Part of the granites and of similar rocks of the Highlands of Padang may very possibly be pre-carboniferous. But the missing of contactphenomena in adjoining limestones cannot count as a proof of a pre-carboniferous age, if part of these limestones is mesozoic; moreover the walls of limestone continuing sometimes uninterruptedly without transgression-conglomerates and without veins of granite or contactphenomena over the limit of granites and the surrounding sediments, may possibly be brought into contact with the granites by overthrusts, after the intrusion of these latter rocks; so that still these granites could very possibly be younger than carboniferous limestones resting upon them.

As to the rocks of upper-triassic limestone, which on the isle of Timor give to the landscape a similar peculiar character as many limestones of Sumatra and the "Klippen" of the Alps and the Carpathian mountains, Molengraaff [1] and Wanner [2] suppose them to be the erosion-rests of an overthrust sheet. For Djambi Tobler [3] has already suggested that the carboniferous rocks of Sangir-Pulau-Bajor, together with the mass of the Pulasian-Plepat-granites, is to be considered as an overthrust sheet. In the Highlands of Padang too there are signs which make possible a geological structure characterised by large overthrusts; however, to bear out this supposition we want further information concerning the age of the different rocks (for a good deal mesozoic and originally taken together as "old slates"), and concerning different facies of the carboniferous and these mesozoic sediments.

Physics. — *"The clustering-tendency of the molecules in the critical state and the extinction of light caused thereby"*. By Dr. F. Zernike. (Communicated by Prof. Lorentz).

(Communicated in the meeting of February 26, 1916).

1. In a former paper by Ornstein and the author [4] an improved formula was derived for the scattering of light by a single substance in the neighbourhood of the critical point. In that paper the inaccuracy of former approximate formulae, which gave an infinitely strong

[1] G. A. F. Molengraaff. Overschuivingen in overschuivingsbladen op de eilanden Timor en Leti Voordracht gehouden in de Bestuursvergadering van 18 Januari 1913. Tijdschr. Kon. Ned. Aardr. Gen. 2. XXX. p. 273. 1913.

[2] J. Wanner. Geologie von West-Timor, Geologische Rundschau Bd. IV. Heft 2.

[3] A. Tobler. l.c. p. 29.

[4] These Proceedings XVII, p. 798. This paper will here be indicated by l. c.

scattering for the critical state, had been corrected by exactly taking into account the influence of the arrangement of the molecules in space, after a new probability-method. In fact, the intensity of the scattered light could be given for the critical point itself.

The former difficulty however has not quite disappeared in that way. For when we compute from the given formula the total scattered energy, in order to find in this way the extinction of the incident light, then a logarithmic expression arises which still becomes infinite in the critical point.

The performance of a better approximation also appeared of practical importance, because it has already been shown l.c. that the magnitude of the sphere of attraction might experimentally be found from measurements of the opalescence. Therefore a theoretically accurate formula for the extinction is of great importance.

It could be seen beforehand that the inaccuracy of the given derivation had to be ascribed to the optical treatment, which was far too rough. Presently I have succeeded in finding a better value of the extinction, by deriving it directly from the theory of electrons. To this end it appeared necessary to calculate explicitly the formerly introduced function g, which indicates the course of the mean density of the molecular clusters. I will start with this calculation. The result thereof is by itself interesting, as I think the problem of the "Neigung zur Schwarmbildung", which SMOLUCHOWSKI[1]) posed in 1904, is solved definitively by it.

2. Two different functions were introduced l.c. : the first, $f(x,y,z)$, represents the influence which a known deviation of density in the point x, y, z has on the mean density in the origin, if the mean density obtains at the same time at all other points of the neighbourhood. The second $g(x, y, z)$ represents the mean density in (x, y, z), when it is only known that there exists a certain deviation of density in the origin. Of these functions it has been shown l. c. that they are connected by an integral-equation, which can be put in the following form

$$g(x_1,y_1,z_1) - \iiint_{-\infty}^{+\infty} g(x_1-x,y_1-y,z_1-z) f(x,y,z)\,dx\,dy\,dz = f(x_1,y_1,z_1) \quad (1)$$

From this integral-equation we also derived a simple relation between the volume-integrals of g and f, for which we respectively write G and F:

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[1]) M. SMOLUCHOWSKI. Boltzmann-Festschrift 1904.

$$1 + G = \frac{1}{1 - F} \quad \cdots \cdots \cdots \quad (2)$$

It will be seen from the meaning of f and g that these functions will not depend upon the direction of the radius-vector $r = \sqrt{x^2 + y^2 + z^2}$. It gives, however, no simplification to introduce this in (1). Now develop g under the integral sign, according to powers of x, y and z. The differential quotients of g can then be brought before the integral-sign, and there remain integrals of the general form

$$\int\int\int_{-\infty}^{+\infty} x^r y^s z^t f(x,y,z) \, dx \, dy \, dz$$

As f only depends upon r, these integrals will vanish when one, or more of the numbers r, s, t are odd; and their value will remain unchanged by transposition of r, s and t. If we only go to terms of the second order, besides the value F for $r = s = t = 0$, we also meet the integral with $r = 2$ $s = t = 0$, which has already been represented l. c. by ε^2. (The function f will only differ from zero for the small value of r, for which the molecular attraction is yet sensible. Therefore we could call the magnitude $\sqrt{3}.\varepsilon$, in analogy with the "mean error", the mean radius of the sphere of attraction). The just mentioned manipulation makes the integral-equation pass into a differential equation for g:

$$g(x_1, y_1, z_1) - Fg - \frac{\varepsilon^2}{2}\left(\frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 g}{\partial y_1^2} + \frac{\partial^2 g}{\partial z_1^2}\right) = f(x_1, y_1, z_1)$$

or, when we now introduce r

$$\frac{d^2 g}{dr^2} + \frac{2}{r}\frac{dg}{dr} - \frac{2(1-F)}{\varepsilon^2} g = -\frac{2}{\varepsilon^2} f \quad \cdots \cdots \quad (3)$$

The general solution of the equation (3) without second member is

$$A r^{-1} e^{-\varkappa r} + B r^{-1} e^{\varkappa r}$$

in which $\varkappa^2 = \frac{2(1-F)}{\varepsilon^2}$ and from this we easily find by variation of constants the solution of (3). The two constants in this solution may be determined from the two following conditions. In the first place g must remain finite for $r = \infty$, a condition which was already required with the integral-equation (1). If we take for the second condition, that g remains finite also for $r = 0$, then we shall find

$$g = \frac{2e^{-\varkappa r}}{\varepsilon^2 \varkappa r}\int_0^r s f(s) \sinh \varkappa s \, ds + \frac{2 \sinh \varkappa r}{\varepsilon^2 \varkappa r}\int_r^\infty s f(s) e^{-\varkappa s} \, ds \quad \cdots \quad (4)$$

It may be verified that the second condition has been well chosen by integrating this expression (4) from 0 to ∞. Reducing the integral by partial integration, it will be found that the condition (2) is accurately satisfied. In the critical point F becomes 1, as found l.c., whence $\varkappa = 0$. We have chosen the form of (4) in such a way that in it we can easily take $\varkappa = 0$. Thence

$$g = \frac{2\,r^{-1}}{\varepsilon^2} \int_0^r s^2 f(s)\, ds + \frac{2}{\varepsilon^2} \int_r^\infty s f(s)\, ds$$

a solution which we might have written down directly from (3) after well-known theorems of the potential-theory. Now nearly all the simplifications which arise for $\varkappa = 0$, can also be used approximately for small values of $\varkappa$. Besides, we must keep in mind that the equation (3) is also but an approximate one, which holds the more accurately, as will be seen by a closer consideration, as $\varkappa$ is smaller. Therefore I used the following formula for the further calculations

$$g = \frac{2}{\varepsilon^2} r^{-1} e^{-\varkappa r} \int_0^r s^2 f\, ds + \frac{2}{\varepsilon^2} \int_r^\infty s f\, ds. \quad \ldots \quad (5)$$

If we call σ the radius of the sphere outside which $f = 0$, then (5) gives for $r > \sigma$

$$g = \frac{F}{2\pi\varepsilon^2} r^{-1} e^{-\varkappa r}.$$

It appears clearly from this form that the function g is sensible over a far greater distance than f, and that outside the sphere of attraction the way in which g vanishes for increasing r no longer depends upon the course of f.

3. The considerations from the theory of electrons to be given now, may be understood as an extension of the treatment which Lorentz[1]) gave for the scattering of light by an ideal gas. In the first place it appears that this treatment may be applied unchanged to our case, if it is allowed to consider separately a space with dimensions small with respect to the wave-length, surrounding a given molecule (and especially if we may assume the mean density in the space outside). According to the above calculation, this will be the case if the quantity $\varkappa$, and therefore also the distance from the critical point, is not too small. In the final formula of Lorentz

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1) H. A. Lorentz, These Proc. XIII, p. 92.

there appears in the damping-term a coefficient which represents the excess of the mean number of molecules in the small space considered, if it is known that one molecule lies within that space. This number is 1 for a random distribution, but in our case it is, as will be seen easily, $1 + G$. Thus this way also leads to the known formula for the extinction, according to which this quantity is inversely proportional to dp/dv.

Thence it is clear that we must take into account the influence of the molecule considered on the density of its surroundings, even at distances comparable with the wave-length. To this end it will no longer be possible to use the developments in power-series, which LORENTZ applies repeatedly. In order to make the calculation possible without that, I introduce the following simplification.

The electrons which are brought into vibration in the molecules by the incident light, (for simplicity we imagine one electron in every molecule) will in reality perform a somewhat irregular vibration, with an amplitude not always equal for everyone. I neglect these differences in calculating the resistance which the neighbouring oscillators exert on the electron particularly considered. LORENTZ also makes use of this approximation at a later point. Of course this is much more acceptable, however, for a diluted gas than for the rather dense nearly critical state.

First consider one molecule in the origin of coordinates, which bears a variable moment in the direction of z:

$$\mathbf{p}_z = p_0 \sin kt . \quad \ldots \cdots \ldots \ldots \quad (6)$$

With the aid of well-known formulae for the potentials φ and $\mathbf{a}$ we then find for a point (x, y, z) at a distance r

$$\frac{\partial \varphi}{\partial z} = \frac{p_0}{4\pi} \left[\left(\frac{k^2 z^2}{c^2 r^3} + \frac{r^2 - 3z^2}{r^5} \right) \sin k \left(t - \frac{r}{c} \right) + \frac{k}{c} \frac{r^2 - 3z^2}{r^4} \cos k \left(t - \frac{r}{c} \right) \right]$$

$$\frac{\dot{\mathbf{a}}}{c} = - \frac{p_0}{4\pi} \frac{k^2}{c^2 r} \sin k \left(t - \frac{r}{c} \right)$$

The sum of these two expressions with the negative sign gives the z-component of the electric force. Now if the incident light comes from the negative x-axis, the phase of the variable moment of a molecule in (x, y, z) will be so much in advance of one for which $x = 0$, as corresponds to a difference of way μx, in which μ is the refractive index of the material. Thus

$$\mathbf{p}_{x,y,z} = p_0 \sin k \left(t - \frac{\mu x}{c} \right).$$

The electric force which this moment exerts in the origin is evi-

dently equal to the already calculated one in (x, y, z) from a moment in O. Terms with $\sin kt$ in that force will not yield a resistance to the motion (6). Therefore we further only need consider the terms with $\cos kt$. For the coefficient of $\dfrac{p_0}{4\pi}\cos kt$ we find

$$\left[-\frac{k^2}{c^2}\frac{r^2-z^2}{r^3}+\frac{r^2-3z^2}{r^5}\right]\sin\frac{k}{c}(\mu x + r) - \frac{k}{c}\frac{r^2-3z^2}{r^4}\cos\frac{k}{c}(\mu x + r) \quad (7)$$

4. We might now add the expression (7) for every molecule, except the one in the origin. We can also find at once the mean value of that sum, by multiplying by the mean density in (x, y, z) and afterwards integrating through the whole space. But here the difficulty arises that the integrals do not converge. Therefore we make use of the known fact that no extinction arises, and no resistance against the motion of the electrons, if the oscillators are quite regularly distributed through the whole space. The phenomenon which we want to calculate can thus only be caused by the deviation from the homogeneous distribution, as it is caused by the action of the molecule in the origin. The mean deviation of density at a distance r is exactly $\dot{g}(r)$. So we multiply (7) by g and by the element of volume $dx\, dy\, dz$ and integrate through the whole space. To this end we introduce spherical coordinates, such that

$$x = r\cos\vartheta \qquad y = r\sin\vartheta\sin\varphi \qquad z = r\sin\vartheta\cos\varphi$$

then the integration over φ may be performed without difficulty and there remains

$$-\pi\int_0^\pi \sin\vartheta\, d\vartheta \int_0^\infty g\, dr\left[\left\{\frac{k^2}{c^2}r(1+\cos^2\vartheta)-\frac{3\cos^2\vartheta-1}{r}\right\}\sin\left\{\frac{k}{c}(1+\mu\cos\vartheta)r\right\}+\right.$$

$$\left.+\frac{k}{c}(3\cos\vartheta-1)\cos\left\{\frac{k}{c}(1+\mu\cos\vartheta)r\right\}\right] \quad (8)$$

In this I put for g the value (5) and then performed the integrations over r. About the rather complicated calculations I will only mention that the integrations from 0 to δ — the region for which the integrals appearing in (5) are variable — were separately performed, and that approximate values for a small δ could be used for the integrals from δ to ∞, which are of the type of the cosine-integral.

The expression in $\cos\vartheta$ thus obtained, contains many terms which vanish afterwards by the integration with respect to ϑ. The other terms can be reduced to an integral of a rational fraction in $\cos\vartheta$, and these give finally the following result for (8)

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$$-\frac{F}{16\varepsilon^2}\frac{k}{c}\Bigg[-4-4\mu^{-2}-4a\mu^{-2}+$$

$$+\Big\{\mu+6\mu^{-1}+\mu^{-3}+2a(\mu^{-1}+\mu^{-3})+a^2\mu^{-3}\Big\}l\,\frac{(\mu+1)^2+a}{(\mu-1)^2+a}\Bigg]\quad.\quad(9)$$

in which we have put $a=\dfrac{c^2\varkappa^2}{k^2}$.

The form (9) yields the force of resistance by multiplying by $\dfrac{p_0}{4\tau}\cos kt$ and from this we further find the energy used up. The quantity p_0, however. still remains unknown. We can find this by considering that the refractive index μ also depends upon the vibrations performed by the electrons. Thus we find e.g. for the critical point where $a=0$ and $F=1$:

$$h=\frac{\pi v}{N\varepsilon^2\lambda^4}\frac{(\mu^2-1)^2}{16\mu}\Big[-4-4\mu^{-2}+2(\mu+6\mu^{-1}+\mu^{-3})\,l\,\frac{\mu+1}{\mu-1}\Big]$$

being an extinction proportional to λ^2, and depending directly upon the quantity ε, just as was found l. c. for the opalescence. Now I especially wanted to find the way in which the extinction-coefficient increases on approaching to the critical point. For this it will suffice to consider the variability of the expression between square brackets in (9). The quantity

$$a=\frac{c^2}{k^2}\frac{2(1-F)}{\varepsilon^2}$$

may be used as a measure of the "distance to the critical point."

For large values of a^2 we may expand the logarithm in that expression into descending powers of a^2. On reduction it is seen that the term with a^{-2} is the first that has a coefficient differing from zero, and that for large values of a^2 we can take for the whole expression

$$\frac{64}{3}a^{-1}-\frac{64}{3}a^{-2}(\mu+\mu^{-1})\quad.\quad.\quad.\quad.\quad.\quad.\quad(10)$$

Only preserving the first term, this will duly give an extinction inversely proportional to $1-F$.

In my thesis I have graphically represented extinction-measurements of a liquid mixture, by plotting the reciprocal value of h as a function of $T-T_k$. The points lay rather accurately on a straight line, which cut the temperature-axis below T_k. Using this same method with the quantity (10), we find for the development of the reciprocal value of (10)

$$\frac{3}{64}\, a + \frac{3}{64}\, (\mu + \mu^{-1}) + - \ldots \ldots \ldots \quad (11)$$

This is a straight line which cuts the axis for $a = - (\mu + \mu^{-1})$.

If the difference of temperature which corresponds to this value is found by extrapolating from observations, the quantity ε can thus be deduced. The expansion used in (10) and (11) would nearly always be sufficient in practice. This appears from the following values P of the expression in (9), computed rigorously with $\mu = 1,1$, which is close to the true value for many substances. In the next column the reciprocal value has been inserted; and in the fourth

Extinction P (arbitrary unit).

a	P	$\dfrac{10}{P}$	$10 \times$ form. (11)
0	18.58	0.54	1.88
0.05	12.34	0.81	1.93
0.1	10.38	0.96	1.98
1	4.244	2.36	2.82
2	2.871	3.48	3.76
3	2.213	4.52	4.70
7	1.186	8 43	8.45
10	0.885	11.30	11.26
20	0.483	20.72	20 63

column the same quantity, as it would be deduced from the straight line (asymptote of the curve). If necessary one could, of course, also use the rigorous value for comparison with experimental data.

Groningen, February 24, 1916.

Chemistry. — *"Glutaconic acid."* (II). By Dr. P. E. VERKADE. (Communicated by Prof. J. BÖESEKEN.)

(Communicated in the meeting of February 28, 1916.)

In the first communication[1]) of this series I have shown that in all the known methods of preparing glutaconic acid (and these are very divergent) *exactly the same acid* is obtained; hence this acid is distinguished from possible isomerides by an extraordinary stability.

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[1]) These Proc. **18.** 981 (1915).