the moment of the frictional forces, is given by (comp. § 3)

$$C = -\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} F\,.\,2\pi R^3 \cos^2\varepsilon\, d\varepsilon = \tfrac{8}{3}\,\pi R^4 \eta \left(\frac{\partial\omega}{\partial r}\right)_{R'} \quad . \quad . \quad (23')$$

According to (10) and (12) we may write

$$\frac{\partial\omega}{\partial r} = -\frac{ke^{kt}}{R^4}\left[P\,(b^2r^2+3br+3)\,e^{-b(r-R)} - Q\,(b^2r^2-3br+3)\,e^{b(r-R)}\right],$$

and therefore

$$\left(\frac{\partial\omega}{\partial r}\right)_R = -\frac{ke^{kt}}{R^4}\left[P\,(b^2R^2+3bR+3) - Q\,(b^2R^2-3bR+3)\right] =$$

$$= -\frac{1}{aR^4}\left[P\,(b^2R^2+3bR+3) - Q\,(b^2R^2-3bR+3)\right]\frac{d\alpha}{dt},$$

so that for the case of a damped harmonic motion we may write

$$K\frac{d^2\alpha}{dt^2} + L\frac{d\alpha}{dt} + M\alpha = 0,[2] \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where

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[1]) The equation once more expresses the fact that the sphere oscillates freely.

[2]) In the case of a not purely harmonic damped motion the proportionality of C with $\frac{d\alpha}{dt}$ no longer exists. As far as I can see, it is in that case impossible to say, how in general C depends on the motion, so that it will then probably be impossible to establish a general differential equation for α.

$$L = \tfrac{8}{3}\frac{\pi R^3\eta}{D}[(b^2R^2+3bR+3)(bR'-1)e^{b(R'-R)}+(b^2R^2-3bR+3)(bR'+1)e^{-b(R'-R)}] \quad (24')$$

D being the form given in (17'). L is again a complex quantity.[1])

When the liquid is (practically) unbounded and the motion periodic (i.e. $Q=0$), we have simply:

$$L = \tfrac{8}{3}\pi R^3\eta \frac{b^2R^2 + 3bR + 3}{bR+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

9. The expression (8') actually satisfies the equation (24), when k satisfies the equation

$$Kk^2 + Lk + M = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If we put again $L = L' + L''i$, we find:

$$K(k'^2 - k''^2) + L'k' - L''k'' + M = 0 \text{ and } 2Kk'k'' + L'k'' + L''k' = 0, \quad (26')$$

or according to (9') and (1),

$$\delta^2 - 4\pi^2 - \delta\frac{L'T}{K} - 2\pi\frac{L''T}{K} + 4\pi^2\frac{T^2}{T_0^2} = 0 \text{ and } 4\pi\delta = 2\pi\frac{L'T}{K} - \delta\frac{L''T}{K} \quad (27)$$

These are therefore the equations which determine k' and k'', and thus also δ and T, under the given experimental conditions; conversely they enable us to compute L' and L'' from the experimental values of T and δ and thereby by the aid of (24') to calculate η.

From (27) it follows that:

$$\frac{L'}{K} = \frac{\delta}{T}\left[\frac{T^2}{T_0^2}\frac{4\pi^2}{4\pi^2+\delta^2} + 1\right], \quad \frac{L''}{K} = \frac{2\pi}{T}\left[\frac{T^2}{T_0^2}\frac{4\pi^2}{4\pi^2+\delta^2} - 1\right]. \quad (28)$$

When δ is a small number, as also $\psi = \dfrac{T-T_0}{T_0}$ (as is usually the case), we may write:

$$\left.\begin{aligned} \frac{L'}{K} &= \frac{2\delta}{T}[1 + \psi + \tfrac{1}{2}(\psi^2 - \chi^2) + \dots] = \frac{2\delta}{T_0}[1 + \tfrac{1}{2}(\psi^2 - \chi^2) + \dots] \\ \frac{L''}{K} &= \frac{4\pi}{T}y\left[1 + \frac{\psi^2 - \chi^2}{\psi} + \dots\right] = \frac{4\pi}{T_0}\psi\left[1 - \frac{\psi^2+\chi^2}{2\psi} + \dots\right] \end{aligned}\right\} . \quad (28')$$

10. As we have been using complex quantities all along, we

[1]) The meaning of this is as follows: the real angle α satisfies equation (23), where everything is real, even C, the moment of the frictional forces, which is determined by (23') with ω still real. If, however, a complex angle α is introduced, the real part of which is the real α, C will be the real part of (23'), where ω must be taken as a complex quantity, and this is at the same time the real part of an expression of the form $L\dfrac{d\alpha}{dt}$, where L is then similarly a complex quantity.

have not come across the fictitious addition to the moment of inertia which usually occurs in problems of this kind. This addition does not show itself, until the real part is extracted from equation (24).

This real part is equal to

$$K\frac{d^2\alpha'}{dt^2}+L'\frac{d\alpha'}{dt}-L''\frac{d\alpha''}{dt}+M\alpha'=0\,, \quad . \quad . \quad (29)$$

having put $\alpha=\alpha'+\alpha''i$; and as is easily found from (8')

$$\frac{d\alpha''}{dt}=-\frac{1}{k''}\frac{d^2\alpha'}{dt^2}+\frac{k'}{k''}\frac{d\alpha'}{dt},$$

so that

$$\left(K+\frac{L''}{k''}\right)\frac{d^2\alpha'}{dt^2}+\left(L'-L''\frac{k'}{k''}\right)\frac{d\alpha'}{dt}+M\alpha'=0, \quad . \quad . \quad . \quad (29')$$

which means an apparent increase of the moment of inertia by the amount $K'=\frac{L''}{k''}$. [1])

Substituting the expression (2) in (29) and again expressing the fact that for all values of t the equation must be satisfied, by equating to zero the coefficients of *cos* and *sin*, the same equations (26') are arrived at.

11. The separation of the general expression (24') into its real and imaginary parts is a troublesome performance, which is of no practical value; the general expressions for L' and L'' are so involved, that they are practically useless for the computation of η from the observed values of T and δ by means of the equations (28). As a matter of fact it is only under simplified conditions, that the determination of η by observation of the oscillations of a sphere is practically possible. Now the whole problem becomes most simple, when the liquid may be considered as unbounded; in that case it follows from (25) which may also be written as

$$L=\tfrac{8}{3}\pi R^3\eta\left(bR+2+\frac{1}{bR+1}\right),$$

[1]) From (29') it also follows, that even in the case of friction in a liquid the well-known equation

$$\frac{T^2-T_0'^2}{T_0'^2}=\frac{\delta^2}{4\pi^2}$$

still holds, on condition that for T_0 is taken the fictitious periodic time T_0' given by

$$T_0'=2\pi\sqrt{\frac{K+K'}{M}}.$$

that

$$L' = \tfrac{8}{3}\,\pi R^3 \eta \left(b'R + 2 + \frac{b'R+1}{(b'R+1)^2 + b''^2 R^2} \right)$$
$$L'' = \tfrac{8}{3}\,\pi R^4 b'' \eta \left(1 - \frac{1}{(b'R+1)^2 + b''^2 R^2} \right) \quad [1) \quad . \quad . \quad (30)$$

For a further approximation in the case, that the liquid may by approximation be considered as unbounded, (24′) can be developed in the form of a series. For this purpose we write first:

$$L = \tfrac{8}{3}\pi R^3 \eta_i \,.\, \frac{b^2R^2 + 3bR + 3}{bR+1} \,.\, \frac{1 + \dfrac{b^2R^2 - 3bR + 3}{b^2R^2 + 3bR + 3} \cdot \dfrac{bR'+1}{bR'-1} \cdot e^{-2b(R'-R)}}{1 - \dfrac{bR-1}{bR+1} \cdot \dfrac{bR'+1}{bR'-1} \cdot e^{-2b(R'-R)}} \quad (24'')$$

when $e^{-2b(R'-R)}$ is sufficiently small[2]), formula (25) will hold as a first approximation; if necessary a first correction-term may be added of the form

$$L_1 = \tfrac{16}{3}\,\pi R^6 b^3 \eta \frac{bR'-1}{(bR'+1)(bR+1)^2}\, e^{-2b(R'-R)}, \quad . \quad . \quad . \quad (31)$$

the value of which can be computed fairly easily, when an approximate value has been found for η.

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1) If $k\,(k_1)$ is replaced by the conjugate imaginary quantity k_2, it is clear, that the real part of α and also of z_r do not undergo any change (b_1 and b_2 are similarly conjugate), so that exactly the same results must be obtained, in particular the same equations (30). That this is actually true may be easily seen from the fact that L_1 and L_2 according to (24′) are also conjugate imaginary.

We might even, in general, have represented the damped harmonic oscillation by the real part of

$$\alpha = \alpha_1 + \alpha_2 = a_1 e^{k_1 t} + a_2 e^{k_2 t}.$$

We should then have obtained

$$\omega = k_1 u_1 e^{k_1 t} + k_2 u_2 e^{k_2 t},$$

and have found, that α must satisfy the equation

$$K\frac{d^2\alpha}{dt^2} + L_1 \frac{d\alpha_1}{dt} + L_2 \frac{d\alpha_2}{dt} + M\alpha = 0,$$

which, owing to $L'_2 = L'_1$ and $L''_2 = -L''_1$ may also be written as:

$$K\frac{d^2\alpha'}{dt^2} + L'_1 \frac{d\alpha'}{dt} - L'_1 \frac{d(\alpha''_1 - \alpha''_2)}{dt} + M\alpha' = 0.$$

By putting $a_1 = a_2$ α may then be real (form. (2)).

2) The coefficients of this factor in (24″) cannot become infinite in this case, on the contrary they do not differ much from unity.

12 In our experiments we intend to choose the conditions such that the liquid may, at least approximately, be considered as unbounded; moreover we shall arrange to make σ small. It is easily found, what conditions these simplifications are subject to.

Clearly it is necessary that the factor $e^{b'(R'-R)}$ obtains so high a value, that the terms containing this factor are sufficiently preponderant; this condition does not necessarily involve a specially high value of b', for if e.g. $R'-R=1$ i.e. if the distance of the two spherical surfaces is only 1 cm. (and this will be about the case in our experiments) still even for $b'=10$, the value of $e^{b(R-R)}$ will be as high as 10000 about. For water in C. G. S. units $\eta=0{,}01$ and $\mu=1$, so that even with $T=3$, i.e. a time of swing of 3 second, b' will reach the value 10, so that even in that case the desired condition will be fulfilled of the wave-motion, which starts from the oscillating sphere, when arriving at the external sphere, being practically completely damped out (§7). If it is further taken into account, that the oscillating sphere can only undergo an influence from the bounding wall by the waves reflected on the wall returning to the sphere and that the returning waves again undergo a damping process, it becomes clear, that the damping on the way from the inner sphere to the outer wall does not need to be so very complete, in order to be able to consider the liquid as being practically unbounded.

This fact is also expressed in our equations (24'') and (31). Practically (24'') is identical with (25), or $L_1=0$, when $e^{-2b'(R-R)}$ is sufficiently small, i.e. when the damping over a distance $2(R'-R)$ is sufficiently strong; in order that $e^{-2b'(R'-R)}$ may be say $\frac{1}{1000}$ with $R'-R=1$, even $b'=3$ would be sufficient and this would still be the case for water with T as high as 30. A somewhat large time of swing of about that magnitude is favourable to the readings from which the logarithmic decrement must be determined and it is accordingly intended in our experiments to make the periodic time about that size.

With $R'-R=1$ and $T=30$ even when working with water the liquid can thus approximately be considered as unbounded. But, moreover, it appears from (20) that with a given time of swing b' and b'' become greater, and therefore the conditions more favourable, according as the ratio $\frac{\eta}{\mu}$ is smaller; for very mobile liquids, like ether and benzene, they would therefore be even more favourable than with water, and, as the available data show, most favourable of all for liquified gases. The oscillation-method appears thus a particularly suitable one for liquid gases[1]).

13. With a view to our experiments it appeared to us desirable to have a rough idea as to the value of the viscosity for liquid hydrogen, say at the boiling point; an estimate may be obtained by the application of the law of corresponding states. KAMERLINGH ONNES[2]) has shown that for two different substances obeying this law the expressions

$$\eta \sqrt[6]{\frac{T_k}{p_k^4 M^3}},$$

must have the same value at corresponding temperatures, where T_k and p_k are the critical temperature and pressure and M the molecular weight. It is therefore possible by the application of this rule, which will be at least approximately valid, to calculate η for hydrogen by comparison with a substance whose viscosity is known over a somewhat wide range of temperatures, such as methyl-chloride according to measurements by DE HAAS[3]). For methyl-chloride $T_k = 416$, $p_k = 66$ (atm.), $M = 50$, and therefore $\sqrt[6]{\frac{T_k}{p_k^4 M^3}} = 0{,}024$; for hydrogen similarly $T_k = 31$, $p_k = 11$, $M = 2$, so that $\sqrt[6]{\frac{T_k}{p_k^4 M^3}} = 0{,}40$. The boiling point of hydrogen is 20° K. and the corresponding temperature for methyl chloride is $20 \times \frac{416}{31} = 268°$ K., or about 0° C., at which temperature η for methyl chloride is 0,0022; it follows that for hydrogen at 20° K. $0{,}40\ \eta = 0{,}024 . 0{,}0022$, which gives $\eta = 0{,}00013$. As at this temperature the density of liquid hydrogen is about 0,071[4]), we have $\frac{\eta}{\mu} = 0{,}0018$.

[1]) On the other hand, in ZEMPLÉN's experiments (Ann. d. Phys., 19, 783, 1906) on the viscosity of air in which concentric spheres were used of 5 and 6 cms. radius the condition of nearly complete damping of the reflected wave is not satisfied by a long way; with $\nu = 0{,}0002$, $\mu = 0{,}00012$ and $T = 30$, $b' = 0{,}8$ i.e. $e^{-2b(R'-R)} = \frac{1}{4}$ about. The damping is thus so weak in this case that the first correction-term (31) is not sufficient: we have therefore been obliged to abandon our intention originally formed, of recalculating ZEMPLÉN's experiments by means of our formulae.

[2]) Comm. phys. Lab. Leiden, n⁰. 12, p. 9.

[3]) Comm. phys. Lab. Leiden, n⁰. 12, p. 1

[4]) Comm. phys. Lab. Leiden, n⁰. 137*d*.

14. In all the above calculations it is assumed that the oscillations of the sphere are only weakly damped; this condition can in any case be satisfied, independently of the specific properties of the liquid. For, even when L' obtains a high value, the logarithmic decrement, by formula (28) can be made as small as desired by giving the oscillating system a high moment of inertia; this does not necessarily involve a corresponding increase of the time of swing, because the rotational moment M may still be chosen at will.

It is, moreover, easily seen, that for substances with a small value of $\frac{\eta}{\mu}$ the circumstances must again be the most favourable: according to (28) and (30) it is exactly for these substances, that under otherwise equal circumstances the oscillations of the sphere will be least damped.

15. When equation (25) holds, the calculation of η, the quantities μ, R, K, T_0, T and δ being known from the experiment, can be made in a fairly simple manner. First L' and L'' are calculated with the aid of equations (28) or, as the case may be, (28'). An approximate value of η having been found, b' and b'' can be obtained in first approximation by means of (20) and using these values a sufficiently accurate value can in general be calculated from the terms $p = \frac{b'R+1}{(b'R+1)^2 + b''^2R^2}$ and $q = \frac{1}{(b'R+1)^2 + b''^2R^2}$ in equations (30). Finally it only remains to solve the following quadratic equation in $\sqrt{\eta}$:

$$(2+p)\,\eta + \gamma' R \sqrt{\mu\eta} = \frac{3L'}{8\pi R^3} \quad \ldots \quad (a)$$

An alternative method of calculation would be from

$$\sqrt{\eta} = \frac{3L''}{8\pi R^4 \gamma' \sqrt{\mu}(1-q)} \quad \ldots \quad (b)$$

but in general this will yield a much less accurate value owing to the smaller accuracy with which $\vartheta = \frac{T-T_0}{T_0}$ is determined as compared with δ. Equation (b) ought rather to be looked upon as a kind of check on the result obtained; but it may also render excellent service for the purpose of obtaining an approximate value for η, if this should not be known; in that case it is even sufficient to neglect q with respect to 1.

16. As an example of a calculation the results of a preliminary experiment made by Mr. CH. NICAISE in water of 20° C. may be given here. A brass sphere of 1,927 cm. radius and weighing 250,8 grms. was suspended from a wire of phosphorbronze, such that in air the time of swing was 12,05 sec.; immersed in a large vessel with pure water the sphere had a periodic time of 12,24 sec. the amplitude of the oscillations diminishing per time of swing in a constant ratio, the natural logarithm of which was 0,1148 (it was found that this did not increase appreciably, until much narrower vessels were used, which shows that the liquid could be considered as being practically unbounded). For this experiment we have therefore $R=1{,}927$, $K=372{,}5$, $T_0=12{,}05$ (properly speaking the time of swing ought to have been measured in vacuo, but this would not have made a difference within the limits of accuracy of the observation) $T=12{,}24$, $\delta=0{,}1145$ (freed from the internal friction of the wire) [1]) and $\mu=0{,}998$.

This gives $\dfrac{\delta}{4\pi}=0{,}0091$ and $\psi=\dfrac{T-T_0}{T_0}=0{,}016$, and therefore within the limits of accuracy of the observation

$$L'=\frac{2\delta K}{T_0}=7{,}08 \quad , \quad L''=\frac{4\pi\psi K}{T_0}=6.$$

A first approximation with $\eta=0{,}01$, gives $b'=b''=\sqrt{\dfrac{\pi\mu}{\eta T}}=5$, therefore $b'R=b''R=10$, so that $p=0{,}05$, $q=0{,}004$. The viscosity is now given by

$$2{,}05\eta+0{,}966\sqrt{\eta}=0{,}1181,$$

hence:

$$\eta_{20}=0{,}01014,$$

a value which agrees very well with the known data. The equation with L'' gives as a very rough verification $\eta=0{,}010$.

17. The formulae become even simpler, if $b'R$ and $b''R$ are large numbers (say of the order 1000); in that case we have:

$$u=a\frac{R^2}{r^2}e^{-b(r-R)} \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$L'=\tfrac{8}{3}\pi R^4 b'\eta \quad , \quad L''=\tfrac{8}{3}\pi R^4 b''\eta. \quad . \quad . \quad . \quad . \quad (33)$$

1) Observation gave $\delta=0{,}1148$; in air $\delta=0{,}0011$, of which, according to a calculation of L' with $\eta=0{,}0002$ and $\mu=0{,}0012$, the fraction 0,0008 is due to the friction of the air, so that 0,0003 is left for the internal friction of the suspension.

If δ is small at the same time, we have in first approximation

$$L' = L'' = \tfrac{8}{3}\,\pi R^4 \sqrt{\frac{\pi\mu\eta}{T}}\,, \quad . \quad . \quad . \quad . \quad . \quad (34)$$

from which, by (28')

$$\delta = \tfrac{4}{3}\frac{\pi R^4}{K}\sqrt{\pi\mu\eta T_0}\ , \quad \frac{T-T_0}{T_0} = \frac{\delta}{2\pi}\,. \quad . \quad . \quad . \quad (35)$$

This extreme case is discussed by KIRCHHOFF in his Vorlesungen uber mathematische Physik, N°. 26, it occurs when $\frac{\eta T}{\mu R^2}$ is a very small number[1]). This case would be realized, if in a liquid with small $\frac{\eta}{\mu}$ (say a liquid gas) a large sphere was made to swing quickly; taking say $\frac{\eta}{\mu} = 0{,}001$, in order to have $b'R = 1000$ with $R = 10$, it would be necessary for T to be 0,3. Apart from the not very practical nature of these conditions, it may be considered very doubtful, whether with the comparatively high velocities, involved in a rapid vibration of that kind the preceding theory would still hold. It seems to me, therefore, that the extreme case in question has no experimental physical importance.

When $b'R$ and $b''R$ are only moderately large L' and L'' may be developed according tot ascending powers of $\frac{1}{b'R}$ and $\frac{1}{b''R}$; if in addition the series (20) and (28'), are introduced, and the development is stopped at a definite point, formulae such as those of LAMPE[1]), KLEMENCIC[2]), BOLTZMANN[3]) and KONIG[2]) are obtained.

1) KIRCHHOFF assumes η to be very small, which must of course be taken to mean: under otherwise normal circumstances, for, taken absolutely, it has no sense to suppose a quantity which is not dimensionless to be very small, seeing that the value depends on the choice of units. For the rest, the liquid need not necessarily have a very small viscosity in order to obtain the simple case in question; a small friction would even be a disadvantage, if combined with a small density, as in the case of gases. For air for instance $\frac{\eta}{\mu}$ is about 0,2, and thus much larger than for water, notwithstanding the much smaller value of η (comp. 12 note).

2) loc. cit.

3) Vid. LAMPE, Wien Ber. II. 93, 291, 1886. These formulae are as a rule, not very suitable for accurate calculations, because a sufficient accuracy cannot be obtained with only a few terms; as an instance, KÖNIG's experiments can be calculated much more simply and accurately in the manner of section 15 of this paper, than by KÖNIG's own method. From one of KÖNIG's experiments (the last

18. The opposite extreme case is that, in which bR and $b(R'-R)$ are very small numbers; in that case R' cannot of course be infinite, i. e. the liquid must be bounded. With normal dimensions of the spheres and usual times of swing this case might be realized with liquids of very high viscosity; for ordinary liquids the time of swing would have to be much greater than practice allows.

In that case (24') leads to:

$$L = L' = 8\pi R^3 \eta \frac{R'^3}{R'^3 - R^3} \text{ and } L'' = 0 \;, \quad . \; . \; . \quad (36)$$

therefore $$\delta = \frac{4\pi}{K} \frac{R'^3 R^3}{R'^3 - R^3} \eta T \text{ , and } \frac{T^2 - T_0^2}{T_0^2} = \frac{\delta^2}{4\pi^2}. \quad . \; . \; . \quad (37)$$

Seeing that by (22)

$$b'' R = \frac{2\pi R}{VT} = \frac{2\pi R}{\lambda} \;, \quad . \; . \; . \; . \; . \; . \; . \quad (38)$$

λ being the wave-length in the liquid, the physical meaning of the given simplifying condition is thus, that the radii R and R' are small as compared to the wavelength. In that case all the spherical shells in the liquid swing practically in the same phase [1]) (φ_r and y are nearly zero, so that u becomes real; in that case $u \doteq x$ (sect. 4) and equation (11) reduces to the first equation (7)); at the same time approximately $e^{-b'(R'-R)} = e^{b'(R'-R)} = 1$, i.e. the waves are propagated without being appreciably damped, as they move forward. The resulting equation is this time:

$$u = a \frac{R^3}{r^3} \frac{R'^3 - r^3}{R'^3 - R^3} \; ^{2)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (39)$$

with sphere 3) I find for water of 15° $\eta = 0{,}01103$, whereas König himself found 0,01140

[1]) This is the simplifying condition used by Zemplén (Ann. d. Phys. (4) 19, 783, 1906) as the basis in the deduction of the formulae which served for the calculation of the results of his experiments; thereby he overlooked the fact, that in that case his coefficient m (our factor b') is very small, so that $\cos m\,(R-r')$ and $\sin m\,(R-r')$ ought to have been developed according to powers of $m\,(R-r')$; carrying out this development, his equation (14) leads to our equation (39) (it may be noted here, thot a small error has crept into his equation (14); the terms m^2Rr^2 and $m^2Rr_2^2$ should be m^2Rr and m^2Rr_2 respectively). As a matter of fact in Zemplén's experiments the assumed approximation is not applicable, for in his case $\lambda = 9$, and thus not large as compared to the radii of the spheres($R = 5$, $R' = 6$); his result is, therefore, very doubtful. Later on (Ann. d. Physik. 29, 899, 1909) he discovered this himsolf and gave a more accurate treatment of the problem; but owing to the very complicated nature of the correct formulae he did not submit his experiments to a new calculation.

[2]) This distribution of velocities is the same as the one found for uniform rotation (comp. for instance Brillouin l.c. p. 89); this explains itself by the consi-

When bR and bR' are only moderately small numbers, L' and L'' can be developed according to powers of those quantities; the equations (36) are the first terms of the series which are obtained in that manner. Probably η might be found by that method for ordinary liquids at low temperature.

19. The formulae become also very simple, when $R'-R$ is small with respect to R, a case which may possibly be of some importance experimentally. In that case:

$$u = a\frac{R'-r}{R'-R}\text{ [1]}, \quad \ldots \ldots \quad (40)$$

$$L = \tfrac{8}{3}\pi R^3 \eta \cdot \frac{R}{R'-R} \quad \ldots \ldots \quad (41)$$

20. Although probably not of any practical utility I will for the sake of completeness discuss the case, in which the oscillating sphere is hollow, contains the liquid and swings about a smaller fixed sphere. Seeing that our general discussion of the state of motion in the liquid is not altered thereby, the preceding treatment retains in general its validity; the boundary-conditions also remain the same, so that equations (17) and (17') remain valid. Only owing to the fact that $R > r > R'$, it is now more logical to write

$$u = \frac{1}{r^3}\left[P' e^{-b(R-r)}(br-1) + Q' e^{b(R-r)}(br+1)\right], \quad \ldots \quad (42)$$

and the conditions at the boundaries now give

$$P' = \frac{aR^3(bR'+1)e^{b(R-R')}}{D} \quad , \quad Q' = -\frac{aR^3(bR'-1)e^{-b(R-R')}}{D} \quad (43)$$

where

$$D = (bR-1)(bR'+1)\,e^{b(R-R')} - (bR+1)(bR'-1)\,e^{-b(R-R')}, \quad (44)$$

As regards L, the expression given in (24') still holds for it, except that it has to be provided with the negative sign, because now that the sphere undergoes friction on the inside, the tangential force is not F but $-F$ (comp. sect. 3 and 8); we thus have [2]:

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deration that, when the wave-length is large as compared to the radii of the spheres, the condition may at any moment be considered as stationary.

1) This distribution of velocities agrees with that between two parallel planes, which move with respect to each other at constant speed; this result could have been expected.

2) All the formulae for this case are obtained from the corresponding ones in 5 and 8 by giving R, R' and r everywhere the opposite sign; this is quite intelligible from a mathematical point of view.

$$L = \frac{8}{3}\frac{\pi R^3 \eta}{D}[(b^2R^2-3bR+3)(bR'+1)e^{b(R-R')}+(b^2R^2+3bR+3)(bR'-1)e^{-b(R-R')}] \quad (45)$$

For the rest no alterations have to be made to section 9 and the calculation of η would proceed in the same manner as with an internal oscillating sphere.

21. Another case which is of practical importance and has found experimental application [1]), is that of a hollow sphere completely filled with liquid which is made to swing. It may be expected that this case can be derived as a special case from our general formulae by putting $R' = 0$. In that case according to (53):

$$P'e^{-bR} = Q'e^{bR} = \frac{aR^3}{(bR-1)e^{bR}+(bR+1)e^{-bR}} \quad . \quad . \quad . \quad (46)$$

and

$$u = a\frac{R^3}{r^3}\frac{(br-1)e^{br}+(br+1)e^{-br}}{(bR-1)e^{bR}+(bR+1)e^{-bR}} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Physically, however, this is only possible, if for $r = 0$, u does not become infinite and, as a matter of fact, it does not, for with $r = 0$, u becomes

$$u_0 = \tfrac{2}{3}\, ab^3 \frac{R^3}{(bR-1)\, e^{bR} + (bR+1)\, e^{-bR}} \quad . \quad . \quad . \quad . \quad . \quad (48)$$

In the general case the liquid cannot be at rest at the centre: the wave-motion starting from the oscillating sphere passes through the centre and expands again beyond it; this may also be formulated by saying, that the waves are reflected at the centre, this time as upon a free boundary, i. e. without reversal of phase. Only when bR is so large, that the motion is damped out before reaching the centre, $u_0 = 0$ practically and further

$$u = a\frac{R^3}{r^3} \cdot \frac{br-1}{bR-1}\, e^{-b(R-r)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

22. In the case of a sphere filled with a liquid we have further (by putting $R' = 0$ in (45)):

$$L = \tfrac{8}{3}\, \pi R^3 \eta \frac{(b^2R^2-3bR+3)\, e^{bR} - (b^2R^2+3bR+3)\, e^{-bR}}{(bR-1)\, e^{bR} + (bR+1)\, e^{-bR}} \quad . \quad (50)$$

If the wave-motion is damped out when arriving at the centre, i. e. if e^{-bR} may be put very small, the value of λ is given by

$$L = \tfrac{8}{3}\, \pi R^3 \eta \frac{b^2R^2 - 3bR + 3}{bR - 1} \quad , \quad . \quad . \quad . \quad . \quad (51)$$

which is obtained from (25) by reversing the sign of R; in the same

[1]) H. v. Helmholtz und G. v. Piotrowski, Wien. Ber. 40 (2), 607, 1860. H. v. Helmholtz, Wissensch. Abh., 1, p. 172.
G. Zemplén, Ann. d. Phys., 19, 791, 1906; 29, 902, 1909.
Vid. also Lamb, Hydrodynamics, p. 578.

manner (30) will then give:

$$L' = \tfrac{8}{3}\,\pi R^3\eta\left(b'R - 2 + \frac{b'R-1}{(b'R-1)^2 + b''^2R^2}\right),$$

$$L'' = \tfrac{8}{3}\,\pi R^4 b''\eta\left(1 - \frac{1}{(b'R-1)^2 + b''^2R^2}\right) \quad . \quad . \quad . \quad . \qquad (52)$$

The calculations are to be carried out as in § 15.

When $b'R$ and $b''R$ are very large, the same formulae (33) are arrived at as before, which means that, when the motion is completely extinguished at a very short distance from the oscillating sphere, it makes no difference whether the friction is internal or external; this might of course have been foreseen. [1])

23. When bR, and therefore also br, are very small, that is: when the wavelength is very large compared to the radius of the sphere, as would probably be the case with very viscous liquids (comp. § 18), it follows from (49) that $u = a$, i. e. the sphere swings as a completely solid mass, as might have been expected a priori. There will thus be no damping and the time of swing must be that of a system the moment of inertia of which is equal to K with the addition of the moment of inertia of the liquid.

This actually follows from the above formulae, for (50) then reduces to

$$L = \tfrac{8}{15}\,\pi\, b^2R^5\eta = \tfrac{8}{15}\,\pi\mu R^5 k,$$

and introducing this into (26'), we find that

$$k' = 0 \quad \text{and} \quad \frac{1}{k''^2} = \frac{T^2}{4\pi^2} = \frac{1}{M}(K+K'),$$

where $K' = \tfrac{8}{15}\,\pi\mu R^5$, the moment of inertia of the liquid. [2])

[1]) In PIOTROWSKI's experiments the aforesaid condition was not fulfilled, no more than in KÖNIG's experiments; R was $= 12{,}5$, $T = 30$, and hence $b'R = 7{,}5$ about. Still this value is sufficiently large to make the application of (51) allowable, and as in KÖNIG's experiments, this leads without difficulty to the value of η Similarly in ZEMPLÉN's experiments with air equation (51) is applicable to the inside-friction on the oscillating sphere, for with $\mu = 0{,}0012$, $\eta = 0{,}0002$, $T = 30$ and $R = 5$ one finds $b' = \sqrt{\dfrac{\pi\mu}{\eta T}} = 0{,}8$, hence $e^{-2b'R} = e^{-8} = \dfrac{1}{2000}$ about.

[2]) This result may be expressed as follows; L is imaginary in this case and

$$L' = 0 \quad \text{and} \quad L'' = \tfrac{8}{15}\,\pi\mu R^5 k'',$$

showing that the addition to the moment of inertia (comp § 10), is here equal to the actual moment of inertia of the liquid, and the equation of motion of the sphere becomes (29'):

$$(K+K')\frac{d^2\alpha'}{dt^2} + M\alpha' = 0.$$