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tioned old steel magnet e.g. amounted to 12,9%, near the middle however to 8,5%; for a magnet of hardened steel the change near the end was 2,1%, at a distance from the end equal to $\frac{1}{3}$ of the length 1,2%. In both cases the coefficient α was positive. For a bundle of piano-strings which each had a diameter of 1,2 mm and a length of 17,5 cm the change between 20° and 100° was -3,9% at the end and -6,1% at a distance from the end equal to $\frac{1}{3}$ of the length. After partial demagnetisation, by which the magnetisation was diminished to $\frac{1}{3}$ of its original value the change between the same temperature limits at the end was only -1,5%. Magneto-metrically (now we are principally concerned with the action of the end of the magnet) there was found before the demagnetisation -3,3% and afterwards -1,6% (here the temperature limits were 10° and 100°). From this we might conclude that for steel the coefficient α depends on the magnetisation in this way that it increases in the positive sense according as the latter decreases. This agrees with what was found in the investigation of different parts of one and the same steel magnet, viz. that near the end α is greater in the positive sense than in the middle; for because of the demagnetising force the magnetisation at the ends is much smaller than in the middle.

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*Physical Laboratory of the
Teyler Institute.*

Physics. — “*Some remarks on the hydrogen-molecule of BOHR — DEBIJE.*” By Miss H. J. VAN LEEUWEN. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of December 18, 1915).

§ 1. BOHR has been the first who supposed the hydrogen-molecule to be formed by two nuclei which carry a positive elementary charge and in which nearly the whole mass of the molecule is concentrated, together with two electrons which in the normal state circulate with a constant angular velocity ω diametrically on a circle that has its centre in the middle between the nuclei and its plane perpendicular to their line of connection, the “axis” of the molecule¹⁾. BOHR supposes that such molecular systems do not obey the laws of classic mechanics, that on the contrary all motions of the electrons are bound by the condition that for each single electron the moment of

¹⁾ N. BOHR. On the constitution of atoms and molecules III, Phil. Mag. 6, 26, 1913 p. 857.

momentum with respect to the axis of the molecule must have the value $\frac{h}{2\pi}$, where h is PLANCK'S constant.

Under the same suppositions as BOHR made on the constitution of the molecule, DEBIJE has investigated the forced vibrations which the system can execute under the influence of the electric force of a transmitted light-beam ¹⁾. DEBIJE supposes the mass of the nuclei to be so great that their motion may be neglected and further that these vibrations can be calculated according to the laws of ordinary mechanics. If the forced vibrations are known, the mean electric moment of the unit of volume is found at the same time and from this follows again the index of refraction n .

For this DEBIJE finds:

$$\frac{n^2-1}{4\pi N} = \frac{e^2}{m\omega^2} \left[\frac{2,97429}{1 - \left(\frac{s}{0,412375\omega}\right)^2} + \frac{2,15347}{1 - \left(\frac{s}{0,556397\omega}\right)^2} + \frac{0,0276447}{1 - \left(\frac{s}{2,412375\omega}\right)^2} + \frac{0,246581 + 0,579918 \left(1 - \frac{s}{\omega}\right)}{0,304388 + \left(1 - \frac{s}{\omega}\right)^2} + \frac{0,246581 + 0,579918 \left(1 + \frac{s}{\omega}\right)^2}{0,304388 + \left(1 + \frac{s}{\omega}\right)^2} \right] \quad (1)$$

where: N is the number of molecules per unit of volume,

— e the charge of the electron

m the mass of the electron.

By comparison with experimental data of J. KOCH or C. and M. CUTHBERSON DEBIJE found:

$$\frac{e}{m} = 4,64 \cdot 10^{17} \text{ or } 5,01 \cdot 10^{17},$$

which value agrees very well with the data for the electrons from cathode rays ($5,58 \cdot 10^{17}$), and for the moment of momentum of each electron $\frac{h}{6,32}$ that is, within the limits of error, just what BOHR had assumed.

Now we will consider, however, how the system behaves, firstly

¹⁾ P. DEBIJE. Die Konstitution des Wasserstoff-Moleküls. Sitz. Ber. München 1915 p. 1.

if, as to its small vibrations, it obeys the laws of mechanics, secondly if it is restricted by BOHR's condition.

§ 2. Let the distance between the nuclei be $2a$, the radius of the path of the electrons r . The conditions that the nuclei are at rest and that the electrons circulate with the angular velocity ω , are

$$r = \sqrt{3}a \quad , \quad \omega^2 = \frac{e^2}{4mr^3} (3\sqrt{3}-1) \dots \dots \dots (2)$$

For the case that no external forces act on the system, the equations, which according to DEBIJE hold for the forced vibrations, become those for the free vibrations of the system. If r_1 , ϑ_1 , r_2 and ϑ_2 are polar coordinates for the two electrons in the plane of their path and z_1 , z_2 their distances to that plane, X_1 , Y_1 , Z_1 , X_2 , Y_2 , Z_2 the components of the electrostatic attractions and repulsions, all with respect to a system of coordinates that also rotates with the angular velocity ω , then these equations are for the first electron

$$\begin{aligned} m \ddot{r}_1 - 2m \omega r_1 \dot{\vartheta}_1 - m \omega^2 r_1 &= X_1 \cos \vartheta_1 + Y_1 \sin \vartheta_1 \\ m r_1 \ddot{\vartheta}_1 + 2m \omega \dot{r}_1 &= -X_1 \sin \vartheta_1 + Y_1 \cos \vartheta_1 \\ m \ddot{z}_1 &= Z_1 \end{aligned}$$

For the second electron we find similar equations. If we introduce

$$r_1 = r + \varrho_1 \quad , \quad r_2 = r + \varrho_2 \quad , \quad \vartheta_2 = \vartheta_1 + \pi + \varphi$$

then ϱ_1 , ϱ_2 , φ , z_1 and z_2 may be treated as small quantities, of which we need only take the first powers (this is not allowed for the deviations of ϑ_1 and ϑ_2 themselves from the values they have in the stationary motion, because by a small change of the angular velocity ϑ_1 and ϑ_2 may obtain great deviations). The equations then become

$$\begin{aligned} \ddot{\varrho}_1 - 2\omega r \dot{\vartheta}_1 - \omega^2 (r + \varrho_1) &= \frac{e^2}{m} \left(-\frac{2r}{\sqrt{r^2 + a^2}} + \frac{4r^2 - 2a^2}{\sqrt{r^2 + a^2}} \varrho_1 + \frac{1}{4r^2} - \frac{\varrho_1 + \varrho_2}{4r^3} \right) \\ r \ddot{\vartheta}_1 + 2\omega \dot{\varrho}_1 &= \frac{e^2}{m} \frac{\varphi}{8r^2} \\ \ddot{\varrho}_2 - 2\omega r \dot{\vartheta}_2 - \omega^2 (r + \varrho_2) &= \frac{e^2}{m} \left(-\frac{2r}{\sqrt{r^2 + a^2}} + \frac{4r^2 - 2a^2}{\sqrt{r^2 + a^2}} \varrho_2 + \frac{1}{4r^2} - \frac{\varrho_1 + \varrho_2}{4r^3} \right) \\ r \ddot{\vartheta}_2 + 2\omega \dot{\varrho}_2 &= -\frac{e^2}{m} \frac{\varphi}{8r^2} \\ \ddot{z}_1 &= \frac{e^2}{m} \left(-\frac{3\sqrt{3}z_1}{16r^3} + \frac{z_1 - z_2}{8r^3} \right) \\ \ddot{z}_2 &= \frac{e^2}{m} \left(-\frac{3\sqrt{3}z_2}{16r^3} + \frac{z_2 - z_1}{8r^3} \right) \end{aligned}$$

If, further, we put

$q_1 + q_2 = \alpha$, $q_1 - q_2 = \beta$, $\vartheta_1 + \vartheta_2 = \mu$, $z_1 + z_2 = \gamma$, $z_1 - z_2 = \delta$,
and use the equations (2) we obtain

$$\left. \begin{aligned} \ddot{\alpha} - 2\omega r\dot{\mu} - \omega^2 \alpha &= \frac{15\sqrt{3}-8}{4(3\sqrt{3}-1)} \omega^2 \alpha \\ r\ddot{\mu} + 2\omega \dot{\alpha} &= 0 \end{aligned} \right\} \dots \dots (3)$$

$$\left. \begin{aligned} \ddot{\beta} - 2\omega r\dot{\varphi} - \omega^2 \beta &= \frac{15\sqrt{3}}{4(3\sqrt{3}-1)} \omega^2 \beta \\ r\ddot{\varphi} + 2\omega \dot{\beta} &= -\frac{r\omega^2 \varphi}{3\sqrt{3}-1} \end{aligned} \right\} \dots \dots (4)$$

$$\ddot{\gamma} + \frac{3\sqrt{3}}{4(3\sqrt{3}-1)} \omega^2 \gamma = 0 \dots \dots (5)$$

$$\ddot{\delta} + \frac{3\sqrt{3}-4}{4(3\sqrt{3}-1)} \omega^2 \delta = 0 \dots \dots (6)$$

From this we shall deduce the six principal modes of vibration of the system. We suppose that all quantities contain the time only in the factor e^{int} , by which the differential equations (3)—(6) become ordinary linear equations which we shall indicate in the same order with (3')—(6'). The determinants of the systems (3') and (4') and the equations (5') and (6') give us then the six values of n . From the obtained equations (3') and (4') we calculate for each value of n the complex ratios between $r\mu$ and α , $r\varphi$ and β and then know the form of vibration. So we obtain

(A) from equations (6) the mode of vibration (A) in which both electrons execute linear vibrations along the Z -axis, in such a way that they always have the same and opposite deviations; the frequency is $n_A = 0,27 \omega$.

(B) from equation (5) the mode of vibration (B), in which they always have equal deviations in the same sense parallel to the Z -axis with the frequency $n_B = 0,556 \omega$.

(C) from the equations (3) $n_{C_1} = 0$ and $n_{C_2} = \pm 1,47 \omega$. To n_{C_1} belongs a mode of vibration c_1 in which both electrons are displaced over the circle in the same sense; to which disturbance the system is of course indifferent. To n_{C_2} belongs a vibration C_2 , in which both electrons describe equal and congruent ellipses of the same position, but in such a way that the radii vectors are always oppositely directed: $\frac{\alpha}{r\mu} = \mp 20,64 i$.

(D) from the equations (4) finally $n_{D_1} = \pm 1,41 2\omega$. In this mode the electrons describe equal and congruent ellipses of the same position

with radii vectors that always have the same direction: $\frac{\beta}{r\varphi} = \pm 0,62 i$; this we call the motion D_1 . There is another mode of motion D_2 , in which the electrons describe straight lines and always have the same and equally directed deviations, but in which the frequency is $m_{D_2} = \pm 0,55 i\omega$. To the value $-0,55 i\omega$ belongs the direction of motion $\frac{\beta}{r\varphi} = 0,49$, which is indicated in the figure and in which the deviations decrease exponentially; to $+0,55 i\omega$ belongs one, in which the deviations increase exponentially; the lines of motion being equally but oppositely inclined with respect to the circle in the two cases. In the second case

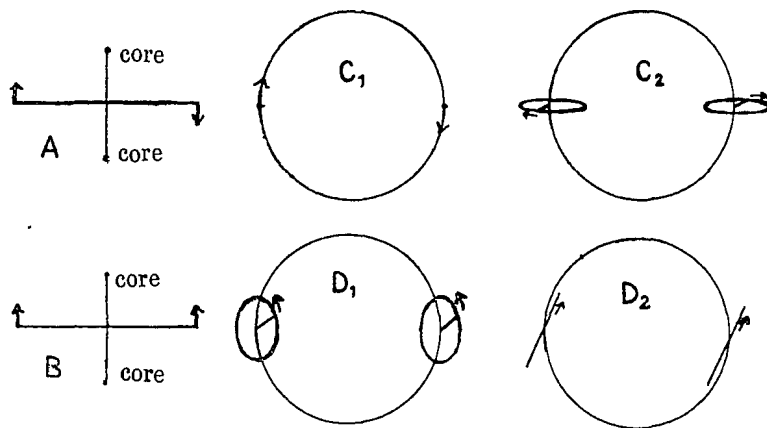


Fig. 1.

$$\frac{\beta}{r\varphi} = -0,49.$$

The system therefore is unstable, as has already been remarked by ВОНН¹⁾ for all systems consisting of two nuclei with a ring of electrons between them.

§ 3. In the vibrations A , C_1 and C_2 the molecule does not possess any resultant electric moment. Therefore an external electric force will never set the molecule in vibration, in this way and if the molecule had this motion, it would not exert an influence on the propagation.

It is otherwise with the motions B , D_1 and D_2 . In DEBIJE's dispersion formula the second term corresponds to the motion B , the 1st and 3rd terms together correspond to the motion D_1 , the 4th and 5th together to the motion D_2 . This becomes evident by the following reasoning. The periodic electric force, which itself is proportional to $\cos(st+p)$,

¹⁾ N. ВОНН, loc. cit.

has along the Z -axis a component also with the period $\frac{2\pi}{s}$, but along the rotating axes in the plane of the path components proportional to $\cos(st+p)\cos(\omega t+q)$, which may be split up into terms with the periods $\frac{2\pi}{s+\omega}$ and $\frac{2\pi}{s-\omega}$. The electric moment which the molecule has under the influence of the electric force will depend linearly on γ , β and φ . If we write D for the sign of differentiation $\frac{d}{dt}$, the

equations (4) and (5) for these quantities may be brought into the form

$$(D^2 + n^2_{D1})(D^2 - n^2_{D2})\beta = 0$$

$$(D^2 + n^2_{D1})(D^2 - n^2_{D2})\varphi = 0$$

$$(D^2 + n^2_B)\gamma = 0.$$

(The integration constants in the solutions β and φ must then be determined with regard to the equations (4)).

For the forced vibrations we obtain instead of these equations:

$$\left. \begin{aligned} (D^2 + n^2_{D1})(D^2 - n^2_{D2})\beta &= F_\beta + F_{\beta'} \\ (D^2 + n^2_{D1})(D^2 - n^2_{D2})\varphi &= F_\varphi + F_{\varphi'} \\ (D^2 + n^2_\beta)\gamma &= F_\gamma \end{aligned} \right\} \dots \dots (7)$$

where F_β , $F_{\beta'}$, F_φ , $F_{\varphi'}$, F_γ are homogeneous, linear functions of the components of the electric force in the beam of light, taken with respect to the rotating system of coordinates, so that F_β and F_φ have the period $\frac{2\pi}{s+\omega}$, $F_{\beta'}$ and $F_{\varphi'}$ the period $\frac{2\pi}{s-\omega}$ and F_γ the period $\frac{2\pi}{s}$.

In virtue of the linear character of the equations (7) we may add the solutions that would be obtained, first if on the right hand sides F_β and $F_{\varphi'}$ stood alone, secondly if only $F_{\beta'}$ and F_φ occurred there. In the first case we assume that β and φ contain the time only in the factor $e^{i(s+\omega)t}$, in the second that they are proportional to $e^{i(s-\omega)t}$ and so we obtain

$$\beta = -\frac{F_\beta}{\{n^2_{D1} - (s+\omega)^2\}\{n^2_{D2} + (s+\omega)^2\}} - \frac{F_{\beta'}}{\{n^2_{D1} - (s-\omega)^2\}\{n^2_{D2} + (s-\omega)^2\}}$$

$$\varphi = -\frac{F_\varphi}{\{n^2_{D1} - (s+\omega)^2\}\{n^2_{D2} + (s+\omega)^2\}} - \frac{F_{\varphi'}}{\{n^2_{D1} - (s-\omega)^2\}\{n^2_{D2} + (s-\omega)^2\}}$$

and in the same way for γ :

$$\gamma = \frac{F_\gamma}{n^2_B - s^2}.$$

In the expression for the moment of a molecule we obtain terms corresponding to each of these, and the first term in β can

be taken together with the first in φ and also the second in β with the second in φ . The mean moment of the molecule and therefore also the moment of the unit of volume with respect to axes fixed in space have corresponding terms, but all with the period $\frac{2\pi}{s}$ ¹⁾. Hence the ratio of this moment to the electric force that excites it

has likewise corresponding terms; as this ratio determines the index of refraction, the latter must be expressed by an equation of the form:

$$n^2 - 1 = \frac{\mu_1}{\{n_{D1}^2 - (s + \omega)^2\} \{n_{D2}^2 + (s + \omega)^2\}} + \frac{\mu_2}{\{n_{D1}^2 - (s - \omega)^2\} \{n_{D2}^2 + (s + \omega)^2\}} + \frac{\mu_3}{n_B^2 - s^2}$$

or if we decompose each of the first two terms into terms with denominators of the second degree:

$$n^2 - 1 = \frac{\mu'_1}{1 - \left(\frac{s}{n_{D1} - \omega}\right)^2} + \frac{\mu'_2}{1 - \left(\frac{s}{n_B}\right)^2} + \frac{\mu'_3}{1 - \left(\frac{s}{n_{D1} + \omega}\right)^2} + \frac{\mu''_2}{\left(\frac{n_{D2}}{\omega}\right)^2 + \left(1 - \frac{s}{\omega}\right)^2} + \frac{\mu''_1}{\left(\frac{n_{D2}}{\omega}\right)^2 + \left(1 + \frac{s}{\omega}\right)^2}.$$

As $n_{D1} = 1,41 \omega$, $n_B = 0,556 \omega$, $n_{D2}^2 = 0,304 \omega^2$, the terms of this equation correspond to those of (1) in the same order. The determination of the numerators would only lead to an unnecessary repetition of the long calculations DEBIJE had to make. Without it we can see already that the second term points to resonance if $s = n_B$, the first if $s + \omega = n_{D1}$, the third if $s - \omega = n_{D1}$, the fourth if $s - \omega = in_{D2}$, the fifth if $s + \omega = in_{D2}$. Professor LORENTZ pointed out to me that by attending to the dispersion formula we should expect the unstable character of the hydrogen molecule of BOHR—DEBIJE.

If now an electric force has worked on this system in the way assumed by DEBIJE, the electrons remain in positions and with velocities which necessarily must give rise to the modes of motion B , D_1 and D_2 . Hence, after having been exposed for some time to an external electric force the system would be destroyed.

§ 4. Now we can try to limit the freedom of motion of the system in the plane of the path in such a way that the motion

¹⁾ As soon as the mean value has been determined with respect to the different phases of the electronic motion, this result is obtained.

D_2 is excluded. It is however not easy to do so for the motion D_2 alone. We could prescribe for the centre of mass of the electrons an elliptical path of the form belonging to D_1 , but by this we should lose the homogeneousness of the equations for the free vibrations and the possibility of superposing them on the forced vibrations.

It is easily seen, however, what form the dispersion formula for hydrogen will take if the motion of the centre of mass of the electrons is limited to the axis of the molecule. We may as well exclude all disturbances in the plane of the path, as there is no electric moment in the case of the motions C_1 and C_2 . The only moment of the molecule is now in the Z -direction and a calculation similar to that of DEBIJE shows that in his dispersion formula only the second term remains, so that

$$\frac{n^2-1}{4\pi N} = \frac{e^2}{m\omega^2} \left[\frac{2,153}{1 - \left(\frac{s}{0,556 \omega} \right)^2} \right]$$

or

$$n - 1 = \frac{2\pi N e^2}{m\omega^2} \left(2,153 + 6,954 \frac{s^2}{\omega^2} \right)$$

while the observations of KOCH give

$$n - 1 = 1,361 \cdot 10^{-4} + 2,908 \cdot 10^{-37} s^2.$$

From this we obtain:

$$\omega = 3,89 \cdot 10^{10}$$

and using the value $Ne = 1,289 \cdot 10^{10}$

$$\frac{e}{m} = 1,18 \cdot 10^{18}$$

which is just twice what is found for the cathode rays. If the charge of a hydrogen atom were twice that of the particles in the cathode rays or if its mass were half that of these latter corpuscles RYDBERG'S constant as calculated by BOHR from his analogous hypothesis concerning the hydrogen atom would no longer have the value that follows from the experiments¹⁾.

Let us further consider the moment of momentum of each of the electrons. From the formula for this moment

$$mr^2\omega = z\hbar$$

and from that for the uniform rotation

¹⁾ N. BOHR. On the constitution of atoms and molecules I Phil. Mag. 6: 26, 1913 p. 1.

$$r^3 = \frac{3\sqrt{3}-1}{4} \frac{e^2}{m\omega^2}$$

follows

$$z^3 = \left(\frac{3\sqrt{3}-1}{4} \right)^2 \frac{me^4}{\hbar^3\omega}$$

If here we take for e the value $4,69 \cdot 10^{-10}$ that has been found for cathode rays (though the value of $\frac{e}{m}$ for these rays is different from that to which we have been led) we get

$$z = \frac{1}{8,02}$$

If we take $e = 2 \cdot 4,69 \cdot 10^{-10}$ this becomes

$$z = \frac{1}{2,53}$$

which no longer agrees with BOHR's hypothesis: $z = \frac{1}{2\pi}$.

One would be led to the same conclusion if it were possible to obtain DEBYE's dispersion formula without the two last terms, i.e. without those that have called forth our objections. The formula would then be:

$$n - 1 = 2\pi N \frac{e^2}{m\omega^2} \left[5,155 + 24,456 \frac{s^2}{\omega^2} \right]$$

We should have:

$$\omega = 1,712 \cdot 10^{16} \quad , \quad \frac{e}{m} = 7,238 \cdot 10^{17}$$

and if we take $e = 4,69 \cdot 10^{-10}$

$$z = \frac{1}{7,27}$$

We must still mention that it is possible to give to the electrons such a mobility that an electric moment in the plane of the path is possible, while still the system remains stable. For this purpose we may assume that the electrons can only circulate on a circle with prescribed radius. As we must now introduce radial connecting forces, one of the equations 2, that which connects r and ω , must be omitted, a rotation with any angular velocity being now possible. Instead of the equations (3)–(6) we get:

$$\begin{aligned} \ddot{u} &= 0, \\ \ddot{\varphi} + \frac{e^2\varphi}{4mr^3} &= 0, \\ \ddot{\gamma} + \frac{3\sqrt{3}e^2}{16mr^3}\gamma &= 0, \end{aligned}$$

from which we find the solution

$$\mu = 2Mt + n.$$

Further it follows from these equations that φ and γ are periodic functions of the time, so that the system is stable.

If Pe^{ist} , Qe^{ist} and Re^{ist} are the components of the electric force Ee^{ist} in the beam of light, M_x , M_y and M_z those of the electric moment of the molecule, both respectively along the X -, Y -, Z -axis of the rotating system of coordinates and if

$$p = P + iQ, \quad q = P - iQ,$$

we find the relations:

$$M_x = \frac{e^2}{m} \sin(\vartheta_1 + tM) \left[\frac{qie^{i\vartheta_1}}{-(s+M)^2 + \frac{e^2}{4mr^3}} e^{i(s+M)t} - \frac{pie^{-i\vartheta_1}}{-(s-M)^2 + \frac{e^2}{4mr^3}} e^{i(s-M)t} \right],$$

$$M_y = \frac{e^2}{m} \cos(\vartheta_1 + Mt) \left[\frac{qie^{i\vartheta_1}}{-(s+M)^2 + \frac{e^2}{4mr^3}} e^{i(s+M)t} - \frac{pie^{-i\vartheta_1}}{-(s-M)^2 + \frac{e^2}{4mr^3}} e^{i(s-M)t} \right],$$

$$M_z = \frac{2e^2}{m} \frac{R}{-s^2 + \frac{3\sqrt{3}e^2}{16mr^3}} e^{ist}.$$

From this we find for the mean moment \overline{M} of the molecule (in the supposition that for all molecules r and M have the same value):

$$\overline{M} = \frac{e^2}{3m} e^{ist} E \left[\frac{1}{-(s+M)^2 + \frac{e^2}{4mr^3}} + \frac{1}{-(s-M)^2 + \frac{e^2}{4mr^3}} + \frac{2}{-s^2 + \frac{3\sqrt{3}e^2}{16mr^3}} \right].$$

The index of refraction is then determined by

$$n^2 - 1 = \frac{8\pi Ne^2}{3m} \left[\frac{1}{\frac{e^2}{4mr^3} - M^2} + \frac{s^2 \left(\frac{e^2}{4mr^3} - M^2 \right) + 4s^2 M^2}{\left(\frac{e^2}{4mr^3} - M^2 \right)^3} + \frac{1}{\frac{3\sqrt{3}e^2}{16mr^3}} + \frac{s^2}{\left(\frac{3\sqrt{3}e^2}{16mr^3} \right)^2} \right].$$

Now, while giving to $\frac{e^2}{m}$ the value that belongs to cathode rays, we are still free to choose a value for r and M and we may do this in such a way that the equation changes into that of J. KOCH:

$$n - 1 = 1,361 \cdot 10^{-4} + 2,908 \cdot 10^{-37} s^2.$$

Indeed, if we put:

$$x = 16mr^3, \quad y = M^2, \quad z = 4e^2 - xy.$$

we have to solve the equations:

$$x = \frac{3\sqrt{3}e^2az}{3\sqrt{3}e^2+z}$$

and

$$z^3(a^2-b) - 6\sqrt{3}be^2z^2 - (81a^2 + 27b)e^4z + 432a^2e^6 = 0$$

where

$$a = \frac{1,361}{30,1} \cdot 10^{-31} \quad b = \frac{2,903}{30,1} \cdot 10^{-64}.$$

This cubic equation has two positive roots, the first of which lies between 10^{-17} and 10^{-18} and the second between 10^{-18} and $9 \cdot 10^{-19}$; these are of no use as they do not give a positive value for xy . The third root is negative viz:

$$z = -2,6 \cdot 10^{-18}.$$

With this we find

$$x = 9,0 \cdot 10^{-51}, \quad y = 3,9 \cdot 10^{32},$$

so that

$$r = 0,875 \cdot 10^{-8}, \quad M = 2,0 \cdot 10^{16}.$$

If with these values we calculate the moment of momentum mr^2M , we find for it: $\frac{h}{5,1}$. We are again led to the conclusion that

it is impossible to obtain agreement with the value $\frac{h}{6,28}$ introduced by BOHR.

§ 5. It would certainly have been better if the way in which we limit the freedom of motion of the electrons agreed more closely with the restriction which BOHR has introduced for the electrons in the atoms, and which is one of his assumptions on which the deduction of BALMER'S formula rests, viz. that for each electron

$$mr^2 \dot{\varphi} = \frac{h}{2\pi}.$$

If in this way we find stable configurations, we have to solve the difficult problem to establish the equations of motion for a system, for which the equations of constraint contain velocities as well as coordinates. In dynamics we are taught how to form the equations of motion for those systems only in which the same connections exist between the velocities and between infinitesimal displacements (e. g. rolling without sliding) and this is not the case here.

In order to investigate the stability we shall apply to this hydrogen-molecule the criterion, introduced by BOHR instead of that of ordinary mechanics. This criterion is as follows:

If in the motion considered the total energy of the system is a minimum with respect to all those disturbances of the system in which the moment of momentum of each of the electrons remains unchanged, the system will be stable. It must be remarked, however that, both for his hypothetic atoms and e. g. for the hydrogen-molecule BOHR examines only disturbances, which consist in an increase or a decrease of the radius of the circle on which the electrons continue circulating uniformly and in which the moment of momentum keeps the same value as in the undisturbed motion. Such a disturbance, however, is no longer possible now, for in the case of uniform rotation the values of $r_1 (= r_2)$ and those of $\dot{\vartheta}_1 (= \dot{\vartheta}_2)$ and a may be deduced from equations similar to (2); if besides we give the value of the moment, the value of r_1 will be determined. The electrons cannot therefore describe a larger or a smaller circle with the same value of the moment of momentum as in the undisturbed motion.

For the atoms, i. e. for systems with one nucleus, about which electrons circulate in a circle, L. FÖPPL ¹⁾ has investigated disturbances of a more general kind. We shall now apply his mode of reasoning to the hydrogen molecule. Thus we have to take the sum of the kinetic energy T and the potential energy P viz.

$$P + T = \frac{m}{2} \left[\dot{r}_1^2 + \dot{r}_2^2 + r_1^2 \dot{\vartheta}_1^2 + r_2^2 \dot{\vartheta}_2^2 + \dot{z}_1^2 + \dot{z}_2^2 \right] +$$

$$+ \frac{e^2}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\vartheta_2 - \vartheta_1) + (z_1 - z_2)^2}} - \frac{e^2}{\sqrt{r_1^2 + (z_1 - a)^2}}$$

$$- \frac{e^2}{\sqrt{r_1^2 + (z_1 + a)^2}} - \frac{e^2}{\sqrt{r_2^2 + (z_2 - a)^2}} - \frac{e^2}{\sqrt{r_2^2 + (z_2 + a)^2}}$$

Replacing $\dot{\vartheta}_1$ and $\dot{\vartheta}_2$ by the values found from

$$mr_1^2 \dot{\vartheta}_1 = \frac{h}{2\pi}, \quad mr_2^2 \dot{\vartheta}_2 = \frac{h}{2\pi},$$

and expanding in powers of the small quantities that determine the deviations from the stationary motion, we obtain:

$$P + T = \frac{e^2}{r} \left\{ \frac{3\sqrt{3}-1}{4} - 2\sqrt{3} + \frac{1}{2} \right\} + \frac{m}{2} (\dot{\varrho}_1^2 + \dot{\varrho}_2^2 + \dot{z}_1^2 + \dot{z}_2^2)$$

$$+ \frac{e^2}{r} \left[\frac{3\sqrt{3}(z_1^2 + z_2^2)}{32r^2} + \frac{\varphi^2}{16} - \frac{(z_1 - z_2)^2}{16r^2} + (21\sqrt{3} - 12) \frac{\varrho_1^2 + \varrho_2^2}{32r^2} + \frac{(\varrho_1 + \varrho_2)^2}{8r^2} \right]$$

or

¹⁾ L. FÖPPL: Ueber die Stabilität des BOHR'schen Atommodelles. Phys. Zeitschr. XV, 1914, p. 707.

$$P + T = \frac{e^2}{r} \left\{ \frac{3\sqrt{3}-1}{4} - 2\sqrt{3} + \frac{1}{2} \right\} + \frac{m}{2} (\dot{q}_1^2 + \dot{q}_2^2 + \dot{z}_1^2 + \dot{z}_2^2) + \frac{e^2}{r} \left[\frac{1,196(z_1^2 + z_2^2)}{32r^2} + \frac{(z_1 + z_2)^2}{16r^2} + \frac{q^2}{16} + 24,372 \frac{q_1^2 + q_2^2}{32r^2} + \frac{(q_1 + q_2)^2}{8r^2} \right], \quad (8)$$

After introduction of the condition for stationary motion

$$\frac{h^2}{4\pi^2} = (mr^2 \omega)^2 = m r e^2 \frac{3\sqrt{3}-1}{4}$$

the first power terms have vanished as they ought to.

Formula (8) shows that the hydrogen molecule of BOHR-DEBIJE satisfies BOHR's criterion of stability. The question remains in how far a system with a limitation of its freedom of motion consisting in the constancy of moments of momentum can execute forced vibrations under the influence of external forces which without that restriction would change those moments.

§ 6. Finally we may mention some remarks of others on the hypotheses of BOHR-DEBIJE. SOMMERFELD¹⁾ has applied to molecules of a more complicated structure the method of deduction of the dispersion formula used by DEBIJE. In his paper he says that DEBIJE has shown that the frequencies of the free vibrations of the system need not be real. Such a proof however is not to be found in DEBIJE's paper, only a remark about the good agreement between the experimental and theoretical dispersion formulae exactly on account of the terms that cannot become infinite for real frequencies of the transmitted light.

OSEEN²⁾ shows that MAXWELL's equations cannot hold for the space between the atoms and molecules, if we suppose the electrons not to radiate and therefore give up the validity of those equations for the space within the atom or molecule. However it is on their validity that DEBIJE's deduction rests.

KEESOM³⁾ has investigated the magnetic properties and has been led to the conclusion that with this constitution of the molecules hydrogen would be paramagnetic with a susceptibility only little below that of oxygen.

In conclusion, I should like to express my indebtedness to Prof. LORENTZ for his interest and many helpful suggestions.

¹⁾ A. SOMMERFELD: Die allgemeine Dispersionsformel nach dem BOHR'schen Modell: Sonderabdruck aus den Arbeiten aus den Gebieten der Physik, Mathematik und Chemie J. ELSTER u. A. GEITEL gewidmet p. 577.

²⁾ C. W. OSEEN: Das BOHR'sche Atommodell und die MAXWELL'schen Gleichungen Phys. Zeitschr. XVI, 1915, p. 395.

³⁾ W. H. KEESOM: The second virial coefficient etc. These Proc. XVIII p. 657, note 2.