Mathematics. -- "*The quadruple involution of the cotangential points of a cubic pencil.*" By Professor JAN DE VRIES.

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1. We consider a pencil of cubics (φ^3), with the nine base-points B_k. On the curve φ^3, passing through an arbitrary point P, lie three points P', P'', P''', which have the *tangential point* [1]) in common with P; in this way the points of the plane may be arranged in quadruples of an *involution* (P^4) *of cotangential points*. We shall suppose, that the pencil is general, consequently contains *twelve* curves with a node D_h. On such a curve δ^3 all the groups of the (P^4) consist of two cotangential points and the point D, which must be counted twice. Apparently the 12 points D are the only *coincidences* of the involution; as the connector of the neighbouring points of D is quite indefinite, the coincidences have no definite support. The points D_h are at the same time to be considered as *singular points*; to each of them an involution of pairs P,P' is associated, lying on the curve $\delta_h{}^3$, which has D_h as node.

2. The nine base-points B_k are also *singular*; to each point B_k a triple involution of points P', P'', P''' is associated, lying on a curve β_k, of which we are going to determine the order.

To each curve φ^3 we associate the line b, which touches it in B; in consequence of which a projectivity arises between the pencil of rays (b) and the cubic pencil (φ^3). The curve τ^4 produced is the locus of the tangential points of B (*tangential curve* of B).

The line b, which touches a φ^3 in B, cuts it moreover in the tangential point of B; this is apparently the only point that b has in common with τ^4 apart from B. So τ^4 has a *triple point* in B; there are three lines b, which have in B three points in common with the corresponding curve φ^3; i. e. B is *point of inflection* of *three* curves φ^3.

Let us now consider the tangential curves τ^4_1 and τ^4_2, belonging to B_1 and B_2. Both pass through the remaining seven base-points, consequently have apart from the points B, three points in common; so there are three curves φ^3, on which B_1 and B_2 have the same tangential point. Hence it ensues that the singular curve β_1 belonging to B_1, has triple points in each of the remaining eight points B; it does not pass through B_1 because (P^4) has coincidences in D_h

[1]) The *tangential point* of P is the intersection of φ^3 with the straight line touching it in P.

only. With an arbitrary φ^3, β_1 has moreover in common the three points which form a quadruple with B_1; consequently 27 points in all. So the triplets of (P^4) belonging to B_1 lie on a curve of *order nine,* which passes *three times* through each of the remaining base-points.

We found that B_1 and B_2 belong to three quadruples; the three pairs, which those quadruples contain besides, belong to the singular curves β_1^9 and β_2^9. They have moreover in the seven remaining points B_k, 63 points in common; the remaining 12 common points are found in the singular points D_h.

3. The locus of the points of inflection I of (φ^3) has *triple points* in B_k, has therefore with an arbitrary φ^3, $9 \times 3 + 9 = 36$ points in common; it is consequently a curve of *order twelve,* ι^{12}. On a curve δ^3 lie only 3 points of inflection; we conclude from this, that ι^{12} has *nodes* in the twelve points D_h; in each of those points ι^{12} and δ^3 have the same tangents.

The points P', P'', P''', which have I as tangential point, lie in a straight line, the *harmonic polar line* h of I. So ι^{12} is the locus of the points, which in (P^4) are associated to *linear triplets.*

The curves β_1^9 and ι^{12} have in the singular points B and D $8 \times 3^2 + 12 \times 2 = 96$ points in common; on β_1^9 lie therefore 12 points I, so that B_1 belongs to 12 linear triplets. From this it ensues by the way, that the involution (P^3) lying on β_1^9 has a *curve of involution* (p) of *class twelve*; for the line $p = P'P''$ will only pass through B_1 if P''' is a point of inflection, while P lies in B_1. As B_1 is point of inflection of three φ^3, (P^3) has three linear triplets, consequently $(p)_{12}$ *three triple tangents.*

The locus λ of the linear triplets has, as was shown, 9 *dodecuple points* B; as φ^3 bears nine points of inflection, therefore 9 linear triplets, it has with λ $9 \times 12 + 9 \times 3 = 135$ points in common.

Consequently the *linear triplets lie on a curve* λ^{45}.

4. We shall now consider the curve ϱ, into which a straight line r is transformed, if a point P of r is replaced by the points P', which form a quadruple with P; for the sake of brevity we shall speak of the transformation (P, P'). If we pay attention to the intersections of r with β_k^9 and with δ_h^3, we arrive at the conclusion that ϱ has *nonuple points* in B_k and *triple points* in D_h. It has therefore with a φ^3 in B_k 81 points in common; further these curves cut moreover in the three triplets which correspond with the intersections of φ^3 and r. Consequently ϱ is a curve of *order thirty.*

On an arbitrary straight line lie therefore *fifteen pairs of cotangential points.*

By the transformation (P, P'), the curve λ^{45}, which contains the linear triplets, is transformed into a figure of order 1350. It consists of twice λ itself, three times ι^{12}, twelve times the curves β^9 and seven times the singular curves δ^3. For $2 \times 45 + 3 \times 12 + 9 \times 12 \times 9 = 1098$; the points D produce therefore a figure of order 252. From this it ensues that λ^{45} has *septuple points* in the 12 singular points D.

The pairs P, P', which are collinear with a point E, lie on a curve ε^{33}, on which E is a triple point; the tangents in E go to the points of the triplet of the (P^4), determined by E. The line EB_k cuts β_k^9 in 9 points P, which form with B_k pairs of the (P^4); hence ε^{33} has *nonuple points* in B_k.

The locus of the pairs P'', P''', belonging to the pairs P, P' of ε^{33}, we shall indicate by ε_*. As E is collinear with 12 pairs of the involution (P^3) lying on β_1^9, B_1 is a *dodecuple point* of ε_*.

On an arbitrary φ^3 the cotangential points form three involutions of pairs and the supports of the pairs of each of those involutions envelop a curve of class three (curve of CAYLEY). Consequently E is collinear with 9 pairs P, P' of φ^3, and this curve contains 9 pairs of ε_*. As the two curves in B_k have moreover 9×12 points in common, consequently 126 points in all, ε_* is a *curve of order* 42.

The curves ε^{33} and β_1^9 have in the points $B_k (k \neq 1)$ $8 \times 9 \times 3$ points in common; moreover they meet in 9 points of EB_1 and in the 12 pairs P, P' mentioned above. The remaining 48 common points must lie in D_h; so ε^{33} has *quadruple points* in the 12 singular points D.

The curves ε_*^{42} and β_1^9 have in $B_k (k \neq 1)$ $8 \times 12 \times 3$ intersections; further they meet in the 9 pairs P'', P''', belonging to the 9 points P' lying on EB_1, and in the 12 points P'', belonging to the 12 pairs P, P' of β_1^9, which are collinear with E. So they must have 60 intersections in D_h; ε_*^{42} has consequently *quintuple points* in the 12 singular points D.

The curves ε_*^{42} and ι^{12} have in B_k $9 \times 12 \times 3$, in D_h $12 \times 5 \times 2$ intersections, together 444; the remaining 60 lie in points of inflection, of which the harmonic polar lines pass through E. In such a point of inflection I, ε_*^{42} will have a *triple point*, for the corresponding polar line h contains a linear triplet, so three pairs of ε^{33}, so that I appears three times as point of ε_*. Consequently E bears 20 straight lines h: the *harmonic polar lines of* φ^3 *envelop a curve of class twenty.*