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§ 5. The relation (4) also follows from equation (19*a*) of the paper by SOMMERFELD (p. 134), quoted in note 2 p. 21, if α occurring there is put equal to $\frac{10}{9}$, as has been supposed in the relations (3) and (4) given above, and if in SOMMERFELD's expression h is replaced by $\frac{1}{2}$ h [1]). The latter change is connected with the fact, that in deriving the expressions given here the supposition was made that in considering the molecular translatory motion in an ideal monatomic gas we have to deal with energy elements of a magnitude $\frac{1}{2}\,h\nu$, as we tried to make probable in Suppl. N°, 30*a* § 2.

The fact that in § 4 a satisfactory agreement with experimental data was obtained, may, if the validity of the other hypotheses is admitted as sufficiently approximate, be regarded as a confirmation of the above supposition concerning the magnitude of the energy elements.

Astronomy. — *"On* SEELIGER's *hypothesis about the anomalies in the motion of the inner planets."* By J. WOLTJER Jr. (Communicated by Prof. W. DE SITTER).

(Communicated in the meeting of April 24, 1914).

To explain the differences between observation and calculation in the secular perturbations of the elements of the four inner planets, SEELIGER [2]) worked out the hypothesis that these are caused by masses of matter, which by reflection of sunlight offer the aspect of the zodiacal light. He imagines these masses to have the form of a flat disc surrounding the sun and extending nearly in the direction of the orbital planes of the planets and reaching outside the orbit of the earth; the density of the matter within the disc has its greatest value in the proximity of the sun, though it is very small even there. For the calculation of the attraction of the mass of matter special hypotheses on its constitution are introduced; we imagine a number of very flattened ellipsoids of revolution with the sun at the centre, the inclinations of the equatorial planes to the orbital planes of the planets being small. It is evident that by the superposition of a number of such ellipsoids we get a flat disc within which the density varies

[1]) This confirms at the same time the fact, that the introduction of the zero point energy does not produce a change in the value of the entropy constant.

[2]) Das Zodiakallicht und die empirischen Glieder in der Bewegung der innern Planeten. Sitzungsberichte der Bayerischen Akademie, XXXVI 1906.

after a certain law from the centre outwards. SEELIGER arrived at the conclusion that two ellipsoids suffice, one of which is wholly contained within the orbit of Mercury, the other reaching outside the orbit of the earth. There appears to exist a certain liberty in choosing the values of the ellipticities and the quantities determining the position of the second ellipsoid. As quantities to be determined so as to account for the differences which are to be explained SEELIGER introduces the densities of both ellipsoids, the inclination and the longitude of the ascending node of the equatorial plane of the first ellipsoid with reference to the ecliptic, and a quantity not connected with the attraction of the masses of matter, but relating to the deviation of the system of coordinates used in astronomy from a so called "inertial system".

Last year Prof. DE SITTER drew my attention to the necessity of testing SEELIGER's hypothesis by calculating the influence of the masses admitted by SEELIGER on the motion of the moon and the perturbation of the obliquity of the ecliptic, which SEELIGER did not consider[1]). I performed the calculations and arrived at the conclusion that the perturbation of the ecliptic changes the sign of NEWCOMB's[2]) residual and makes its absolute value a little larger; further that the perturbations of the motion of the moon are insensible. I may be allowed to thank Prof. DE SITTER for the introduction into this subject and the interest shown in its further development. — One could take the formulae required for the last mentioned purpose from SEELIGER's publication; I did not do so, but developed them anew. I give them here on account of small differences in derivation. First I shall give this derivation and the results; after that I shall do the same for the motion of the moon.

I. *Perturbations of the ecliptic.*

Let x, y, z be coordinates in a system the origin of which is at the centre of the ellipsoid, while the axis of rotation is the axis of z, k^2 the constant of attraction, q the density of the ellipsoid, a, a and c its axes, then the potential V at the point x, y, z is given by the expression:

$$V = k^2 \pi q a^2 c \int_\lambda^\infty \left(1 - \frac{x^2+y^2}{a^2+u} - \frac{z^2}{c^2+u}\right) \frac{du}{(a^2+u)\sqrt{c^2+u}};$$

1) See DE SITTER, the secular variations of the elements of the four inner planets, *Observatory*, July 1913.

2) Astronomical Constants p. 110.

for a point outside the ellipsoid λ is the positive root of the equation $1-\frac{x^2+y^2}{a^2+\lambda}-\frac{z^2}{c^2+\lambda}=0$; for a point inside λ is zero.

Putting $V=k^2\pi q a^2 c\Omega$ and $x^2+y^2+z^2=r^2$ we have:

$$\Omega=\int_{\lambda}^{\infty}\left(1-\frac{r^2}{a^2+u}-\frac{z^2(a^2-c^2)}{(a^2+u)(c^2+u)}\right)\frac{du}{(a^2+u)\sqrt{c^2+u}}$$

$$1-\frac{r^2}{a^2+\lambda}-\frac{z^2(a^2-c^2)}{(a^2+\lambda)(c^2+\lambda)}=0.$$

Perturbations caused by the first ellipsoid.

I develop in powers of $z^2=\zeta$, ζ being a small quantity; for that purpose we need (neglecting terms of the third order):

$$\left(\frac{\partial\Omega}{\partial\zeta}\right)_0=-(a^2-c^2)\int_{r^2-a^2}^{\infty}\frac{du}{(a^2+u)^2(c^2+u)^{3/2}}$$

$$\left(\frac{\partial^2\Omega}{\partial\zeta^2}\right)_0=\frac{(a^2-c^2)^2}{r^4(r^2-a^2+c^2)^{5/2}}.$$

I put $r=a_1(1+\xi)$ and develop the part of Ω independent of ζ besides the coefficients of the different powers of ζ in powers of ξ.

Introducing the quantities:

$$C_1=\int_{a_1^2-a^2}^{\infty}\frac{du}{(a^2+u)\sqrt{c^2+u}}\qquad C_2=\int_{a_1^2-a^2}^{\infty}\frac{du}{(a^2+u)^2\sqrt{c^2+u}}\qquad C_3=\int_{a_1^2-a^2}^{\infty}\frac{du}{(a^2+u)^2(c^2+u)^{3/2}}$$

$$a_1^2-a^2+c^2=p^2\qquad \frac{a_1^2}{p^2}=\gamma$$

we get:

$$\Omega=C_1-a_1^2C_2-2a_1^2C_2\,\xi+(2-pa_1^2C_2)\frac{\xi^2}{p}+\left(-\frac{2}{3}-\frac{2}{3}\gamma\right)\frac{\xi^3}{p}+$$
$$+\left(\frac{1}{2}+\frac{1}{3}\gamma+\frac{1}{2}\gamma^2\right)\frac{\xi^4}{p}+\zeta\,\frac{a^2-c^2}{a_1^2p^3}\Big\{-C_3a_1^2p^3+2\xi+(-3-3\gamma)\xi^2+$$
$$+(4+5\gamma+5\gamma^2)\xi^3+\left(-5-\frac{27}{4}\gamma-\frac{15}{2}\gamma^2-\frac{35}{4}\gamma^3\right)\xi^4\Big\}+$$
$$+\frac{1}{2}\zeta^2\,\frac{(a^2-c^2)^2}{a_1^4p^5}.$$

Let v be the true anomaly of the planet, ψ the angular distance between the ascending node of the equatorial plane of the ellipsoid on the orbital plane of the planet and the perihelion of the orbit,

J the inclination of the equatorial plane to the orbital plane, then we have:

$$z = - a_1 (1 + \xi) \sin (v + \psi) \sin J$$
$$\zeta = a_1^2 (1 + \xi)^2 \sin^2 (v + \psi) \sin^2 J.$$

For the calculation of the secular portion of the perturbative function we thus need the secular portions of ξ^p, $\xi^p \sin^2 (v + \psi)$ and $\sin^4 (v + \psi)$ for different values of p. I get (denoting the secular portion by the letter S):

$$S\xi = \frac{e^2}{2} \qquad S\xi^2 = \frac{e^2}{2} \qquad S\xi^3 = \frac{3}{8} e^4 \qquad S\xi^4 = \frac{3}{8} e^4$$

$$S \sin^2 (v + \psi) = \frac{1}{2} - \left(\frac{3}{8} e^2 + \frac{1}{16} e^4\right) \cos 2\psi$$

$$S \xi \sin^2 (v + \psi) = \frac{1}{4} e^2 \left(1 - \frac{3}{2} \cos 2\psi\right) + \frac{1}{16} e^4 \cos 2\psi$$

$$S \xi^2 \sin^2 (v + \psi) = \frac{1}{4} e^2 \left(1 - \frac{1}{2} \cos 2\psi\right) - \frac{1}{16} e^4 \cos 2\psi$$

$$S \xi^3 \sin^2 (v + \psi) = e^4 \left(\frac{3}{16} - \frac{1}{4} \cos 2\psi\right)$$

$$S \xi^4 \sin^2 (v + \psi) = e^4 \left(\frac{3}{16} - \frac{1}{8} \cos 2\psi\right)$$

$$S \sin^4 (v + \psi) = \frac{3}{8}.$$

Substituting in the expression for Ω we find:

$$S\Omega = C_1 - a_1^2 C_2 + \frac{e^2}{p}\left(1 - \frac{3}{2} a_1^2 C_2 p\right) + \frac{e^4}{p}\left(-\frac{1}{16} - \frac{1}{8}\gamma + \frac{3}{16}\gamma^2\right) +$$
$$+ \frac{a^2 - c^2}{p^3} \sin^2 J \Bigg[-\frac{1}{2} C_3 a_1^2 p^3 + e^2 \Bigg\{\frac{3}{4} - \frac{3}{4}\gamma - \frac{3}{4} C_3 a_1^2 p^3 +$$
$$+ \cos 2\psi \left(-\frac{7}{8} + \frac{3}{8}\gamma + \frac{5}{4} C_3 a_1^2 p^3\right)\Bigg\} + e^4 \Bigg\{-\frac{9}{64}\gamma + \frac{45}{32}\gamma^2 -$$
$$- \frac{105}{64}\gamma^3 + \cos 2\psi \left(\frac{1}{16} + \frac{13}{32}\gamma - \frac{25}{16}\gamma^2 + \frac{35}{32}\gamma^3\right)\Bigg\}\Bigg] + \frac{3}{16} \frac{(a^2 - c^2)^2}{p^5} \sin^4 J.$$

Let i, $\tilde{\omega}$ and Ω be the inclination, the longitude of the perihelion and the longitude of the ascending node of the orbital plane of the planet, J_0 and Φ the inclination and the longitude of the node of the equatorial plane of the ellipsoid all with reference to a fixed fundamental plane, e.g. the ecliptic of a certain epoch; then we have:

$$\sin J \cos (\psi - \tilde{\omega} + \Omega) = - \cos J_0 \sin i + \sin J_0 \cos i \cos (\Omega - \Phi)$$
$$\sin J \sin (\psi - \tilde{\omega} + \Omega) = \sin (\Omega - \Phi) \sin J_0.$$

From these expressions we can determine $\frac{\partial J}{\partial \Omega}, \frac{\partial J}{\partial i}, \frac{\partial \psi}{\partial \Omega}, \frac{\partial \psi}{\partial i}$ the quantities required for the computation of the derivatives of Ω with regard to these elements. In view of the calculation of the perturbation of the obliquity of the ecliptic I do not use the elements i and Ω, but the elements p and q thus defined:

$$p = \tan i \sin \Omega \qquad q = \tan i \cos \Omega$$

I get:

$$\frac{\partial J}{\partial p} = \cos i \left\{ \cos^2 \frac{i}{2} \sin (\psi - \tilde{\omega}) + \sin^2 \frac{i}{2} \sin (\psi - \tilde{\omega} + 2\Omega) \right\}$$

$$\frac{\partial J}{\partial q} = - \cos i \left\{ \cos^2 \frac{i}{2} \cos (\psi - \tilde{\omega}) - \sin^2 \frac{i}{2} \cos (\psi - \tilde{\omega} + 2\Omega) \right\}$$

$$\sin J \frac{\partial \psi}{\partial p} = - \sin J \tan \frac{i}{2} \cos i \cos \Omega + \cos J \cos i \left\{ \cos^2 \frac{i}{2} \cos(\psi - \tilde{\omega}) + \sin^2 \frac{i}{2} \cos (\psi - \tilde{\omega} + 2\Omega) \right\}$$

$$\sin J \frac{\partial \psi}{\partial q} = \sin J \tan \frac{i}{2} \cos i \sin \Omega + \cos J \cos i \left\{ \cos^2 \frac{i}{2} \sin(\psi - \tilde{\omega}) - \sin^2 \frac{i}{2} \sin(\psi - \tilde{\omega} + 2\Omega) \right\}.$$

The differential equations for p and q are[1]:

$$\frac{dp}{dt} = \frac{1}{n a_1^2 \sqrt{1 - e^2 \cos^3 i}} \frac{\partial V}{\partial q}$$

$$\frac{dq}{dt} = - \frac{1}{n a_1^2 \sqrt{1 - e^2 \cos^3 i}} \frac{\partial V}{\partial p}.$$

To verify these formulae I have used them for the computation of some of the perturbations of i and Ω, which are given by SEELIGER[2]).

To compute the perturbation of the obliquity of the ecliptic I take:

$$V = - k^2 \pi q a^2 c \frac{\sin^2 J}{2} (a^2 - c^2)\, C_3 a_1^2.$$

According to SEELIGER's data $a = 0.2400$, $c = 0.0239$, $J = 6°57'.0$; I get $C_3 = 0.426$; taking as unit of mass the mass of the sun, as unit of time the mean solar day I get $\log q = 0.7119 - 5$ and I find:

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1) TISSERAND, Traité de Mécanique Céleste I p. 171.

2) For Mercury I get: $\frac{di}{dt} = + 0''.573$; $\sin i \frac{d\Omega}{dt} = - 0''.049$; SEELIGER gives: $+ 0''.574$ and $- 0''.049$. For Venus I get: $\frac{di}{dt} = + 0''.163$; $\sin i \frac{d\Omega}{dt} = + 0''.091$; SEELIGER: $+ 0''.159$ and $+ 0''.088$; the small difference is owing to the value I get for $C_3 = 2.286$, while from SEELIGER's data follows $C_3 = 2.217$.

$$\frac{\partial V}{\partial} = - k^2 \pi q a^2 c (a^2 - c^2) C_3 \sin J \cos J \frac{\partial J}{\partial} = - [0.5986-8] \frac{\partial J}{\partial}$$

where the number within brackets is a logarithm.

Further:

$$\frac{\partial J}{\partial p} = - \sin \Phi ; \quad \frac{\partial J}{\partial q} = - \cos \Phi ; \quad \Phi = 40^\circ 1'.8 ;$$

therefore

$$\frac{\partial J}{\partial p} = - [0.8083 - 1] ; \quad \frac{\partial J}{\partial q} = - [0.8841 - 1] ;$$

therefore

$$\frac{\partial R}{\partial p} = + [0.4069 - 8] ; \quad \frac{\partial R}{\partial q} = + [0.4827 - 8] ;$$

from which follows, taking as unit of time the century:

$$\frac{dp}{dt} = + 0''.065 ; \quad \frac{dq}{dt} = - 0''.054.$$

Perturbations caused by the second ellipsoid.

Here the calculation is much simpler. Introducing:

$$E_1 = \int_0^\infty \frac{du}{(a^2+u)\sqrt{c^2+u}} \quad E_2 = \int_0^\infty \frac{du}{(a^2+u)^2 \sqrt{c^2+u}} \quad E_3 = \int_0^\infty \frac{du}{(a^2+u)^2 (c^2+u)^{3/2}}$$

we find:

$$S\Omega = E_1 - a_1{}^2 E_2 - \frac{3}{2} a_1{}^2 E_2 e^2 - (a^2 - c^2) a_1{}^2 E_3 \sin^2 J \left\{ \frac{1}{2} + \frac{3}{4} e^2 - \frac{5}{4} e^2 \cos 2\psi \right\}.$$

As a verification I have here also computed the perturbations of the inclination and longitude of the node for some of the other planets [1]).

To compute the perturbation of the obliquity of the ecliptic I take:

$$V = - k^2 \pi q a^2 c (a^2 - c^2) E_3 a_1{}^2 \frac{\sin^2 J}{2}.$$

According to SEELIGER's data $a = 1.2235$ and $c = 0.2399$; I get

1) For Mercury I find: $\frac{di}{dt} = - 0''.060$; $\sin i \frac{d\Omega}{dt} = - 0''.013$; SEELIGER gives: $- 0''.057$ and $- 0''.016$. For Venus I find: $\frac{di}{dt} = + 0''.007$; $\sin i \frac{d\Omega}{dt} = + 0''.153$; SEELIGER: $+ 0''.009$ and $+ 0''.144$; the results differ somewhat; however, calculating according to SEELIGER's formulae, for Venus I find: $\sin i \frac{d\Omega}{dt} = + 0''.154$.

$E_3 = 2.445$; $\log q = 0.8582-9$;

$$\frac{\partial V}{\partial} = -[0.3401-7]\frac{\partial J}{\partial}; \quad \Phi = 74°22' \,(1900.0), \quad J = 7°15';$$

therefore

$$\frac{\partial J}{\partial p} = -[0.9836-1]; \quad \frac{\partial J}{\partial q} = -[0.4305-1];$$

therefore

$$\frac{\partial V}{\partial p} = +[0.3237-7]; \quad \frac{\partial V}{\partial q} = +[0.7706-8];$$

from which, taking as unit of time the century, I get:

$$\frac{dp}{dt} = +0''.125; \quad \frac{dq}{dt} = -0''.447$$

Therefore the perturbation caused by both ellipsoids together is:

$$\frac{dp}{dt} = +0''.190; \quad \frac{dq}{dt} = -0''.501.$$

Let ε be the obliquity of the ecliptic for the time t, ε_0 the same for the time t_0, i and Ω inclination and longitude of the node of the ecliptic for t with reference to the ecliptic for t_0, then:

$$\cos\varepsilon = \cos i \cos\varepsilon_0 - \sin i \sin\varepsilon_0 \cos\Omega,$$

from which, differentiating, we get:

$$-\sin\varepsilon\frac{d\varepsilon}{dt} = -\sin i\cos\varepsilon_0\frac{di}{dt} - \sin\varepsilon_0\frac{d}{dt}(\sin i\cos\Omega)$$

therefore for $t = t_0$:

$$\frac{d\varepsilon}{dt} = \frac{dq}{dt}.$$

The perturbation of the obliquity of the ecliptic thus is $\frac{d\varepsilon}{dt} = -0''.501$. The difference between observation and theory given by NEWCOMB is $-0''.22 \pm 0.18$ (pròbable error); this thus becomes $+0''.28$. The addition to the planetary precession a is given by:

$$\frac{da}{dt} = \frac{1}{\sin\varepsilon}\frac{dp}{dt} = +0''.478.$$

II. *Perturbations of the motion of the moon.*

We shall now proceed to the formulae for the computation of the perturbation of the motion of the moon. As the perturbative force in the motion of the moon we have to take the difference between the attractions of the ellipsoid on the moon and on the earth. Suppose a system of coordinates, the sun at the origin, the axis of z perpendicular to the eliptic; let x, y, z be the coordinates of the earth in this system, $x+\xi$, $y+\eta$, $z+\zeta$ those of the moon, then the projections of the perturbative force on the three axes are given by the expressions:

$$\left(\frac{\partial V}{\partial x}\right)_{x+\xi}-\frac{\partial V}{\partial x}, \quad \left(\frac{\partial V}{\partial y}\right)_{y+\eta}-\frac{\partial V}{\partial y}, \quad \left(\frac{\partial V}{\partial z}\right)_{z+\zeta}-\frac{\partial V}{\partial z};$$

The ratio of the distances sun-earth and earth-moon being very large, I develop in powers of ξ, η, ζ, neglecting second and higher powers. Then the expressions for the perturbative forces are:

$$\frac{\partial^2 V}{\partial x^2}\xi+\frac{\partial^2 V}{\partial x \partial y}\eta+\frac{\partial^2 V}{\partial x \partial z}\zeta, \quad \frac{\partial^2 V}{\partial x \partial y}\xi+\frac{\partial^2 V}{\partial y^2}\eta+\frac{\partial^2 V}{\partial y \partial z}\zeta, \quad \frac{\partial^2 V}{\partial x \partial z}\xi+\frac{\partial^2 V}{\partial y \partial z}\eta+\frac{\partial^2 V}{\partial z^2}\zeta$$

and one can introduce as the perturbative function the function

$$R=\tfrac{1}{2}\left[\xi^2\frac{\partial^2 V}{\partial x^2}+\eta^2\frac{\partial^2 V}{\partial y^2}+\zeta^2\frac{\partial^2 V}{\partial z^2}+2\xi\eta\frac{\partial^2 V}{\partial x \partial y}+2\xi\zeta\frac{\partial^2 V}{\partial x \partial z}+2\eta\zeta\frac{\partial^2 V}{\partial y \partial z}\right].$$

Here for x, y, z are to be substituted their expressions in elliptic elements and then the secular portion of R is to be taken. Since the powers and products of ξ, η, ζ, contain only the elements of the orbit of the moon, the coefficients on the contrary only the elements of the orbit of the earth we can take the secular portion of each separately and multiply these together.

Besides the system just mentioned suppose another system x', y', z', the sun also being at the origin, but the axis of z' perpendicular to the equatorial plane of the ellipsoid. Then we have

$$z'=x \sin \Phi \sin J_0-y \cos \Phi \sin J_0+z \cos J_0,$$

therefore

$$\frac{\partial z'}{\partial x}=\sin \Phi \sin J_0; \quad \frac{\partial z'}{\partial y}=-\cos \Phi \sin J_0; \frac{\partial z'}{\partial z}=\cos J_0.$$

Perturbations caused by the first ellipsoid

From the expression given for $\Omega=\dfrac{V}{k^2 \pi q a^2 c}$ we deduce, neglecting the terms having $\sin^2 J$ as a factor:

$$\frac{\partial^2 \Omega}{\partial x^2}=-2\int_\lambda^\infty \frac{du}{(a^2+u)^2(c^2+u)^{1/2}}+\frac{4x^2}{(a^2+\lambda)^2(c^2+\lambda)^{1/2}}$$

$$\frac{\partial^2 \Omega}{\partial x \partial y}=\frac{4xy}{(a^2+\lambda)^2(c^2+\lambda)^{1/2}}$$

$$\frac{\partial^2 \Omega}{\partial x \partial z}=\frac{4xz'}{(a^2+\lambda)^2(c^2+\lambda)^{3/2}}(a^2-c^2)-2(a^2-c^2)\sin \Phi \sin J_0\int_\lambda^\infty \frac{du}{(a^2+u)^2(c^2+u)^{3/2}}$$

$$\frac{\partial^2\Omega}{\partial y^2} = -2\int_\lambda^\infty \frac{du}{(a^2+u)^2(c^2+u)^{1/2}} + \frac{4y^2}{(a^2+\lambda)^2(c^2+\lambda)^{1/2}}$$

$$\frac{\partial^2\Omega}{\partial y\partial z} = \frac{4yz'}{(a^2+\lambda)^2(c^2+\lambda)^{3/2}}(a^2-c^2) + 2(a^2-c^2)\cos\Phi\sin J_0\int_\lambda^\infty \frac{du}{(a^2+u)^2(c^2+u)^{3/2}}$$

$$\frac{\partial^2\Omega}{\partial z^2} = -2\int_\lambda^\infty \frac{du}{(a^2+u)^2(c^2+u)^{1/2}} - 2(a^2-c^2)\int_\lambda^\infty \frac{du}{(a^2+u)^2(c^2+u)^{3/2}}.$$

Substituting the elements of the orbit of the earth for x, y, z and neglecting the second and higher power of the excentricity I get:

$$\frac{\partial^2\Omega}{\partial x^2} = -2C_2 + \frac{2}{a_1{}^2p} = \frac{\partial^2\Omega}{\partial y^2};\ \frac{\partial^2\Omega}{\partial x\partial y} = 0;$$

$$\frac{\partial^2\Omega}{\partial x\partial z} = \frac{2(a^2-c^2)}{a_1{}^2p^3}\sin\Phi\sin J_0 - 2(a^2-c^2)C_3\sin\Phi\sin J_0$$

$$\frac{\partial^2\Omega}{\partial y\partial z} = -\frac{2(a^2-c^2)}{a_1{}^2p^3}\cos\Phi\sin J_0 + 2(a^2-c^2)C_3\cos\Phi\sin J_0$$

$$\frac{\partial^2\Omega}{\partial z^2} = -2C_2 - 2(a^2-c^2)C_3.$$

Let ϱ be the radius vector, v the true anomaly, $\tilde\omega$ the longitude of the perigee, ☊ the longitude of the node, i the inclination of the orbit of the moon, then we have:

$$\xi = \varrho\,[\cos(v+\tilde\omega-☊)\cos ☊ - \sin(v+\tilde\omega-☊)\sin ☊\cos i]$$

$$\eta = \varrho\,[\cos(v+\tilde\omega-☊)\sin ☊ + \sin(v+\tilde\omega-☊)\cos ☊\cos i]$$

$$\zeta = \varrho\sin(v+\tilde\omega-☊)\sin i.$$

I write these expressions thus:

$$\xi = \varrho\,(A\cos v + B\sin v)$$

$$\eta = \varrho\,(C\cos v + D\sin v)$$

$$\zeta = \varrho\,(E\cos v + F\sin v),$$

A, B, C, D, E, F being expressions not containing the true anomaly.

For the formation of the required products we need the secular portion of $\varrho^2\cos^2 v$ and $\varrho^2\sin^2 v$; I get:

$$S\varrho^2\cos^2 v = a'_1{}^2(\tfrac{1}{2}+2e^2) \qquad S\varrho^2\sin^2 v = \tfrac{1}{2}a'_1{}^2(1-e^2)$$

a'_1 being the semi-major axis of the lunar orbit.

Thus we get expressions as:

$$\frac{\xi^2}{a'_1{}^2} = A^2(\tfrac{1}{2}+2e^2) + B^2(\tfrac{1}{2}-\frac{e^2}{2}).$$

Neglecting terms like $e^2 \sin^2 \frac{i}{2}$, $e^2 \sin^4 \frac{i}{2}$ we get:

$$\frac{\xi^2}{a'^2_1} = \frac{1}{2} - \frac{1}{4}\sin^2 i(1 - \cos 2\Omega) + e^2\left(\frac{3}{4} + \frac{5}{4}\cos 2\tilde{\omega}\right)$$

$$\frac{\xi\eta}{a'^2_1} = \frac{1}{4}\sin^2 i \sin 2\Omega + \frac{5}{4} e^2 \sin 2\tilde{\omega}$$

$$\frac{\xi\zeta}{a'^2_1} = -\frac{1}{2}\sin i \sin \Omega + e^2 \sin\frac{i}{2}\left\{\frac{5}{2}\sin(2\tilde{\omega} - \Omega) - \frac{3}{2}\sin \Omega\right\}$$

$$\frac{\eta^2}{a'^2_1} = \frac{1}{2} - \frac{1}{4}\sin^2 i(1 + \cos 2\Omega) + e^2\left(\frac{3}{4} - \frac{5}{4}\cos 2\tilde{\omega}\right)$$

$$\frac{\eta\zeta}{a'^2_1} = \frac{1}{2}\sin i \cos \Omega + e^2 \sin\frac{i}{2}\left\{-\frac{5}{2}\cos(2\tilde{\omega} - \Omega) + \frac{3}{2}\cos \Omega\right\}$$

$$\frac{\zeta^2}{a'^2_1} = \frac{1}{2}\sin^2 i.$$

Substituting in R these expressions we get:

$$\frac{k^2\pi q a^2 c}{R} = \frac{1}{2}\frac{a_1'^2}{a_1^2}\Big[-2C_2 a_1^2 + \frac{2}{p} + 3e^2\left(\frac{1}{p} - C_2 a_1^2\right) + 4\sin^2\frac{i}{2}\left(-\frac{1}{p} - C_3(a^2 - c^2)a_1^2\right)$$
$$+ 2(a^2 - c^2)\sin J \sin i \cos(\Omega - \Phi)\left(a_1^2 C_3 - \frac{1}{p^3}\right)\Big].$$

The only perturbations to be considered are those of the longitude of the perigee and of the node.

The differential equations required are:

$$e\frac{d\tilde{\omega}}{dt} = \frac{1}{na_1'^2}\frac{\partial R}{\partial e} \qquad \sin i\frac{d\Omega}{dt} = \frac{1}{na_1'^2}\frac{\partial R}{\partial i}.$$

One easily perceives that the last term in the expression for R gives no sensible perturbation on account of the factor $a^2 - c^2$, the value of which is about $\frac{6}{100}$, and of the fact that Ω has a period of $18^1/_2$ years so that the coefficient we get by integration is about thirty times as small as would have been the case if Ω had been absent. In the same way I omit the term $C_3(a^2 - c^2)a^2_1$ in the coefficient of $\sin^2\frac{i}{2}$ and thus we have the following expression for R:

$$\frac{R}{k^2\pi q a^2 c} = \frac{1}{2}\frac{a_1'^2}{a_1^2}\left[3e^2\left(\frac{1}{p} - C_2 a_1^2\right) - \frac{4}{p}\sin^2\frac{i}{2}\right].$$

I get $C_2 = 0.678$; $\frac{1}{p} = 1.030$ from which follows taking as unit of time the century:

$$\frac{d\tilde{\omega}}{dt} = +\,2''.28 \quad ; \quad \frac{d\Omega}{dt} = -\,2''.22$$

Perturbations caused by the second ellipsoid.

I find:

$$\frac{\partial^2 \Omega}{\partial x^2} = \frac{\partial^2 \Omega}{\partial y^2} = -2E_2 ; \frac{\partial^2 \Omega}{\partial x \partial y} = 0;$$

$$\frac{\partial^2 \Omega}{\partial x \partial z} = -\,2\,(a^2-c^2)\,E_3 sin\,\Phi\,sin\,J; \quad \frac{\partial^2 \Omega}{\partial y \partial z} = 2(a^2-c^2)\,E_3\,cos\,\Phi\,sin J;$$

$$\frac{\partial^2 \Omega}{\partial z^2} = -\,2E_2 - 2(a^2-c^2)E_3$$

from which follows·

$$\frac{R}{k^2 \pi q a^2 c} = \frac{1}{2}\frac{{a_1'}^2}{{a_1}^2}\,[-2E_2 {a_1}^2 - 3E_2 {a_1}^2 e^2 - E_3(a^2-c^2){a_1}^2 \,sin^2\, i$$
$$+\,2(a^2-c^2)\,{a_1}^2\,E_3 sin\,J\,sin\,i\,cos\,(\Omega - \Phi)].$$

Although the term $a^2 - c^2$ is not small, yet it is allowed to omit the periodic term.

I get $E_2 = 0.684$, $E_3 = 2.445$ from which follows taking as unit of time the century:

$$\frac{d\tilde{\omega}}{dt} = -\,0''.16 \quad ; \quad \frac{d\Omega}{dt} = -\,0''\,28$$

Thus both ellipsoids together give:

$$\frac{d\tilde{\omega}}{dt} = +\,2''.12 \quad ; \quad \frac{d\Omega}{dt} = +\,2''\,50\,;$$

both insensible amounts.

Astronomy. — "*Remarks on Mr.* WOLTJER'*s paper concerning* SEELIGER'*s hypothesis.*" By Prof. W. DE SITTER.

(Communicated in the meeting of April 24, 1914).

SEELIGER'S explanation of NEWCOMB'S anomalies in the secular motions of the four inner planets consists of three parts, viz·

a. The attraction of an ellipsoid entirely within the orbit of Mercury The light reflected by this ellipsoid is, on account of the neighbourhood of the sun, invisible to us.

b. The attraction of an ellipsoid which incloses the earth's orbit. The light reflected by this ellipsoid appears to us as the zodiacal light.

c. A rotation of the empirical system of co-ordinates with reference