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Mathematics. — "*Characteristic numbers for nets of algebraic surfaces.*" By Professor JAN DE VRIES.

(Communicated in the meeting of January 29, 1916).

§ 1. We consider a general net of surfaces Φ^n, with n^3 base-points B, represented by the equation

$$\alpha a_x^n + \beta b_x^n + \gamma c_x^n = 0.$$

For a base-point B, with coordinates y_k, is

$$a_y^n = 0, \qquad b_y^n = 0 \text{ and } c_y^n = 0.$$

By the substitution $\varrho x_k = y_k + \lambda z_k$ we find

$$\alpha\,(n\lambda\, a_y^{n-1} a_z + n_2 \lambda^2 a_y^{n-2} a_z^2 + \ldots + \beta(n\lambda b_y^{n-1} b_z + \ldots) + \gamma(n\lambda c_y^{n-1} c_z + \ldots) = 0$$

If the point Z is taken on a straight line, which has in Y four coinciding points in common with a Φ^n, we have

$$\alpha a_y^{n-1} a_z + \beta b_y^{n-1} b_z + \gamma c_y^{n-1} c_z = 0,$$

$$\alpha a_y^{n-2} a_z^2 + \beta b_y^{n-2} b_z^2 + \gamma c_y^{n-2} c_z^2 = 0,$$

$$\alpha a_y^{n-3} a_z^3 + \beta b_y^{n-3} b_z^3 + \gamma c_y^{n-3} c_z^3 = 0.$$

Eliminating α, β, γ we obtain from this the *locus of the tangents* t_4, *which have in B with a* Φ^n *a four-point contact*; it is a *cone of order six*, which will be indicated by $(t_4)^6$.

If to the three equations just considered is associated the condition

$$\alpha a_y^{n-4} a_z^4 + \beta b_y^{n-4} b_z^4 + \gamma c_y^{n-4} c_z^4 = 0$$

we have for the determination of the tangents with five-point contact in B

$$M \equiv \begin{vmatrix} a_y^{n-1} a_z & a_y^{n-2} a_z^2 & a_y^{n-3} a_z^3 & a_y^{n-4} a_z^4 \\ b_y^{n-1} b_z & b_y^{n-2} b_z^2 & b_y^{n-3} b_z^3 & b_y^{n-4} b_z^4 \\ c_y^{n-1} c_z & c_y^{n-2} c_z^2 & c_y^{n-3} c_z^3 & c_y^{n-4} c_z^4 \end{vmatrix} = 0.$$

If the fourth and the third column are respectively cancelled, the equations thus obtained represent two surfaces [1]), of the orders 6 and 7. To their intersection belong the straight lines which are obtained by equating to zero the matrix of the first two columns. The number of those straight lines apparently amounts to $3^2-2=7$.

[1]) The first surface is the cone $(t_4)^6$ already mentioned, the second has in B a sextuple point, is therefore a monoid. In order to see this, the substitution $\varrho z_k = y_k + \mu x_k$ may be done in the equation of the monoid; we then find $\mu^6 = 0$. If a vertex of the tetrahedron of coordinates is laid in B the monoid is replaced by a cone.

These straight lines do not belong to the surfaces, of which the equations arise from $M=0$ by leaving out the first or second column; therefore we have found:

Through every base-point pass 35 *tangents which have in that base-point a five-point contact.*

So the locus of the groups of $(n-4)$ points S, which every t_4 with point of contact B has moreover in common with the corresponding surface Φ^n, has a 35-fold point in B. A plane passing through B contains $6(n-4)$ points S on the generatrices of $(t_4)^6$ lying in it; the locus in question is therefore a twisted curve of order $(6n+11)$.

We can now find the number of tangents $t_{4,2}$, which have with a Φ^n in B a four-point contact and elsewhere a two-point contact. To that end we consider the correspondence between the planes, which project two points S and S', lying on the same surface Φ^n out of an arbitrary axis a. Any plane σ passing through a contains $(6n+11)$ points S, is therefore associated to $(6n+11)(n-5)$ planes σ', which each project a point S'. On a generatrix of the cone $(t_4)^6$ which is intersected by a, lie $(n-4)(n-5)$ pairs S, S'; the plane (at_4) replaces therefore $(n-4)(n-5)$ coincidences of the correspondence (σ, σ'). The remaining coincidences arise apparently from coincidences $S \equiv S'$; their number amounts to $2(6n+11)(n-5)-6(n-4)(n-5)=$ $=(6n+46)(n-5)$.

There are consequently $(6n+46)(n-5)$ *tangents* $t_{4,2}$, which have in B *a four-point contact.*

The method followed here will, for the sake of brevity, be indicated as "process (a)".

§ **2.** The straight lines t_3, which have in B three coinciding points in common with the surface

$$\alpha a_x^n + \beta b_x^n + \gamma c_x^n = 0\,,$$

are indicated by the two equations

$$\alpha a_y^{n-1} a_x + \beta b_y^{n-1} b_x + \gamma c_y^{n-1} c_x = 0\,,$$
$$\alpha a_y^{n-2} a_x^2 + \beta b_y^{n-2} b_x^2 + \gamma c_y^{n-2} c_x^2 = 0\,.$$

By elimination of α, β, γ from these equations we find, that the points Q, in which each surface Φ^n is moreover intersected by its two principal tangents t_3, are lying on a surface of order $(n+3)$. As any t_3 bears a group of $(n-3)$ points Q, the surface $(Q)^{n+3}$ has a sextuple point in B; the tangents in B apparently form the cone $(t_4)^6$.

By a plane φ containing B, $(Q)^{n+3}$ is intersected in a curve φ^{n+3}. We shall now consider the pairs of points Q, Q' lying on the straight lines t_3 and pay attention to the correspondence between the rays m, m', connecting them with an arbitrary point M of φ.

Any ray m bears $(n+3)$ points Q, is therefore associated to $(n+3)(n-4)$ rays m'. The straight line MB contains $(n-3)(n-4)$ pairs Q, Q', represents therefore $(n-3)(n-4)$ coincidences $m \equiv m'$. The remaining coincidences pass through points $Q \equiv Q'$; their number amounts to $2(n+3)(n-4) - (n-3)(n-4)$ or $(n+9)(n-4)$.

The method followed here will, for the sake of brevity, be indicated as "process (M)".

From the result found it ensues, *that the locus of the tangents* $t_{3,2}$, *which osculate in* B, *is a cone of order* $(n+9)(n-4)$.

We arrive at the same result by paying attention to the tangents of φ^{n+3}, which meet in B. Since B is a sextuple point, the number of those tangents amounts to $(n+3)(n+2) - 6.7 = n^2 + 5n - 36 = (n+9)(n-4)$.

§ **3**. On a straight line t passing through B an involution of order $(n-2)$ is determined by the surfaces Φ^n, which touch t in B, and therefore form a pencil. There are consequently $2(n-3)$ surfaces which touch t moreover in another point. On a generatrix of the cone $(t_4)^6$ that second point of contact, R_2, is united with B. The locus of the points R_2 has therefore in B a sextuple point and is a surface of order $2n$.

The surface Φ^n, which touches t in R_2, intersects that straight line moreover in a group of $(n-4)$ points S. If S coincides with B, t is one of the straight lines $t_{3,2}$ considered above, the locus of S has therefore in B an $(n+9)(n-4)$-fold point and is a surface of order $(n+9)(n-4) + 2(n-3)(n-4) = (3n+3)(n-4)$.

We now lay again through B a plane φ; it intersects the surfaces (R_2) and (S) along two curves of the orders $2n$ and $(3n+3)(n-4)$. We again apply the process (M) to the pairs of points (R_2, S) and find for the number of coincidences $R_2 \equiv S$

$$2n(n-4) + (3n+3)(n-4) - (2n-6)(n-4) \quad \text{or} \quad (3n+9)(n-4).$$

The straight lines $t_{2,3}$, *which have their point of contact* R_2 *in* B, *form therefore a cone of order* $(3n+9)(n-4)$.

If the same process is applied to the pairs of points S, S', belonging to one and the same point R, *a cone of order* $(2n+6)(n-4)(n-5)$ *is found*, of which the generatrices $t_{2,2,2}$ *have a point of contact in* B.

§ **4**. A straight line t passing through B is intersected by the net $[\Phi^n]$ in the groups of an involution of the second rank I^2_{n-1}. In each of the $3(n-3)$ triple points (R_2) t is osculated by a Φ^n;

this surface intersects t moreover in a group of $(n-4)$ points S. If a point S comes in B, t becomes a generatrix of the cone $(t_{2,3})$ found above; the surface (S) has therefore in B a $(5n+2)(n-4)$-fold point, and is of order $(3n+9)(n-4)+3(n-3)(n-4)$ or $6n(n-4)$. If R_3 comes in B, t has in that point a four-point contact, is therefore a generatrix of $(t_4)^6$; the surface (R_3) is consequently of order $(3n-3)$. Applying the process (M) to the pairs (R_3, S) we find now, that there are $(3n-3)(n-4)+6n(n-4)-(3n-9)(n-4)$ or $(6n+6)(n-4)$ coincidences $R_3 \equiv S$.

The tangents t_4, passing through B, which have their point of contact not in B, form therefore a cone of order $(6n+6)(n-4)$..

Analogously it ensues from $12n(n-4)(n-5)-3(n-3)(n-4)(n-5)=$ $=(9n+9)(n-4)(n-5)$, that *the tangents $t_{3,2}$ passing through B, which have their points of contact not in B, form a cone of order* $(9n+9)(n-4)(n-5)$.

§ **5**. As an involution I^2_{n-1} contains $2(n-3)(n-4)$ groups with two double-points, there are as many surfaces of the net, which touch the straight line t passing through B in two points R_2 and R_2' and intersect it moreover in $(n-5)$ points S.

If t is a straight line, osculating a Φ^n in B, and touching it elsewhere, it will touch the surface (R_2) of the points R_2, R_2' in B. Consequently (R_2) has in B an $(n+9)(n-4)$-fold point (§ 2), and is of order $(n+9)(n-4)+4(n-3)(n-4)$ or $(5n-3)(n-4)$.

If S comes in B, t is a tangent $t_{2,2,2}$, of which one of the points of contact lies in B. On the surface (S) therefore B is a $(2n+6)$ $(n-4)(n-5)$-fold point (§ 3); the order of S amounts therefore to $(2n+6)(n-4)(n-5)+2(n-3)(n-4)(n-5)$ or $4n(n-4)(n-5)$.

By means of the process (M) we find from the correspondence (R_2, R'_2) again the order $(6n+6)(n-4)$ of the cone of the tangents t_4, passing through B, which have not their point of contact in B (§ 4).

If applied to the correspondence (R_2, S), we find again the order $(9n+9)(n-4)(n-5)$ of the cone of the $t_{3,2}$, passing through B without touching in that point (§ 4).

We can finally apply it to the pairs (S, S') belonging to the pairs (R_2, R'_2). From $8n(n-4)(n-5)(n-6) - 2(n-3)(n-4)(n-5)$ $(n-6)=(6n+6)(n-4)(n-5)(n-6)$ we find that the *tangents $t_{2,2,2}$, passing through B, form a cone of order* $2(n+1)(n-4)(n-5)(n-6)$.

§ **6**. We found that the tangents $t_{3,2}$, which osculate in B, form a cone of order $(n+9)(n-4)$. On each of the 35 straight lines which have a five-point contact in B, the point of contact R_2 is united with the point of contact B_3. Consequently the locus of the points R_2 has in B a 35-fold point; as a plane passing through B contains $(n-9)(n-4)$ points R_2, (R_2) is a curve of order (n^2+5n-1).

Any $t_{3,2}$ intersects the surface Φ^n,' which it osculates in B, moreover in $(n-5)$ points S. If one of these points comes in B, $t_{3,2}$ passes into a $t_{4,2}$ with point of contact B_4. The number of these tangents amounts to $(6n+46)(n-5)$; the order of the curve (S) is therefore $(6n+46)(n-5)+(n+9)(n-4)(n-5)$ or $(n^2+11n+10)(n-5)$.

To the curves (R_2) and (S) we now apply the process (a). A plane ϱ passing through a intersects (R_2) in (n^2+5n-1) points R_2, is therefore associated to the $(n^2+5n-1)(n-5)$ planes σ, which project the corresponding points S out of a. To a plane σ $(n^2+11n+10)(n-5)$ planes ϱ evidently correspond. The axis a meets $(n+9)(n-4)$ generatrices of the cone $(t_{3,2})$; in the plane passing through a and such a generatix lie a point R_2 and $(n-5)$ points S associated to it, so that this plane is an $(n-5)$-fold coincidence $\varrho\equiv\sigma$. As the remaining coincidences must arise from coincidences $R_2\equiv S$, it ensues from $(n^2+5n-1)(n-5)+(n^2+11n+10)(n-5)-$ $-(n^2+5n-36)(n-5)$, that $(n^2+11n+45)(n-5)$ *twice osculating tangents* $t_{3,3}$ *have one of their points of contact in* B.

If the process (a) is applied to the pairs of points S, S' of the straight lines $t_{3,2}$, we find from $2(n^2+11n+10)(n-5)(n-6)-$ $-(n^2+5n-36)(n-5)$, that the number of the *straight lines* $t_{3,2,2}$ *osculating in* B *amounts to* $\frac{1}{2}(n^2+17n+56)(n-5)(n-6)$.

§ **7**. Let us now consider the locus (R_3) of the points R_3 on the tangents $t_{2,3}$, which have a point of contact B_2. As we found that they form a cone of order $(3n+9)(n-4)$, and (R_3) will evidently pass 35 times through B, the order of this curve is equal to $(3n^2-3n-1)$. On each tangent $t_{3,3}$ with points of contact B_3 and R_3 one of the points S is united with B; the curve (S) is consequently of order $(n^2+11n+45)(n-5)+(3n+9)(n-4)(n-5)$ or of $(4n^2+8n+$ $+9)(n-5)$.

By means of the process (a) we now find from $(3n^2-3n-1)$ $(n-5)+(4n^2+8n+9)(n-5)-(3n^2-3n-36)(n-5)$, that there *will be* $(4n^2+8n+44)(n-5)$ *straight lines* $t_{2,4}$ *with point of contact* B_2.

And from $2(4n^2+8n+9)(n-5)(n-6)-(3n+9)(n-4)(n-5)(n-6)$ it ensues that $(5n^2+19n+54)(n-5)(n-6)$ *tangents* $t_{2,2,3}$ *have a point of contact* B_2.

The straight lines $t_{2,2,2}$ with points of contact B_2, R_2 and R'_2, form a cone of order $2(n+3)(n-4)(n-5)$. The points R_2 lying on it form a curve, which passes $(6n+46)(n-5)$ times through B

(number of the $t_{4,2}$), consequently is of order $(4n^2+2n-2)(n-5)$.

The process (a) again produces now from $2(4n^2+2n-2)(n-5)$ $-4(n+3)(n-4)(n-5)$ the number of $t_{2,4}$ with point of contact B_2 found above.

The locus (S) of the points S lying on the tangents $t_{2,2,2}$, passes $\frac{1}{2}(n^2+17n+56)(n-5)(n-6)$ times through B, is therefore of order $\frac{1}{2}(5n^2+13n+8)(n-5)(n-6)$.

If the process (a) is applied to the pairs S,S', we find from $(5n^2+13n+8)(n-5)(n-6)(n-7) - 2(n+3)(n-4)(n-5)(n-6)$ $(n-7)$ that $\frac{1}{3}(3n^2+15n+32)(n-5)(n-6)(n-7)$ *tangents* $t_{2,2,2,2}$ *have a point of contact in* B.

Extension of these considerations on the cone of the tangents t_4 with point of contact R_4, which tangents pass through B, and on the corresponding loci (R_4) and (S) makes clear that through B $(10n^2-10n+55)(n-5)$ *tangents* t_5 may be drawn, *of which the point of contact* R_5 *does not lie in* B, and $(14n^2-2n+64)(n-5)(n-6)$ *tangents* $t_{4,2}$, *with points of contact* R_4, R_2.

Analogously the cone of the tangents $t_{3,2}$, passing through B produces *the number of* $(15n^2+3n+63)(n-5)(n-6)$ *of the straight lines* $t_{3,3}$ *and the number of* $\frac{1}{2}(19n^2+11n+72)(n-5)(n-6)(n-7)$ *of the straight lines* $t_{3,2,2}$ *passing through* B, having no point of contact there.

We finally find that through B $\frac{1}{3}(12n+22)(n-5)(n-6)(n-7)$ $(n-8)$ *tangents* $t_{2,2,2,2}$ may be drawn, *of which none of the points of contact lie in* B.

§ **8**. The net $[\Phi^n]$ is intersected by an arbitrary plane along a net of plane curves φ^n. Making use of the results at which I have arrived elsewhere for a similar net [1]), the order of the cone formed by the tangents t_4 coinciding in an arbitrary point P, may be determined.

Then it will be evident, that the *tangents* t_4 *form a complex of order* $6n(n-3)$.

For a base-point B the cone of the complex degenerates into the cone of order $(6n+6)(n-4)$ on whose generatrices the point of contact R_4 does *not* lie in B and the cone $(t_4)^6$ of the straight lines t_4, which have their point of contact in B; the latter is to be counted four times.

The tangents $t_{2,3}$ *form a complex of order* $9n(n-3)(n-4)$. The cone of the complex of B consists of three parts: a cone of order

1) See my communication on "Characteristic numbers for nets of algebraic curves". (Proceedings XVII, p. 935).

$9(n+1)(n-4)(n-5)$, formed by the straight lines $t_{2,3}$, on which neither point of contact in B is lying, the twice to be counted cone of order $3(n+3)(n-4)$ with B_2, and the thrice to be counted cone of order $(n+9)(n-4)$ with B_3.

The tangents $t_{2,2,2}$ *form a complex of order* $2n(n-3)(n-4)(n-5)$. Here the cone with vertex B is composed of the twice to be counted cone of order $2(n+3)(n-4)(n-5)$ on which B is one of the points of contact, and the cone of order $2(n+1)(n-4)(n-5)(n-6)$, for which this is not the case.

§ **9.** We can also determine the order and the class of the congruence, formed by the tangents with five-point contact t_5.

Any point P is point of contact of eleven four-point tangents. For the surfaces Φ^n passing through P form a pencil of which the base-curve passes through P. Let the pencil be represented by

$$\alpha a_x^n + \beta b_x^n = 0 ,$$

the point P by (y_k), then we find, analogously with § 1, that the straight lines t_4 with the point of contact P are indicated by

$$\begin{vmatrix} a_y^{n-1} a_z & a_y^{n-2} a_z^2 & a_y^{n-3} a_z^3 \\ b_y^{n-1} b_z & b_y^{n-2} b_z^2 & b_y^{n-3} b_z^3 \end{vmatrix} = 0.$$

They are obtained as intersections of a cubic cone with a monoid of order four, which have the tangent at base-curve in common; there are consequently 11 straight lines t_4 with point of contact P.

On the cone of order $6n(n-3)$, which the complex $\{t_4\}$ associates to P, the points of contact R lie consequently on a curve of order $(6n^2-18n+11)$.

Each generatrix contains moreover $(n-4)$ points S. The locus of S has in P a multiple point, the order of which is equal to the number of straight lines t_4 passing through P at surfaces Φ^n of the pencil determined by P. An arbitrary point lies on $(4n^2-6n+4)(n-3)$ straight lines t_4 of that pencil [1]). As the 11 straight lines t_4, which touch in P, are each to be counted four times, P bears $(4n^2-2n+14)(n-4)$ straight lines t_4, which have their point of contact outside P.

The above mentioned curve (S) is therefore of order $(4n^2-2n+14)(n-4)+6n(n-3)(n-4)$ or $(10n^2-20n+14)(n-4)$.

[1]) See my paper "On pencils of algebraic surfaces". (These Proceedings VIII, p. 29). The *class* of the congruence $[t_4]$ has been given wrongly there; the exact number is found in another communication (These Proceedings VIII, p. 817). The same observation holds good with regard to the class of the congruence $[t_{3,2}]$.

By means of the process (a) we find now that $(6n^2-18n+11)(n-4)+(10n^2-20n+14)(n-4)-6n(n-3)(n-4)$ straight lines pass through P, which have a five-point contact elsewhere. *The order of the congruence* $[t_5]$ *amounts therefore to* $(10n^2-20n+25)(n-4)$.

We found above (§ 7) that the base-point B lies on $(10n^2-10n+55)(n-5)$ straight lines t_5, which have their point of contact R_5 not in B, whereas there are 35 straight lines t_5, on which R_5 coincides with B. From this it ensues that each of these 35 straight lines must be counted five times.

The class of the congruence $[t_5]$ agrees with the number of those curves φ^n of a net, which possess a tangent t_5 [1]; it is therefore equal to $15(4n-5)(n-4)$.

§ **10**. Through B pass $(15n^2+3n+63)(n-5)(n-6)$ tangents $t_{3,3}$, of which none of the points of contact lies in B, and $(n^2+11n+45)(n-5)$ straight lines $t_{3,3}$, which osculate in B. As they must be counted thrice, we find for the *order of the congruence* $[t_{3,3}]$ the number of $(15n^3-84n^2+78n-243)(n-5)$. *The class amounts to* [2] $\frac{9}{2}(n^2+7n-9)(n-4)(n-5)$.

In a similar way we find, that the *congruence* $[t_{4,2}]$ is of *order* $(14n^3-78n^2+116n-112)(n-5)$ and of *class* $6(n^2+11n-14)(n-4)(n-5)$ [2], the *congruence* $[t_{2,2,3}]$ of *order* $\frac{1}{2}(19n^3-99n^2+122n-120)(n-5)(n-6)$ and of *class* $\frac{3}{2}(5n^2+23n-30)(n-4)(n-5)(n-6)$ [3], the *congruence* $[t_{2,2,2,2}]$ of *order* $\frac{1}{3}(18n^2-44n-112)(n-5)(n-6)(n-7)$ and of *class* $(n-1)(n+4)(n-4)(n-5)(n-6)(n-7)$ [3].

Mathematics. — "*Tangential curves of a pencil of rational cubics*". By Professor JAN DE VRIES.

(Communicated in the meeting of January 29, 1916).

§ **1.** We consider a pencil (φ^3) of rational cubics which all have a node in A; each of the remaining base-points will be indicated by B or by C.

The tangent b in B at φ^3 intersects the latter in the tangential point B'; the locus τ of the points B' is called *tangential curve* of B. It is generated by the projective pencils (φ^3) and (b).

One of the figures of (φ^3) consists of the straight line AB and

[1]) See the communication quoted above in volume XVII of these Proceedings (p. 938).

[2]) L.c. p. 942.

[3]) L.c. p. 941.