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Astronomy. — "*The Motions of the Lunar Perigee and Node. and the Figure of the Moon.*" By Prof. W. DE SITTER.

1. The motions of the perigee and node of the moon have been derived from the observations by different investigators.

For the perigee the resulting sideral motions are:

E. W. BROWN[1])	146435″.35
P. H. COWELL[2])	.37
E. J. DE VOS VAN STEENWIJK[3])	.29
NEWCOMB[4])	.30

All these values have been reduced to the value 50″.2500 of the constant of precession (see the preceding paper). The first three depend on meridian observations. The agreement between COWELL and BROWN is excellent, but the result of DE VOS deviates rather more than can be explained by the mean errors (which are about ± 0″.02 for each result). It is, however, in perfect agreement with the value derived by NEWCOMB from the discussion of occultations.

The theoretical motion due to other causes than the figures of the earth and moon is by BROWN's theory:

$$146428''.77.$$

There thus remains for these two causes

I.	BROWN—COWELL	$d\tilde{\omega} = +\, 6''.59$
II.	NEWCOMB—DE VOS	$d\tilde{\omega} = +\, 6\,.53.$

For the node the results derived by NEWCOMB[4]) and BROWN[5]) are in perfect agreement. They both find

$$-\,69679''.44_5$$

The theoretical value, as above, is

$$-\,69673''.22,$$

The part due to the figures of the earth and moon is thus

$$d\Omega = -\,6''.22_5.$$

The mean errors of both values of $d\tilde{\omega}$ and of $d\Omega$, so far as it is due to the observations, is ± 0″.02. The theoretical value, however, in both cases is the sum of a large number of terms, each

[1]) Monthly Notices, Vol. LXXIV, p. 419.

[2]) Monthly Notices, Vol. LXV, p. 275.

[3]) These Proceedings, Vol. XVI, p. 891.

[4]) Researches on the motion of the Moon. (Second paper), p. 224. The corrections indicated by BROWN, M.N. Vol. LXXIV, pp. 420 and 562 have been applied.

[5]) Monthly Notices, Vol. LXXIV, p. 563.

of which was computed to two decimals only, and may thus be 0″.005 in error. The mean error of the sum can be assumed on this account to be about $\pm 0''.02$. The mean error of the differences $d\tilde{\omega}$ and $d\Omega$ thus becomes $\pm 0''.03$.

2. The terms due to the figure of the earth are, by BROWN's theory, the factors being given as logarithms.

$$d\tilde{\omega} = [3.5907]\, J,$$
$$d\Omega = -[3.5620]\, J,$$

With $\varepsilon^{-1} = 295.96 \pm 0.20$ (see the preceding paper), we have $J = 0.0016502$, from which

$$d\tilde{\omega} = 6''.430 \pm 0''.008,$$
$$d\Omega = -6.019 \pm 0.007.$$

There thus remains for the figure of the moon

$$\begin{matrix} I & d\tilde{\omega} = +0''.16 \pm 0''.03, \\ II & d\tilde{\omega} = +0.10 \pm 0.03, \end{matrix} \qquad d\Omega = -0''.20^5 \pm 0''.03 \quad . \quad (1)$$

The values used in BROWN's theory are

$$d\tilde{\omega} = +0''.03, \qquad d\Omega = -0''.14.$$

The contradiction is apparently very great. It will be shown, however, that the values (1) can very well be ascribed to the figure of the moon. BROWN's values depend only partially on actually determined constants, from which they are derived by means of the hypothesis that the ratio $g = \frac{3}{2} \frac{C}{Mb^2}$ has the same value for the moon as for the earth. It will be seen below that the values (1) lead to a different value g'.

Let A', B', C' be the moments of inertia, M' the mass, and b' the largest radius of the moon. Further, in analogy with the notation used for the earth

$$J' = \tfrac{3}{2}\, \frac{2C' - A' - B'}{2M'b'^2} \quad , \quad K' = \tfrac{3}{2} \cdot \frac{B' - A'}{M'b'^2}$$

Then the theoretical expressions for the motions of the perigee and the node are

$$\begin{aligned} d\tilde{\omega} &= +390''\, J' - 1027''\, K', \\ d\Omega &= -470\ J' - 235\, K'. \end{aligned} \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \quad (2)$$

The coefficients are easily derived from BROWN's theory, Chapter V, § 378 [1]), where however $db_2 = 6''.57$, $db_3 = -6''.15$, must be

[1]) Memoirs R. Astr. Soc. Vol. LIX, Part I, p 81.

substituted for $+6''.41$ and $-6''.00$ respectively. The numerical coefficients in the next line of formulas then become

$$8''.62, -45''.4 \text{ and } -8''.07.$$

Then, discarding the assumption regarding $C/_M : C/_E$, and introducing J' and K', the formulas (2) are easily derived.

Comparing (1) and (2) we find

	I	II	$\frac{1}{2}(I+II)$	m. e.	
$J' =$	0.000435	0 000410	0 000422	± 0.000055	(3)
$K' =$	0.000009	0.000057	0.000033	± 0.000032	

3. The ratios

$$\alpha = \frac{C'-B'}{A'}\ ,\quad \beta = \frac{C'-A'}{B'}\ ,\quad \gamma = \frac{B'-A'}{C'}.$$

are, in the case of the moon, so small that we may neglect the difference of the numerators, and take $\beta = \alpha + \gamma$.

These ratios appear in the theory of the libration of the moon [1]), where they are analogous to $H = \frac{C-A}{C}$ in the theory of precession and nutation. Generally β and $f = \frac{\alpha}{\beta}$ are introduced as unknown quantities to be determined from the observations. The constant β is derived with great accuracy from the mean inclination of the moon's equator on the ecliptic. The equation determining this mean inclination θ_0 as a function of β is given by TISSERAND, Vol. II, p. 472, and also, more exactly, by HAYN, Selenographische Koordinaten, I [2]), p. 900, with a further addition on p. 909. The values of θ_0 derived by different investigators are:

FRANZ,	from observations by SCHLÜTER	$\theta_0 = 1°31'22''.1 \pm$	$7''.3$
STRATTON,	„ „ „ „	1 29 37 ±	7 1
HAYN		1 32 6 ±	15

I adopt

$$\theta_0 = 1°31'40'' \pm 20''.$$

Introducing this into HAYN's equation, I find

$$\beta(1 + 0.0047 f) = 0.0006286 \pm .0000022.$$

For $f = 0$ this gives $\beta = 0.0006286$,
and for $f = 1$ . . . . . $\beta = 0.0006257$.

Now we have from (3)

$$J' + \tfrac{1}{2} K' = 0.000439 \pm .000066$$

[1]) See e g. TISSERAND, Mécanique céleste, Tome II, Chapter XXVIII.
[2]) Abh. der K. Sachs. Ges. der Wiss. Band XXVII, Nr. IX, 1902.

Referring to the definitions given above we have

$$J' + \tfrac{1}{2} K' = g' . \beta$$

Taking now

$$\beta = 0.000626 \pm .000002,$$

we find

$$g' = 0.70 \pm 0.11,$$

which differs considerably from the value for the earth ($g = 0.502$).

If f is brought into evidence in the expressions for $d\tilde{\omega}$ and $d\Omega$ we have,

$$d\tilde{\omega} = [-832'' + 1222'' f] (J' + \tfrac{1}{2} K'),$$
$$d\Omega = -470'' (J' + \tfrac{1}{2} K').$$

From $d\Omega$ we find, of course, the value of $J' + \frac{1}{2} K'$ stated above, and then from $d\tilde{\omega}$:

	I	II	½ (I + II)	m. e.	
$f =$	0.98	0.87	0.92^5	$\pm 0.06^5$	. . (5)

Generally f is determined from the coefficients of certain terms in the libration in longitude, which depend on γ [1]), and of which the largest are, for $f = \frac{1}{2}$, $-156''$ *sin S* and $+22''$ *sin M*, where S and M are the mean anomalies of the sun and moon respectively. The geocentric amplitudes of these oscillations are 1''.4 and 0''.2 respectively. It is hardly surprising that the determinations of such small quantities by different observers are not very accordant. The results are

Franz	$f = 0.48777$	± 0.0278
Stratton	0.50	± 0.03
Hayn	0.75	± 0.04.

The results of Franz and Stratton are both derived from the observations by Schlüter. The results of the different observers are very discordant amongst themselves as well as with the value (5). It seems certain that the mean errors of the values derived from the observations of the libration are no true measure of the real accuracy. The true value of f is certainly much nearer to unity than to $\frac{1}{2}$. The value found by Schlüter and others for the coefficient of the principal term of the libration in longitude must then be due to systematic errors in the observations with a period of a year [2]).

[1]) It would thus be more natural to take as unknown $\frac{\gamma}{\beta} = 1 - f$. All writers have however expressed their results in terms of f.

[2]) See also Hayn, Selenographische Koordinaten II, p. 135—136. He there finds $f = 0.85 \pm 0.07$ and explains how the smaller values found by Franz and Hartwig (0.47) can be due to errors in the adopted radius-vector of Mösting *A* which, through optical libration, give rise to a spurious oscillation of yearly period, if the observations are made near the time of full moon.

We may remark that f cannot exceed unity. A value of f larger than 1 would mean that the moment of inertia about the axis pointing to the earth was larger than about the axis which is tangent to the orbit, and this would be an unstable state.

4. The theory of CLAIRAUT would lead to values of J', β, f and g', which are absolutely in contradiction with those found above from the observations.

Although the development of the theory is well known, and also its application to an ellipsoid with three unequal axes introduces no new principles, it is perhaps not devoid of interest to collect the different formulas into a concise summary.

The forces acting on the moon are: its own gravitational attraction, the attraction of the earth, and the centrifugal force. Take a system of coordinate axes, with its origin in the centre of gravity of the moon and the axis of Z along the axis of rotation. We can with sufficient approximation suppose the earth to be situated on the axis of X at a constant distance R from the origin.

The equipotential surfaces are approximately ellipsoids of which the principal axes are situated along the coordinate axes, and have the lengths

$$\beta\,, \qquad \beta\,(1-\nu)\,, \qquad \beta\,(1-\sigma)$$

Further the equipotential surfaces are also surfaces of equal density. The density at any point is denoted by Δ and the mean density within any equipotential surface by D. We have thus

$$D = \frac{1}{\beta^3(1-\sigma)(1-\nu)} \int_0^\beta \Delta \frac{d}{d\beta}\left[\beta^3\,(1-\sigma)\,(1-\nu)\right] d\beta\,.$$

As we will only develop the theory to the first order of ν and σ inclusive, we require D only to the order zero; thus

$$D = \frac{3}{\beta^3}\int_0^\beta \Delta\,\beta^2\,d\beta\,.$$

Further we introduce the integrals

$$S = \frac{1}{\beta^5}\int_0^\beta \Delta \frac{d}{d\beta}\,(\beta^5\,\sigma)\,d\beta\,, \qquad T = \int_\beta^b \Delta \frac{d\sigma}{d\beta}\,d\beta\,,$$

$$P = \frac{1}{\beta^5}\int_0^\beta \Delta \frac{d}{d\beta}\,(\beta^5\,\nu)\,d\beta\,, \qquad Q = \int_\beta^b \Delta \frac{d\nu}{d\beta}\,d\beta\,.$$

If then r, φ, λ are the polar coordinates of any point, the potential V_1 at that point due to the attraction of the moon is given by [1])

$$\frac{3}{4\pi f} V_1 = \frac{D_1\beta^3}{r} + (\tfrac{1}{5}-\tfrac{3}{5}\sin^2\varphi)\left[\frac{\beta^5}{r^3} S + r^2 T\right] +$$
$$+ (\tfrac{1}{5}-\tfrac{3}{5})\cos^2\varphi \sin^2\lambda \left[\frac{\beta^5}{r^3} P + r^2 Q\right].$$

If ω be the velocity of rotation, and if we put

$$\varrho = \frac{3\omega^2}{4\pi f D},$$

then the potential of the centrifugal force is

$$\frac{3}{4\pi f} V_2 = \tfrac{1}{2} D \varrho r^2 \cos^2\varphi .$$

Further if M be the mass of the earth, and

$$\varkappa = \frac{3M}{4\pi R^3 D},$$

the potential of the attraction of the earth is

$$\frac{3}{4\pi f} V_3 = D\varkappa\,[1 - \tfrac{1}{2}\sin^2\varphi - \tfrac{1}{2}\cos^2\varphi\sin^2\lambda] .$$

Along an equipotential surface the sum $V = V_1 + V_2 + V_3$ must be constant. If we are content with the first order of σ and ν we can also take $r = \beta$ in the factors of S, T, P, Q, ϱ and $\varkappa$. The equation to the equipotential surface then becomes, if a is a constant:

$$\frac{r}{a} = D(1 + \tfrac{1}{3}\varrho) + (\tfrac{1}{2} - \tfrac{1}{2}\sin^2\varphi)\,[\tfrac{2}{5}(S + T) + \tfrac{1}{3} D\varrho + D\varkappa]$$
$$+ (\tfrac{1}{2} - \tfrac{3}{2}\cos^2\varphi\sin^2\lambda)\,[\tfrac{2}{5}(P + Q) + D\varkappa].$$

The equation of the ellipsoid is

$$r = \beta\,\{1 - \sigma\sin^2\varphi - \nu\cos^2\varphi\sin^2\lambda\}.$$

Comparing the coefficients of $\sin^2\varphi$ and $\cos^2\varphi\sin^2\lambda$, we find

$$\left.\begin{aligned} D\sigma &= \tfrac{3}{5}(S + T) + \tfrac{1}{2} D\varrho + \tfrac{3}{2} D\varkappa , \\ D\nu &= \tfrac{3}{5}(P + Q) + \tfrac{3}{2} D\varkappa . \end{aligned}\right\} \quad \ldots\ldots (6)$$

The quantities referring to the outer surface will be distinguished by the suffix 1. We then have

$$M' = \tfrac{4}{3}\pi D_1 b'^3 ,$$
$$C' - A' = \tfrac{8}{15}\pi S_1 b'^5, \qquad B' - A' = \tfrac{8}{15}\pi P_1 b'^5,$$
$$T_1 = 0, \qquad Q_1 = 0.$$

1) The constant of the gravitation f in this formula of course is a different thing from the ratio f, which has been defined above.

Consequently for the outer surface we have

$$\left.\begin{aligned} \sigma_1 &= \tfrac{3}{2}\frac{C'-A'}{M'b'^2} + \tfrac{1}{2}\varrho_1 + \tfrac{3}{2}\varkappa_1, \\ \nu_1 &= \tfrac{3}{2}\frac{B'-A'}{M'b'^2} + \tfrac{1}{2}\varkappa_1. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

Putting now

$$\varepsilon_1 = \sigma_1 - \tfrac{1}{2}\nu_1,$$

so that ε_1 is the mean compression of the meridians, we find

$$\left.\begin{aligned} \varepsilon_1 &= J' + \tfrac{1}{2}\varrho_1, \\ \nu_1 &= K' + \tfrac{1}{2}\varkappa_1 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

5. We now put

$$\eta = \frac{\beta}{\sigma}\cdot\frac{d\sigma}{d\beta}, \qquad \theta = \frac{\beta}{\nu}\cdot\frac{d\nu}{d\beta}, \qquad \zeta = -\frac{\beta}{D}\cdot\frac{dD}{d\beta}.$$

From the definition of D we find easily

$$\zeta = 3\left(1 - \frac{\Delta}{D}\right).$$

If now the assumption is made that the density never increases from the centre outwards [1]), we have always $1 \geqq \frac{\Delta}{D} \geqq 0$, or

$$0 \leqq \zeta < 3.$$

We now differentiate the equations (6). If the whole mass rotates as one solid body, then $D\varrho$ is constant. Also $D\varkappa$ is a constant. We thus find easily

$$\left.\begin{aligned} \eta + 3\left(\frac{S}{\sigma D} - 1\right) &= 0 \\ \theta + 3\left(\frac{P}{\nu D} - 1\right) &= 0. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

We have thus

$$\beta^5 \sigma D (\zeta - \eta) = 3\,(\beta^5 S - \beta^5 \Delta\sigma).$$

If for $\beta^5 S$ we write $\int_0^\beta \Delta \frac{d}{d\beta}(\beta^5\sigma)\, d\beta$, and integrate by parts, we find

[1]) It is not necessary to suppose that, for all values of β, $\frac{d\Delta}{d\beta} \leqq 0$. It is sufficient if $\int_0^\beta \beta^3 \frac{d\Delta}{d\beta}\, d\beta \leqq 0$ and $\int_0^\beta \beta^5\sigma \frac{d\Delta}{d\beta}\, d\beta \leqq 0$

$$\beta^5 \sigma D (\zeta - \eta) = -3 \int_0^\beta \beta^5 \sigma \frac{d\Delta}{d\beta} d\beta.$$

Since $\frac{d\Delta}{d\beta}$ is supposed never to be positive, the integral also cannot be positive, and we conclude

$$\zeta \geqq \eta.$$

Similarly we find

$$\zeta \geqq \theta.$$

Now differentiating (9) again, we find

$$\left.\begin{aligned} \beta \frac{d\eta}{d\beta} + 5\eta + \eta^2 - 2\zeta (1 + \eta) = 0, \\ \theta \frac{d\theta}{d\beta} + 5\theta + \theta^2 - 2\zeta (1 + \theta) = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

For $\beta = 0$ we have $\eta = \theta = 0$. For small values of β, η and $\frac{d\eta}{d\beta}$ are therefore necessarily of the same sign. It follows from (10) that this is only possible when η is positive; η and $\frac{d\eta}{d\beta}$ thus begin by being both positive, and η cannot become negative without passing through zero. But, for values of β larger than zero, we find from (10) that, for $\eta = 0$, $\frac{d\eta}{d\beta}$ is positive. It follows that η can never become negative. The same reasoning holds for θ. Collecting the different inequalities, which have been found we can write

$$\left.\begin{aligned} 0 \leqq \eta \leqq \zeta \leqq 3, \\ 0 \leqq \theta \leqq \zeta \leqq 3. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From (10) we find

$$\beta \frac{d}{d\beta} (\eta - \theta) + 5(\eta - \theta) + (\eta + \theta)(\eta - \theta) - 2\zeta (\eta - \theta) = 0.$$

Putting now

$$y = \frac{\eta - \theta}{\beta},$$

we find

$$\beta \frac{dy}{d\beta} + [6 + \eta + \theta - 2\zeta]\, y = 0 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The factor in square brackets is necessarily positive, and is equal to 6 for $\beta = 0$. Putting thus

$$[6 + \eta + \theta - 2\zeta] = 6 + p\beta + q\beta^2 + \ldots,$$
$$y = a + b\beta + c\beta^2 + \ldots,$$

and substituting in (12), we can successively determine the constants $a, b, c \ldots$ We find that all these coefficients are zero. Consequently $y=0$, or

$$\eta = 0.$$

This being so for small values of β, it remains true for all β, since η and θ satisfy the same differential equation.

Referring now to the various definitions given above, we conclude

$$\frac{v}{\sigma} = \frac{P}{S} = \frac{Q}{T} = \frac{P_1}{S_1} = \frac{B'-A'}{C'-A'} = \frac{\frac{1}{2}\varkappa_1}{\frac{1}{2}\varkappa_1 + \frac{1}{2}\varrho_1} = 1 - f. \quad . \quad . \quad (13)$$

Now, since the velocity of rotation equals the mean motion in the orbit, we have by KEPLER's third law, for the average value of R,

$$R^3\omega^2 = AfM(1+\mu),$$

where the factor A is taken from the lunar theory. Therefore

$$\frac{\varrho}{\varkappa} = A(1+\mu) = 1.0095$$

We have thus from (13):

$$1 - f = 0.7482, \qquad f = 0.2518.$$

We found above that for the actual moon the true value of f is probably very near unity. We must thus conclude that the distribution of mass within the moon is *not* approximately in accordance with the theory of CLAIRAUT.

6. Continuing however to trace the consequences of this theory, we now apply RADAU's transformation of the differential equation (10) of CLAIRAUT. Since $\theta=\eta$, it is sufficient to treat the equation for η.

Put

$$\Phi = D\beta^5 \sqrt{1+\eta}.$$

Differentiating, and comparing with (10), we find

$$\frac{d\Phi}{d\beta} = 5D\beta^4 \cdot \frac{1 + \frac{1}{2}\eta - \frac{1}{10}\eta^2}{\sqrt{1+\eta}}$$

Now the function $F = \dfrac{1 + \frac{1}{2}\eta - \frac{1}{10}\eta^2}{\sqrt{1+\eta}}$ is nearly constant for small values of η, as will be seen from the following little table

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$\eta = 0 \ldots$	$F = 1$
$\frac{1}{3}$	1.00074 (maximum)
0.6	0.99928
1	0.98995
3	0.8

Therefore, F_0 being a certain mean value of F, which will never differ much from unity, we have

$$D\beta^5 \sqrt{1+\eta} = \Phi = 5F_0 \int D\beta^4 d\beta, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Now the moment of inertia C' is given by

$$C' = \tfrac{8}{15}\,\pi \int_0^{b'} \Delta \frac{d}{d\beta} \left[\beta^5 (1-\sigma)(1-\nu)^2\right] d\beta$$

$$= \tfrac{8}{3}\,\pi \int_0^{b'} \Delta\beta^4 d\beta - (C'-A') - 2\,(B'-A').$$

If in C' we neglect small quantities of the first order, we can take $\Delta = D\,(1-\tfrac{1}{3}\,\zeta) = D + \tfrac{1}{3}\beta\,\frac{dD}{d\beta}$, and consequently

$$\int \Delta\, \beta^4 d\beta = \int D\beta^4 d\beta + \tfrac{1}{3} \int \beta^5 \frac{dD}{d\beta}\, d\beta.$$

Integrating the second integral in the right hand member by parts, and substituting in the value for C', we find

$$C' = \tfrac{8}{9}\,\pi D_1 b'^5 - \tfrac{16}{9}\,\pi \int_0^{b'} D\beta^4 d\beta.$$

The integral is determined by (14). Introducing the mass $M' = \tfrac{4}{3}\,\pi b'^3 D_1$ we find

$$g' = \tfrac{5}{2}\,\frac{C'}{M'b'^2} = 1 - \tfrac{2}{5}\,\frac{\sqrt{1+\eta_1}}{F_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Since $0 \leqq \eta_1 \leqq 3$, we have

$$\tfrac{3}{5} \geqq g' > 0.$$

The upper limit corresponds to homogeneity, the lower limit to condensation of the whole mass in the centre.

We have found above

$$g' = 0.70 \pm 0.11 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The most probable value of g' is therefore outside the limits of Clairaut, though the mean error does not entirely exclude a value near the upper limit. An excess of g' over the value for homogeneity indicates that in the moon the density *increases* from the centre outwards. A small excess could of course be due to irregularities in the distribution of the mass. But, unless we are prepared to admit a considerable excess of density of the outer layers of the moon over the mean density, we are led to the conclusion that the true value of g' is certainly not larger and probably smaller than the value (16). Now this value was determined from the observed motion of the node combined with the adopted compression of the earth $\varepsilon^{-1} = 296.0$. For $\varepsilon^{-1} = 297.0$ we should have found $g' = 0.85$, and HELMERT's value 298.3 gives $g' = 1.02$. Thus, if the observed motion of the node is accepted, any value of ε appreciably smaller than $1/_{296}$ becomes very improbable.

7. From (7) and (9), combined with (11), we find easily

$$\tfrac{1}{2}\varrho_1 + \tfrac{1}{2}\varkappa_1 \leqq \sigma_1 \leqq \tfrac{5}{4}\varrho_1 + \tfrac{15}{4}\varkappa_1,$$

$$\tfrac{1}{2}\varkappa_1 \leqq \nu_1 \leqq \tfrac{15}{4}\varkappa_1.$$

The numerical value is approximately

$$\varrho_1 = \varkappa_1 = 0.0000078.$$

Therefore

$$0.0000156 \leqq \sigma_1 \leqq 0.0000390$$

$$0.0000117 \leqq \nu_1 \leqq 0.0000292.$$

Take e.g.

$$\sigma_1 = 0.0000300, \quad \nu_1 = 0.0000225.$$

We then have from (6)

$$\tfrac{3}{2}\frac{C'-A'}{M'b'^2} = 0.0000144, \qquad \tfrac{3}{2}\frac{B'-A'}{M'b'^2} = 0.0000108,$$

and consequently

$$J' = 0.000021 \quad , \quad K' = 0.000011.$$

For the limiting case of homogeneity, these values would become

$$J' = 0.000032 \quad , \quad K' = 0.000018.$$

The values derived from the motions of the perigee and the node were

$$J' = 0.000422 \pm .000055 \quad , \quad K' = 0.000033 \pm .000032.$$

Further we have from (9), with the above value of σ_1:

$$\eta_1 = 3\left[1 - \frac{5}{2}\frac{C'-A'}{M'b'^2}\cdot\frac{1}{\sigma_1}\right] = 0.60.$$

Then from (15) taking $F_0 = 1$, we find $g' = 0.494$ and consequently:

$$\beta = \frac{C'-A'}{C'} = 0.000029.$$

For the case of homogeneity this would become

$$\beta = 0\ 000059.$$

The value derived from the mean inclination of the moon's equator was

$$\beta = 0.000626 \pm .000002.$$

Here again we find an enormous difference between the true values and the theory of CLAIRAÛT.

8. The conclusion that the distribution of mass in the body of the moon is not in agreement with the theory of hydrostatic equilibrium, has already been reached by LAPLACE [1]).

The mass constituting the crust of the earth is not in equilibrium either. But below the isostatic surface there is equilibrium. We are naturally led to assume that the depth of the isostatic surface is the depth at which the pressure of the outer layers becomes so large that the material of the earth behaves as a fluid and therefore necessarily is in equilibrium [2]). To form an estimate of the pressure at the isostatic depth we can compute the pressure as it would be if the whole earth, including the crust, were in hydrostatic equilibrium. Then, treating the earth as a sphere, we have

$$p = \int_{b-Z}^{b} \Delta g \, dr,$$

where g is the acceleration of gravity. Now

$$g = \frac{fm}{r^2} \quad , \quad m = \tfrac{4}{3}\,\pi r^3 D.$$

Therefore

$$p = \tfrac{4}{3}\,\pi f \int_{b-Z}^{b} \Delta \,.\, D \,.\, r dr$$

For the earth the interval of integration is relatively small, and we can take Δ and D constant. Then $D = D_1$ and very approximately $\Delta = \frac{1}{2} D_1$. Further if $Z = kb$, we find

$$p = \tfrac{2}{3}\,\pi f D_1{}^2 b^2 [k - \tfrac{1}{2} k^2].$$

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1) Mécanique Céleste, Livre V, Chapitre II, § 18.

2) So far as constant, or slowly varying forces and stresses are concerned. The behaviour of the material with respect to sudden forces is of no importance for our argument.

The material out of which the moon is built up is probably not very different from that of the outer layers of the earth. We will therefore assume that it requires the same pressure to be fluid enough for the state of permanent equilibrium. If now on the moon the depth of the isostatic surface, if there be one, is $Z' = k'b'$, we have

$$p' = \tfrac{4}{3}\pi f \int_{b'-Z'}^{b'} \Delta' . D' . rdr.$$

Now we can put $\Delta' . D' = aD_1'^2$. If the moon were homogeneous, we should have $a = 1$. If the density increases towards the centre, then at the outer surface $a < 1$, and at the centre $a > 1$. If a_0 be a certain mean value of a over the interval of integration, we have

$$p' = \tfrac{4}{3}\pi f a_0 D_1'^2 b'^2 [k' - \tfrac{1}{2}k'^2].$$

Now

$$b' = 0.272\, b\ ,\quad D_1' = 0.610\, D_1.$$

Taking further $k = 0.018$, we find from the condition $p' = p$

$$k' - \tfrac{1}{2}k'^2 = \frac{0.32}{a_0}.$$

If we take $a_0 = 1$, we find

$$k' = 0.40.$$

Most probably the true value of a_0 does not differ much from unity. The isostatic surface in the moon would thus be situated at a depth of about two fifths of the radius, and little more than one fifth of the total volume would be inclosed within it. Of course there can be no question of an isostatic compensation as there is in the earth. The differences of the moments of inertia are almost entirely determined by the irregularities in the "crust", which here contains by far the largest part of the mass, and the small central part has only very little influence.

This reasoning, of course, is not entirely rigorous, but it undoubtedly points out the true reason why the theory of CLAIRAUT, which in the case of the earth agrees so well with the actual facts, is not at all applicable to the moon.