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Mathematics. — *"Circles cutting a plane curve perpendicularly."* I. By Prof. Hk. de Vries.

(Communicated in the meeting of January 29, 1916).

In the "Proceedings of the Royal Academy of Sciences at Amsterdam", section I, volume VIII, N°. 7, 1904, the present writer published a paper, entitled: "Anwendung der Cyklographie auf die Lehre von den ebenen Curven"[1]), in which the circles are investigated cyclographically, which either touch *one* or more plane curves *once* or several times or osculate them.

At the end of that paper the observation is made that by means of a slight alteration in the plan, the circles may also be investigated that cut *one* or more plane curves *once* or several times perpendicularly; the aim of the following paper is to carry out that investigation.

§ 1. As before we start from a plane curve k^{μ} of order μ, class ν, with δ nodes, $\varkappa$ cusps, τ bitangents, ι stationary tangents, and which moreover passes ε-times through each of the two absolute points at infinity, and σ times touches the straight line at infinity of its plane. In an arbitrary point P of the curve we think the tangent t to be drawn, and consider it as the locus of the centra of all the circles cutting the curve perpendicularly in P; if we then bring through t the vertical plane (the plane β of the curve itself, the base, being supposed horizontal), and if we draw in it through P the two straight lines enclosing with t angles of 45°, the cyclographic image circles of the points of those two straight lines are exactly the above mentioned circles cutting the curve k^{μ} perpendicularly.

If we call the two 45°-lines b, and if we repeat the construction indicated for all the points and tangents of the curve, all the straight lines b are the generatrices of a non-developable ruled surface Ω, non-developable, because the two systems of circles cutting the curve perpendicularly in two infinitely near points, have no circle in common. That Ω is symmetrical in regard to the plane of the curve is to be seen at once, while no more proof need be given that the cone of direction is a cone of revolution with vertical axis, and whose generatrices with that axis enclose angles of 45°. This cone cuts the plane at infinity of space along a conic k^2_{∞}, which touches the absolute circle in the two absolute points $I_{1\infty}$, $I_{2\infty}$ of the plane β; for the point at infinity Z_{∞} of the axis of the cone of direction is the pole of the straight line at infinity l_{∞} of the plane of k^{μ}, as

[1]) Henceforth we shall quote this paper for the sake of brevity as *"Anw. Cykl."*

well with regard to k^2_∞ as to the absolute circle, and if both the cone of direction and the isotropic cone are considered for a point of this plane as vertex, they have the two isotropic rays in that plane and passing through that vertex in common, so that $I_{1\infty}$, $I_{2\infty}$, lie on k^2_∞ as well as on the absolute circle.

Consequently the surface Ω may be imagined to have arisen more intuitively in the following way.

Let the tangent t be drawn in a point P of k^μ, and the point at infinity T_∞ of it be connected with Z_∞; the connecting line cuts k^2_∞ in two points $K_{1\infty}$ and $K_{2\infty}$, and if these points are connected with P, the two generatrices b_1, b_2, passing through P have been found.

From this construction the order of Ω ensues at once and that in two ways, if we suppose for the moment that the above mentioned numbers $\delta, \varkappa, \iota, \tau, \varepsilon, \sigma$ are all zero. For, in the first place, the complete intersection of Ω with the plane of k^μ is easy to indicate, it consists of k^μ itself, counted twice, as k^μ is evidently a nodal curve of Ω, and further only of such 45°-lines with regard to this plane as may be esteemed to lie at the same time in this plane, i.e. isotropic straight lines. Through each of the two isotropic points $I_{1\infty}$, $I_{2\infty}$ of β pass $\mu(\mu-1)$ tangents of this curve, and the plane passing through such a tangent and Z_∞ touches k^2_∞ (as Z_∞ is the pole of l_∞ with regard to k^2_∞), and consequently contains of Ω two coinciding generatrices, or rather only one generatrix, which in this plane itself, however, counts for 2, in any other plane passing through that line, as for instance β, for *one*; *the order of* Ω *is therefore*

$$m = 2\mu + 2\mu(\mu-1) = 2\mu^2 \text{ or } = 2\mu + 2\nu.$$

We may, however, also easily determine the intersection of Ω with the plane at infinity of space. If we suppose an arbitrary point K_∞ of k^2_∞ connected with Z_∞, $\mu(\mu-1)$ tangent planes of k^μ will then pass through the connecting line; the lines connecting the points of contact with K_∞ are the generatrices of Ω passing through this point; k^2_∞ *is therefore for* Ω *a* $\mu(\mu-1)$- *or* ν-*fold curve.*

But k^μ possesses further μ points at infinity, whose tangents meet l_∞ in those points themselves; the lines connecting those points with Z_∞ are therefore edges of Ω and that nodal edges, because they cut k^2_∞ in two points. Ω *therefore contains at infinity* μ *nodal edges, viz. the lines connecting* Z_∞ *with the points at infinity of* k^μ, and from this it ensues that we again find for order $m: 2\mu(\mu-1) + 2\mu = 2\mu^2$. *At the same time we observe that* Z_∞ *as intersection of* μ *nodal edges is a* 2μ-*fold point of* Ω.

§ 2. We will now investigate the influence which have the singularities $\delta, \varkappa, \iota, \tau, \varepsilon, \sigma$ provisionally supposed equal to zero in the preceding §; that it is necessary to consider them follows among others from this that already in the two simplest cases imaginable, viz. if k^μ is a straight line or a circle, the number $2\mu^2$ appears to be incorrect for the order of Ω; for the straight line, Ω is evidently the vertical plane passing through that line, so $m = 1$, and for the circle Ω is, as is known, the hyperboloid of revolution of one sheet with that circle as gorge, so $m = 2$, whereas $2\mu^2$ would give 2 and respectively 8. The differences are easy to explain in either case. The plane is apparently to be counted twice, as through each of its points two 45°-lines pass; for the hyperboloid of revolution the same holds good, but there the circle passes moreover through the two points $I_{1\infty}, I_{2\infty}$, so that $\varepsilon = 1$, and consequently the influence of ε must be investigated.

Let us now suppose that a tangent t has been drawn out of $I_{1\infty}$ to k^μ, we then have to connect the point of contact P with $I_{1\infty}$ according to § 1; if, however, k^μ itself passes through $I_{1\infty}$, and if t is the tangent in this point, then the line $PI_{1\infty}$ becomes indefinite *in the plane* passing through t and Z_∞, so that the pencil with vertex $I_{1\infty}$ lying in this plane branches off, and that twice, as the tangent t represents two coinciding tangents of k^μ; every time therefore when k^μ passes through one of the cyclic points, a pencil, counted twice, branches off from Ω. In our example mentioned above, we found $\varepsilon = 1$, consequently two planes, each counted twice, branch off from Ω; the order of the complete surface was 8, and is therefore reduced to 4, as the twice to be counted hyperboloid of revolution requires.

Besides ε the number σ (the number of times that k^μ touches the straight line l_∞ of β) is also of influence on the order of the "true" surface Ω, as easily appears from the following consideration.

According to § 1 the base β can contain beside the nodal curve k^μ only isotropic generatrices of Ω; through $I_{1\infty}$ pass only $\nu-2\varepsilon-\sigma$ tangents, not having their point of contact on l_∞, so only $2(\nu-2\varepsilon-\sigma)$ isotropic generatrices lie in β; if we are therefore able to prove that l_∞ itself does not belong to the "true" surface, *it is proved by this that the order of Ω is:*

$$m = 2(\mu + \nu - 2\varepsilon - \sigma).$$

As to l_∞ we may observe the following. We have to intersect each tangent t of k^μ with l_∞, to connect the intersection with Z_∞ and to connect the two intersections of this connecting line and

k^2_∞ with the point of contact of t; if t now coincides with l_∞, the point of contact remains definite, the intersection with l_∞ does not, and so we can connect the point of contact with any point of k^2_∞ in order to find always a straight line, which does belong to the "true" surface Ω; this is the reason why l_∞ not belongs to Ω either. To Ω does belong, however, the line connecting the point of contact of l_∞ and k^μ with Z_∞, as is easily to be seen if the tangent t is made to approach to l_∞. At the same time we are then convinced that at the limit *two* generatrices coincide in this line, according to its two intersections with k^2_∞, so that it is a double generatrix; but we should moreover consider that even as a double generatrix it is to be taken twice, as k^μ has in common with l_∞ two infinitely near points, and for one point the same obtains that obtains for the other; we may say that it is a double torsal line, whereas the tangent-plane coincides both times with ε_∞.

This becomes still more evident if we just consider an ordinary intersection S_∞ of k^μ with l_∞. By causing a point P of k^μ to approach to S_∞ we are at once convinced that $S_\infty Z_\infty$ is a double generatrix of Ω, and again a double torsal line, with a tangent-plane, however, that contains the tangent S_∞ at k^μ; if now two points S get to lie infinitely near, two double generatrices get to lie infinitely near.

These considerations enable us moreover to control the order m of Ω arrived at above by means of the plane at infinity of space. The intersection of this plane with Ω consists viz. of the following parts:

a. the 2σ double generatrices lying in pairs infinitely near, arising from the σ points of contact of k^μ with l_∞;

b. the $\mu - 2\varepsilon - 2\sigma$ double generatrices arising from the simple intersections of k^μ with l_∞;

c. the conic k^2_∞. This is a $(\nu - \sigma)$-fold curve of the surface, for if an arbitrary point K_∞ of k^2_∞ is connected with Z_∞, and the connecting line is made to intersect l_∞, there pass through the intersection $\nu - \sigma$ tangents at k^μ, whose point of contact does not lie at infinity, consequently pass through K_∞ $\nu - \sigma$ generatrices of Ω.

By means of the plane at infinity of space we find therefore for the order of Ω:

$$m = 4\,\sigma + 2\,(\mu - 2\,\varepsilon - 2\,\sigma) + 2\,(\nu - \sigma) = 2\,(\mu + \nu - 2\varepsilon - \sigma).$$

As the points $I_{1\infty}$, $I_{2\infty}$ lie on k^μ, even as ε-fold points, it might be expected that the two straight lines $Z_\infty I_{1\infty}$, $Z_\infty I_{2\infty}$ lay also on Ω; this, however, is not so, and that because these lines touch k^2_∞ instead of cutting it. If along one of the ε branches of k^μ passing through

$I_{1\infty}$ a point P with tangent t is made to approach to $I_{1\infty}$ direct contemplation teaches that as the intersections with k^2_∞ of the line connecting Z_∞ with the point at infinity of t, simultaneously with P approach to $I_{1\infty}$, the two generatrices passing through P approach to limit positions not coinciding with $I_{1\infty}Z_\infty$. Through $I_{1\infty}$, $I_{2\infty}$ pass therefore every time ν—2ε—σ generatrices lying in β, and 2ε others not lying in β but neither passing through Z_∞; together therefore ν—σ, as well as through any other point of k^2_∞.

Of the μ—2ε nodal edges passing through Z_∞ μ—2ε—2σ lie isolated, while the 2σ remaining ones coincide in pairs; this influences the multiplicity of the point Z_∞, considered as a point of the surface. As namely in general through a point where two nodal edges, or more generally two nodal lines meet, four sheets of the surface pass, and this point consequently becomes a quadruple point for the surface, there pass through the intersection of two infinitely near nodal edges only two sheets, viz. simply those two that touch along those edges; the conseqence of this is that an arbitrary straight line passing through Z_∞ does not cut the surface there in $2(\mu$—$2\varepsilon)$, but only in $2(\mu - 2\varepsilon - 2\sigma) + 2\sigma = 2(\mu - 2\varepsilon - \sigma)$ points, *so that* Z_∞ *is for our surface a* $2(\mu$—2ε—$\sigma)$*-fold point. The* σ *pairs of coinciding nodal edges are torsal lines of* Ω, *and they are to be counted twice, because two sheets of* Ω *touch each other along each of them.*

As the order of Ω is equal to $2\mu + 2\nu - 4\varepsilon - 2\sigma$, and a straight line passing through Z_∞ has, in this point only, already 2μ—4ε—2σ points in common with Ω, only a number 2ν remains for the intersections not lying in this point; they lie in pairs symmetrically in regard to the plane of k^μ, and are represented by the ν-circles, which may evidently be described round the foot point of the straight line as centre in such a way that they intersect k^μ perpendicularly.

At first sight it is somewhat striking that the order of the non-developable surface which is under present observation, corresponds exactly with that of the developable surface in the treatise quoted in the Introduction, which surface we defined at the time as the common circumscribed developable surface of k^μ and k^2_∞; the peculiarity of this phenomenon disappears, however, if we observe that we might have constructed this developable surface as well by applying the construction which we now apply to the *tangents* of k^μ, to the normals of k^μ, which we have not done, however, as in that way its character as a *developable* surface becomes less prominent.

§ 3. In § 1 we found that k^μ is a nodal curve for Ω, and we will now investigate how the two sheets of the surface passing through this nodal curve cut each other. Through a point P of k^{μ} pass two edges b_1, b_2, one lying on one sheet, the other on the other; the tangent plane in P at one sheet contains therefore b_1, and the tangent t in P at k^{μ}; and the tangent plane of the other the lines b_2 and t. Now, however, b_1, b_2 and t lie in one plane; *in each point of k^{μ} the two sheets have consequently the same tangent plane.* More may be said, however, viz. *that the two sheets osculate each other along the whole curve k^{μ}.* Let us namely suppose the normal plane of k^{μ} brought in P, and this plane intersected with Ω, we shall then see two curves having the same vertical tangent in P and being each other's reflected image with regard to the normal n of k^{μ} lying in the base β. The circle of curvature in P of *one* curve has its centre on n, but as this circle is its own reflected image, it is at the same time circle of curvature of the other curve, from which it ensues that both curves osculate in P. And it may be further observed, as to the situation of the two sheets osculating along k^{μ} that, at least in the neighbourhood of k^{μ}, both must lie on the convex side of the cylinder which projects k^{μ} out of point Z_{∞}.

In a node D of k^{μ} meet 4 sheets of Ω, intersecting each other in pairs in 6 branches of the complete nodal curve of Ω; two of them belong, however, to k^{μ}, *so that* 4 *remain belonging to the rest nodal curve, which are in pairs each other's reflected image with regard to β and have all in D the same (vertical) tangent.* As a twisted curve that has a vertical tangent in D projects itself on β as a plane curve with a cusp in D, and the 4 branches of the nodal curve lie in pairs symmetrically with regard to β, *the projection of the rest nodal curve on β in D will show* 2 *cusps, both lying in that part of the plane from which the convex side of the two branches of k^{μ} is to be seen.* Each of the 4 branches of the projection of the rest-nodal curve, meeting in D, is locus of points from where two equally long tangents may be drawn at k^{μ}, and these tangents always touch at *both* branches, not at *one* and the same branch (from which the number 4 of the branches may be easily deduced); if namely two equally long tangents are to touch at *the same* branch, the two sheets of Ω, which pass through that branch, and which as we saw above osculate each other along that branch, must have another intersection in common, and this is only the case, as we shall see, in the neighbourhood of the so-called *vertices* of k^{μ}, and these vertices are generally not situated in the immediate neighbourhood of the nodes.

Let us now investigate the influence of the cusps of k^{μ}. A cusp K causes in Ω 2 cuspidal edges, *one* for each sheet, and lying in the vertical plane passing through the cuspidal tangent, and of course at angles of 45° with regard to β; the acute edges point both to the same side as the acute point of K. In order to discover now the conduct of the rest nodal curve of Ω in the neighbourhood of K, we just replace the cusp by a node D with a little loop and then intersect Ω with a plane lying in the neighbourhood of D (but not on the side of the loop), and for convenience, sake thought vertical. Let us suppose k'' in the environment of D exactly drawn, the intersection of Ω with the plane is also easily and sufficiently exactly to be constructed; two branches 1 and 1* are found lying symmetrically with regard to β, and also two others, 2 and 2*. The branches 1 and 2 intersect each other in 2 points, 1* and 2* in those symmetrical with regard to β, and when the plane of intersection is moved these four points describe two with regard to β symmetrical branches of the nodal curve, which project themselves on β in *one* curve with a cusp in D, as has been explained above. And the same holds good with regard to the branches 1 and 2*, and 1* and 2 respectively.

If, however, the node passes into a cusp, the branches 1 and 2 join (and 1* and 2* symmetrically) into a cusp, lying on *one* of the two 45° lines passing through K, mentioned above, whereas the second intersection remains arbitrary; by removal of the plane of intersection in the direction of K, one intersection describes the 45°-line, however, no farther than K, the other a curve ending in K, and that, as a simple investigation will teach, with an arbitrary inclination with regard to β; the continuous curve passing through D, which had a vertical tangent in D, has therefore passed into a curve showing a break in K, and composed of a true curve and a piece of a 45-line. And the branches 1*, 2*, produce, it is true, of that curve the image, but as the tangent in K, as we shall see, is generally speaking not vertical, a break remains in existence here as well.

As, however, Ω is algebraic, every discontinuity is seemingly done away with again, and this happens here whereas the curve with vertical tangent in the node passes, in the case of the cusp into a curve with a node in K, and of which two branches, which are each other's image with regard to β, are active, the two others parasitic.

Let us take as a simple example the curve $y^2 = x^3$, which has the advantage of possessing an axis of symmetry, so that one of the branches of the nodal curve passing through K (or more exactly two) gets to be situated in the plane of symmetry of Ω. By means of differentiation we find $2yp = 3x^2$, so that the tangent becomes:

$$Y - y = \frac{3x^2}{2y}(X - x);$$

the latter cuts the x-axis in the point $X = \frac{x}{3}$. The length of the tangent between the point of contact and the intersection with the x-axis becomes therefore $\sqrt{\frac{4}{9}x^2 + y^2}$, or $\sqrt{\frac{4}{9}x^2 + x^3}$, and if we now take this length as z-coordinate, and call it ζ, and put thus:

$$\xi = \frac{x}{3} \quad , \quad \zeta = \sqrt{\frac{4}{9}x^2 + x^3},$$

the point (ξ, ζ) is a point of the nodal curve. The equation of this curve becomes therefore:

$$\zeta^2 = 4\xi^2 + 27\xi^3,$$

and this curve has apparently a node in O, while the nodal tangents enclose with β an angle whose tangent is determined by

$$\lim_{\xi=0} \frac{\zeta}{\xi} = \pm 2.$$

Besides in O it cuts the x-axis moreover in the point $\xi = -\frac{4}{27}$; it consists therefore of two infinite branches and a knot, and now the knot is parasitic; the circles representing the points of this knot cyclographically are of course real indeed, but they do not cut the curve $y^2 = x^3$ really, at least not really orthogonally.

§ 4. The point $\xi = -\frac{4}{27}$ has its meaning too, for this simple example as well as in the general case; we will just illustrate it therefore. If the tangent

$$Y - y = \frac{3x^2}{2y}(X - x)$$

will become isotropic, $\frac{3x^2}{2y}$ must be $= i$, consequently $3x^2 = 2yi$. From this equation and $y^2 = x^3$ we find $x = -\frac{4}{9}$, $y = \frac{8}{27i}$, and if the tangent in this point is intersected with the x-axis, we find $x = -\frac{4}{27}$; the point $x = -\frac{4}{27}$ is therefore a focus of $y^2 = x^3$, and the parasitic knot of the nodal curve extends between the cusp and the focus.

From this simple example we may now draw important conclusions for the general case. Even then the sheets 1 and 2 cut each other on one side of β in a 45°-line, on the other in a curve, and the latter is completed by its image and a parasitic part into a curve with a node in K. With the sheets 1 and 2^* it is in so far different that they cut each other both above and below β in branches of curves, both completed again by parasitic parts into a curve with a node in K, and finally the sheets 1^* and 2 of this last curve produce moreover the image. *Apart from the two cuspidal edges* (45°-*lines*) *therefore, the restnodal curve of* Ω *possesses* 3 *nodes in each cusp of* k_p; and as of the three curves in question here one is its own image, whereas the two others are each other's image, *the projection of the restnodal curve in the neighbourhood of K will consist of* 3 *branches which all touch at the cuspidal tangent.* This may again be easily perceived planimetrically. The projection of the two cuspidal edges is the cuspidal tangent of K, and the latter is the locus of the centres of all the circles which cut the two branches of k^p meeting in K perpendicularly in this point. The two branches of the rest-nodal curve, which stereometrically belong according to the considerations put down in the preceding §, to the cuspidal edges, and complete them into curves with a break in them, project themselves into a branch containing all the points out of which two equally long tangents at k^p pass which are both turned away from K; the two other branches contain the points out of which *one* tangent of K is turned away from, the other turned towards K.

Of the two 45°-lines passing through K we found in the preceding § so to say every time only *one* half, but the other halves have their signification too. Let us viz. to that purpose consider a node D with a small knot while the nodal tangents almost coincide already. If we follow this small knot *from* the node *to* the node, we see the circle of curvature decrease at first, but afterwards increase; it has been a minimum in *one* point, and this point is for k^p a vertex, that is to say a point where the circle of curvature touches in 4 points; and it is easy to see now that from this point a branch of the projection of the rest-nodal curve must start; for, if we suppose the 4 points which the circle of curvature has in common with k^p, infinitely near, and then call them 1, 2, 3, 4, there pass through the intersection of the lines 12 and 34 two tangents at k^p, which at the same time touch the circle of curvature, and are therefore equally long. *The two sheets of* Ω *near a vertex of* k^p *cut each other consequently along a with regard to* β

symmetrical curve, which in the vertex itself has a vertical tangent.

And it will be clear now without further demonstration that if the node D passes into a cusp K, the vertex of the small knot gets to lie in K, and the new branch of the rest-nodal curve just found passes into the two halves not yet accounted for of the 45°-lines passing through K.

§ 5. The points of inflexion of k^μ, as is easy to understand, are not directly connected with the rest-nodal curve. The vertical plane passing through an inflexional tangent contains two systems of generatrices, mutually parallel and with regard to β symmetrical, lying infinitely near and they are evidently torsal-lines of Ω, but they are in no way connected with the nodal curve; on the other hand there are in β two groups of points that do belong to the rest-nodal curve, and which we have not yet discussed in the preceding §. According to § 2 there pass through each of the two absolute points at infinity $\nu-2\varepsilon-\sigma$ tangents at k^μ and each of them meets k^μ except in the point of contact and the cyclic point in question, moreover in $\mu-\varepsilon-2$ other points; through the point of contact passes no other generatrix but the isotropic tangent itself, counted twice, so that this point does not belong to the rest-nodal curve (it is a pinch-point of Ω, of course an imaginary one, and along the isotropic tangent two sheets of the surface pass into each other); in each of the $\mu-\varepsilon-2$ other points, however, the sheet, to which that isotropic tangent belongs, cuts the two sheets which pass already through those other points, so that two branches of the rest-nodal curve appear, which in such a point pass through β with a vertical tangent; so we find the following result. *In each of the* $2(\mu-\varepsilon-2)(\nu-2\varepsilon-\sigma)$ *points which the tangents out of the two isotropic points of β have, besides these points and the points of contact, moreover in common with k^μ, passes the rest-nodal curve with two branches through β, which branches osculate each other along a vertical tangent.* These points are of course all imaginary.

But we have further to consider the points in which the $\nu-2\varepsilon-\sigma$ tangents out of the isotropic point $I_{1\infty}$ cut the tangents out of $I_{2\infty}$; these points amount to $(\nu-2\varepsilon-\sigma)^2$, and among them are $\nu-2\varepsilon-\sigma$ real ones; they are the so-called foci of k^μ [1]). Through each of these points pass two single sheets of the surface, and consequently passes *one* single branch of the rest-nodal curve; so we find: *the* $(\nu-2\varepsilon-\sigma)^2$ *foci of k^μ are single intersections of the rest-nodal curve with β.*

1) Cf. Ann. Cykl p. 25.

As to the $\nu-2\varepsilon-\sigma$ *real* foci a peculiar phenomenon is to be observed here; through these points passes, as we have seen, *one* branch of the rest-nodal curve, and the tangents at those points are vertical, consequently real, so that not only the foci themselves, but also the points infinitely near to them, therefore whole branches passing through those points, must be real, and consequently must have real projections on β. Now it is a matter of course (think for instance of the conics) that neither the foci themselves, nor neighbouring points may be centres of circles cutting k^{p} twice really, so that the branches of the nodal curve passing through the real foci are parasitic branches of the nodal curve, and there is nothing particular in this after all, for parasitic branches of the nodal curve separated from the "active" parts by pinch-points, are met with already in the simplest ruled surfaces, as the wedge of WALLIS, the cubic ruled surfaces, the surface of normals, etc.; the peculiarity in our case is that the pinch-points are lying at infinity, and so the branches of the nodal curve passing through the foci nowhere reach the surface in fact.

That this is correct indeed is easy to control on the parabola and the ellipse. For the parabola $y^2 = 2px$ the tangent is $y'y = p(x' + x)$, and consequently the abscissa of the intersection with the x-axis: $x = -x'$, while the distance from this point to the point of contact amounts to: $\sqrt{4x'^2 + y'^2}$, or $\sqrt{4x'^2 + 2px'}$; if this distance is extended vertically upwards and downwards in the intersection of the tangent with the x-axis, 2 points of the nodal curve are evidently found, so that the equation of that curve becomes:

$$z^2 = 4x^2 - 2px.$$

If the origin is removed along the axis of the parabola over a distance of $\frac{1}{4}p$, so that it gets to lie half way between the vertex and the focus, and x' becomes $= x - \frac{1}{4}p$, the equation becomes:

$$4x'^2 - z^2 = \frac{1}{4}p^2,$$

and this is an hyperbola cutting the plane β in the points $x' = \pm\frac{1}{4}p$, i. e. in the vertex and the focus of the parabola. But it is evident that only the branch passing through the vertex is really lying on the surface, whereas the one passing through the focus is parasitic as far as it extends.

Of further importance is the observation that the directions of the asymptotes of the hyperbola are determined by the relation $\frac{z}{x'} = \pm 2$, so that half the asymptotic angle is greater than 45°; if therefore a point moves along the curve towards infinity, the associated circle cutting the parabola twice perpendicularly does not only become greater and greater, but it removes farther and farther from view and disappears at infinity, which proves that the parabola does not possess double normals.

It is different with the ellipse. Here a calculation, as simple as the one just performed produces as equation of the nodal curve in the xz-plane:

$$x^2z^2 = (x^2 - a^2)(x^2 - c^2),$$

a curve of order 4 consequently, cutting the plane β in the vertices $x = \pm a$, and in the foci $x = \pm c$, and being real for $|x| < c$, and $|x| > a$, while the points at infinity must be determined out of the relation $x^2z^2 = x^4$, so $x = 0$ twice. and $z^2 = x^2$.

The two branches passing through the vertices of the ellipse are much like an equilateral hyperbola and form the active part, whereas the points at infinity are represented by the minor axis, in fact therefore by a double normal; the branches passing through the foci on the contrary, which in the finite are in no way connected with the surface and are parasitic as far as they extend, approach the z-axis on both sides asymptotically, and have both a point of inflection in Z_∞, as follows immediately from the symmetry with regard to β. In the vertical plane passing through the *minor* axis of the ellipse lies of course as well a nodal curve of order 4 of which, however, only the hyperbolical branches are real.

§ 6. According to the two preceding sections the intersections of the rest-nodal curve with β consist of the following groups;

a. the δ nodes of k^μ; through each of them pass 4 branches;

b. the $\varkappa$ cusps of k^ν, through each of them pass 6 branches,

c. the $5\mu - 3\nu + 3\iota - 8\varepsilon - 3\sigma$ vertices of k^μ [1]); through each of them passes *one* branch;

d. the $2(\mu-\varepsilon-2)(\nu-2\varepsilon-\sigma)$ points, in which the $\nu-2\varepsilon-\sigma$ tangents at k^μ out of each of the two isotropic points cut the curve; through each of them pass 2 branches;

e. the $(\nu-2\varepsilon-\sigma)^2$ foci of k^μ; through each of them passes *one* branch.

The order of the rest-nodal curve of Ω is therefore:

$d = 4\delta + 6\varkappa + (5\mu-3\nu+3\iota-8\varepsilon-3\sigma) + 4(\mu-\varepsilon-2)(\nu-2\varepsilon-\sigma) + (\nu-2\varepsilon-\sigma)^2.$

For the parabola we find from this $5\mu-3\nu-3\sigma+(\nu-\sigma)^2 = 10-6-3+1=2$, for the other conics $5\mu-3\nu+\nu^2=10-6+4=8$ (which is evidently correct according to what precedes), and for the

1) Anw. Cykl. p. 19.

circle $5\mu - 3\nu - 8\varepsilon = 10-6-8 = -4$, which bears out that the formula may not be applied to the circle. In fact Ω consists in this case of a twice to be counted equilateral hyperboloid of revolution (cf. § 2), and the nodal curve is consequently indefinite. A certain control on the general case we find in the circumstance that the order of the rest-nodal curve must be even, as it is, just as the surface on which it lies, symmetrical with regard to β, and must therefore be cut by a vertical plane in an even number of points. It is true, such a plane contains the point Z_∞, which is its own image with regard to β; it will, however, appear that the multiplicity of Z_∞ is indicated by an even number, and as the *finite* intersections on account of their symmetry are also present in an even number, the complete ordernumber must be even. This now may be proved indeed.

According to the formulae of PLUCKER we have:

$$\varkappa = \iota + 3(\mu - \nu)$$

$$\iota = 3\mu(\mu-2) - 6\delta - 6\varepsilon(\varepsilon-1) - 8\varkappa \text{ [1]}, \text{ so}$$

$$\iota = 3\mu^2 - 6\mu - 6\delta - 6\varepsilon(\varepsilon-1) - 8\iota - 24\mu + 24\nu, \text{ or}$$

$$6\delta = 3\mu^2 - 30\mu - 6\varepsilon^2 + 6\varepsilon - 9\iota + 24\nu, \text{ and consequently}$$

$$4\delta = 2\mu^2 - 20\mu - 4\varepsilon^2 + 4\varepsilon - 6\iota + 16\nu;$$

if these values are substituted, we find for the order of the rest-nodal curve:

$$2\mu^2 + 4\mu\nu - 8\mu\varepsilon - 4\mu\sigma - 8\nu\varepsilon + 8\varepsilon^2 + 8\varepsilon\sigma - 13\nu + 12\varepsilon + 5\sigma + 3\mu + \\ + \nu^2 + \sigma^2 - 2\nu\sigma + 3\iota;$$ even must therefore be:

$$3\mu + 3\iota + \nu^2 - 13\nu + \sigma^2 + 5\sigma, \text{ or}$$

$$3(\mu + \iota) + \nu(\nu - 13) + \sigma(\sigma + 5).$$

It stands to reason that $\nu(\nu-13)$ and $\sigma(\sigma+5)$ are even, and further is

$$\iota + \mu = 3\mu^2 - 5\mu - 6\delta - 6\varepsilon(\varepsilon-1) - 8\varkappa;$$

$3\mu^2 - 5\mu$, or $\mu(3\mu-5)$ is, however, always even again, so d is after all always even.

The multiplicity of Z_∞ as point of the rest-nodal curve we find as follows. According to § 2 there pass through Z_∞:

1. $\mu - 2\varepsilon - 2\sigma$ nodal edges (torsal lines) of Ω, arising from the single intersections of k^μ with l_∞;

2. 2σ nodal edges, lying in pairs infinitely near (also torsal lines) arising from the σ points of contact of k^μ with l_∞.

Through the edges of the first group pass 2 sheets of Ω, touching each other along the whole of that edge, while the common

[1] Anw. Cykl. p. 10.

tangent plane contains the associated asymptote of k^μ; and two of those edges give therefore rise to 4 branches of the nodal curve, which cut ε_∞ in Z_∞ singly; the total number of these branches amounts therefore to:

$$\tfrac{1}{2}(\mu-2\varepsilon-2\sigma)(\mu-2\varepsilon-2\sigma-1)\,.\,4.$$

Through the edges of the second group coinciding in pairs pass 2 sheets, which we can approximately realize if we suppose that two cylinders of revolution of which one lies inside the other rest on a table with the same edge. Let us suppose *two* pairs of such cylinders; each cylinder of one group cuts each of the other group along a curve with a node, because they have the same tangent plane; both the cylinders of one group and both of the other give rise to 4 curves of intersection, each with a node, i. e. the sheets of Ω passing through the edges of the second group, give rise, for each pair of these edges, to 8 branches of the nodal curve that each touch ε_∞ in Z_∞. The total number of these branches amounts therefore to

$$\tfrac{1}{2}\sigma(\sigma-1)\,.\,8.$$

Finally each sheet passing through an edge of the first group cuts the two sheets passing through an edge of the second according to 2 branches which both touch ε_∞ in Z_∞, as, however, 2 sheets pass through an edge of the first group, each edge of the first group gives with each pair of coinciding edges of the second rise to 4 branches, which each touch ε_∞ in Z_∞; in total therefore

$$(\mu-2\varepsilon-2\sigma)\,.\,\sigma\,.\,4.$$

If the three amounts found here are added up, we find *that* Z_∞ *is for the rest-nodal curve of* Ω *a* $(2\mu^2-8\mu\varepsilon-4\mu\sigma-2\mu+8\varepsilon^2+8\varepsilon\sigma+4\varepsilon+4\sigma^2)$-*fold point.*

And from this it is in fact to be seen at once that the multiplicity of Z_∞ for the nodal curve is indicated by an even number, of which we have already made use higher up.

For the general conic we find from this $2.2^2-2.2=4$, for the parabola $2.2^2-4.2-2.2+4=0$, which agrees with the results of § 5.

If the order of the rest-nodal curve is diminished with the multiplicity of Z_∞, we find the number of points that an arbitrary vertical plane outside Z_∞ has moreover in common with that curve; these points are symmetrical in pairs with regard to β, so that *half* of the number in question indicates the order of the projection of the rest-nodal curve out of Z_∞ as centre on β, and this projection is evidently the locus of the points that are centres of circles cutting

k^4 twice perpendicularly, the locus therefore of the points out of which two equally long tangents may be drawn at k^4. If the calculation is carried out, we find:

The locus of the points out of which two equally long tangents may be drawn at k^4 is a curve of order:

$$d^* = \tfrac{1}{2}(4\mu\nu + \nu^2 + 5\mu - 13\nu + 3\iota - 8\nu\varepsilon - 2\nu\sigma - 3\sigma^2 + 8\varepsilon + 5\sigma).$$

And according to the preceding observations this curve has in each node of k^4 2 cusps, while through each cusp of k^4 pass 3 branches, which all three touch at the cuspidal tangent. Through each vertex of k^4 and through each focus the curve passes once.

For the hyperbola and the ellipse we find;

$d^* = \frac{1}{2}(4.2.2 + 2^2 + 5.2 - 13.2) = 2$, viz. the two axes, for the parabola: $\frac{1}{2}(4.2.2 + 2^2 + 5.2 - 13.2 - 2.2.1 - 3.1^2 + 5.1) = 1$, viz. the axis.

We may observe moreover that the curve found here is of course only partly active, and for the rest parasitic, the parasitic parts are, however, of two kinds: some parts of the curve are centres of circles with imaginary radius, others on the other hand of real circles, which, however, do not cut k^4 perpendicularly in a real way, i.e. where exactly those points, where the intersection takes place perpendicularly, are imaginary. So as to the ellipse the parts of the major axis lying outside the ellipse, are active (cf. § 5), the parts between the vertices and the foci are centres of imaginary circles, whereas the part between the two foci contains the centres of real circles, which, however, do not cut k^4 perpendicularly in a real way. As the branches of the nodal curve which pass through the foci extend to either side of β as far as point Z_∞, radii of any greatness must be found in the cyclographic representation of those branches, from the zero circle, which corresponds with the focus, to the straight line at infinity, which represents Z_∞ cyclographically. The circles representing the points of the nodal curve in the close neighbourhood of the focus are very small and lie therefore entirely within the ellipse; but there are also very great circles, and so there must be a circle that meets the ellipse really for the first time. This meeting must of course be contact, and this contact will take place in the vertex nearest to the focus; the circle then touches at the ellipse in its vertex and cuts it perpendicularly in two imaginary points. The two intersections coinciding in the vertex diverge now, describe the ellipse, meet again in the other vertex, and after that the circle will enclose the ellipse entirely.