Mathematics. — *"A cubic involution of the second class."* By Prof. JAN DE VRIES.

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1. By the *class* of a cubic involution in the plane we shall understand the number of pairs of points on an arbitrary straight line [1]). In a paper presented in the meeting of February 28^{th}, 1914 [2]) I considered the cubic involutions of the *first class*, and proved that they may be reduced to *six* principally differing sorts.

The triangles Δ, which have the triplets of an involution of the first class as vertices, belong at the same time to a cubic involution of lines; the sides of each Δ form one of its groups.

The cubic involutions of the *second class* possess the characteristic quality of determining an involution of pairs i. e. an involutive birational correspondence of points. For, let X, X', X'' be a group of an involution (X^3) of the second class; on the line $X'X''$ lies another pair Y', Y''; the point Y, completing this pair into a triplet, is apparently involutively associated to X. In the following sections I shall consider a definite (X^3) of the second class and inquire into the associated involutive correspondence (XY).

2. We start from a pencil of conics φ^2 with the base-points A, B_1, B_2, B_3 and a pencil of cubics φ^3 with the base-points B_1, B_2, B_3, C_h ($h = 1$ to 6). The curves φ^2 and φ^3, which pass through an arbitrary point X, intersect moreover in two points X', X'', which we associate to X. As the involutions I^2 and I^3, which are determined on a straight line by the pencils (φ^2) and (φ^3), have two pairs X', X'' and Y', Y'' in common, a cubic involution (X^3) of the *second class* arises here.

The ten base-points are *singular points*, for they belong each to ∞^1 groups; on the other hand is a singular point certainly a base-point of one of the pencils.

The pairs of points which with the singular point A determine triangles of involution Δ, lie apparently on the curve α^3 of the pencil (φ^3), passing through A. As they are produced by the pencil (φ^2), they form a central involution, i. e. the straight lines $x = X'X''$ pass through a point T of α^3 (*opposite point* of the quadruple $AB_1B_2B_3$).

Analogously the pairs X', X'', which are associated to C_h, lie on

[1]) This corresponds to the denomination introduced by CAPORALI for involutive birational transformations. (*Rend. Acc. Napoli*, 1879, p. 212).

[2]) *"Cubic involutions in the plane"*. These Proceedings vol. XVI, p. 974.

the conic γ_h^2 passing through C_h, which conic belongs to (φ^2); the straight lines x intersect in a point M_h, the centre of the I^2.

In order to find the locus of the pairs, corresponding to B_1, we associate to each φ^3 the φ^2, which touches it in B_1. The pencils being projective on this account produce a *curve of order five*, β_1^5, which has a triple point in B_1, nodes in B_2, B_3 and passes through A and C_h. If the straight line $x = X'X''$ is associated to the straight line, which touches the corresponding curves φ^3 and φ^2 in B_1, a correspondence (1, 1) arises between the *"curve of involution"* enveloped by x and the pencil of rays B_1; from this it ensues that (x) must be a rational curve. As no other lines x can pass through B_1 but the tangents at β_1^5 in the triple point B_1, (x) is a *rational curve of the third class*, has consequently a bitangent; on it lie two pairs of (X^3). To the tangents of $(x)_3$ belong the lines AB_2 and AB_3.

There are *three singular straight lines* $b_k = AB_k$; each of them bears a I^2 of pairs X', X''. The corresponding points X lie on the line $b_{mn} = B_m B_n$.

3. The *curve of coincidences* (locus of the points $X \equiv X'$) has *triple points* in B_k and passes through A and C_h. With the singular curve γ^2_1 it has 10 intersections in A and B_k; as it touches it in C_1 and at the same time contains the coincidences of the involution (X', X'') lying on γ^2_1, it is a curve of *order seven* [1]), which will be indicated by σ^7. It passes through the 12 nodes of (φ^3) and the 3 points $(b_k b_{lm})$.

As σ^7 has six points in common with φ^3, apart from B_k and C_h, the involution I^3 of the Δ inscribed in φ^3 possesses *six* coincidences. In the same way it appears that the involutions I^2 lying on α^3 and β_k^5 possess *four* coincidences each.

The supports of the coincidences envelop a curve (d) of *class eight*; for through A pass in the first place the lines b_k, each bearing two coincidences, and which consequently are bitangents of (d) and further the tangent in A at α^3, which will touch (d) in A.

4. To the points X of a straight line l correspond the pairs of points X' and X'' of a curve λ, which has in common with l the two pairs of the (X^3) lying on l, besides the points of intersection of l and σ^7; hence λ is a curve of *order eleven*. By paying attention to the intersections of l with the singular curves α^3, β_k^5, and γ_h^2, we see that λ^{11} passes *three times* through A, *five times* through B_k and two times through C_h.

[1]) This corresponds to this well known proposition: the locus of the points where a curve φ^m of a pencil is touched by a curve φ^n of a second pencil is a curve of order $2(m+n)-3$.

On λ^{11}, X' and X'' form a pair of an involution; of the straight lines $v = X' X''$ six pass through A. Three of them are indicated by the intersections X of l and α^3; here X' lies every time in A. The remaining three are the lines b_k; for each of them contains a pair X', X'' corresponding to the point $X = (lb_{mn})$.

The curve $(x)_6$ enveloped by x is rational, because we can associate x to X; it has therefore *ten bitangents*. As such a bitangent bears two pairs X', X'' and Y', Y'' it follows that the *involution* (X, Y) contains ten pairs on l, and consequently is of the *tenth class*.

5. Let a straight line l be revolved round a point E; the pairs X', X'' and Y', Y'' lying on it describe then a curve ε^6, which passes twice through E and is touched there by the straight lines EE' and EE''. On EA lie two points X' and Y', each forming with E a pair of the (X^3); so A is a node of ε^6. For the same reason ε^6 has nodes in B_k; it also contains the points C_h. In consequence of the existence of 5 *nodes*, ε^6 is of class 20, so that E lies on 16 of its tangents. Of these 8 contain each a coincidence of the (X^3); the remaining 8 are represented by *four bitangents*, being straight lines s, on which both pairs belonging to (X^3) have coincided. From this it ensues that the lines s envelop a curve $(s)_4$ of the *fourth class*. Apparently the straight lines s, passing through A, are tangents to α^3. In the same way the four tangents out of B_k to β_k^5 are the straight lines s, which may be drawn through B_k.

Apart from the singular points ε^6 and δ^7 have 16 points in common; to them belong the 8 coincidences of which the supports d pass through E. The remaining 8 must be points X', coinciding with the corresponding point X without d's passing through E; i.e. they belong to the locus ε_* of the points X, which complete the pairs lying on ε^6 into groups of (X^3).

As E lies on three of the straight lines $x = X'X''$ belonging to B_k, B_k is a *triple point* of ε_*; analogously A and C_h are simple points of that curve, so that the latter has $2 + 3 \times 2 \times 3 + 6 = 26$ intersections with ε^6 in the singular points. Besides the 8 points of δ^7 indicated above they have moreover the points E', E'' in common; so we conclude that ε_* must be a curve of the *sixth order*. To the intersections X of ε^6_* and l correspond lines x, which pass through E; from this it ensues again that x envelops a curve of the sixth class, when X describes the straight line l.

6. If E is laid in C_1, ε^6 is replaced by the figure composed of the singular conic γ_1^2 and a curve γ_1^4, which has a node in C_1, and passes through the points A, B_k, C_h. The two curves have apart from A and B_k two more points E', E'' in common; the lines C_1E', C_1E'' touch γ_1 in C_1 and are apparently the only possible lines s passing through C_1; hence C_1 is a *node* on the curve $(s)_4$.

The curve ε_*^6 belonging to C_1 is represented by the figure composed of γ_1^2 and a curve $^*\gamma_1^4$, which has nodes in B_k. This may be found independently of what is mentioned above. The transformation replacing a point X by the corresponding points X', X'', transforms a straight line l into a curve γ^{11}, consequently the curve γ_1^4 into a figure of order 44. It consists of γ_1^4 itself (for this curve bears ∞^1 pairs X, X'), twice γ_1^2, the curves α^3, $\beta_k^5\,\gamma_k^2$ and twice the locus of X''; the latter is therefore of order four.

If E is brought into the centre M_1 of the I^2 lying on γ_1^2, ε^6 passes into γ_1^2 and a curve μ_1^4 with node M_1. Of the latter 6 tangents pass through M_1, whereas this point lies on 2 tangents of γ_1^2; from this it ensues anew that the lines d envelop a curve of the *eighth class*. As γ_1^2 apart from A and B_k has with μ_1^4 four points in common, which must form two pairs of the I^2, and so determine two lines s, M_1 too is a *node* of the curve $(s)_4$.

If E lies in A, ε^6 consists apparently of α^3, and the three lines b_k; whereas ε_*^6 is the figure composed of an α^3 and the three lines b_{mn}. For E in T ε^6 is replaced by the figure formed by α^3 and a curve τ^3, also passing through T and having with α^3 besides the four points A, B_k two more pairs collinear with T; consequently T is also a node of $(s)_4$.

For B_k ε^6 consists of β_k^5 and the line B_kA; ε_*^6 of β_k^5 and B_mB_n.

7. Passing on to the consideration of the involutive correspondence (X, Y) we cause X to describe the straight line l, and we try to find the locus of the corresponding points Y. On each line $X'X''$ lies a second pair Y', Y''; the curves φ^2 and φ^3, which intersect in the points Y', Y'' we shall associate to each other. In order to determine the characteristic numbers of this correspondence, we consider the involutions I^3, which are formed on a curve φ^2 or φ^3 by groups of (X^3).

The sides of the Δ described in a φ^2 envelop a conic; among the 12 tangents, which this curve has in common with the curve of involution $(x)_6$ belonging to λ^{11} must be reckoned the two lines X', X'', for which X is one of the intersections of l and φ^2. The remaining 10 contain each a pair Y', Y''; consequently each φ^2 is in the said correspondence associated to 10 curves φ^3.

The involution I^3 on a φ^3 possesses a curve of involution of the third class; for B_1 bears in the first place the line b_1, which contains a pair of the I^3, then the lines joining B_1 to the two points, determined by the φ^2, which touches φ^3 in B_1. The intersections of l and φ^3 procure three common tangents of $(x)_3$ and $(x)_6$; there are consequently 15 straight lines, which bear a pair Y', Y'', so that the said correspondence associates 15 curves φ^2 to φ^3.

By means of this correspondence the points of a straight line r are arranged into a correspondence (30, 30). For to the φ^2 passing through a point R of r correspond the 30 intersections R' of r with the 10 curves φ^3 associated to φ^2; on the other hand the φ^3 passing through R' procures 30 points R, by means of the corresponding 15 φ^2. The intersections of the corresponding curves form therefore a figure of order 60; it consists, however, of two parts: the locus of the pairs Y', Y'', which lie on the tangents of the $(x)_6$, and the locus of the points Y.

The former may also be produced by the pencil (φ^2) and the system of rays $(x)_6$. To each φ^2, in virtue of the consideration mentioned above, a number of ten straight lines is associated, which are each coupled to one φ^2 only; hence a (10, 12) arises now on r, so that the pairs of points Y', Y'' are lying on a figure of order 22.

For the points Y we find therefore a figure of order 38; it is composed of the three lines b_{lmn} and a curve of order 35. For to the intersection X of l and B_1B_2 corresponds a pair X', X'' on AB_3; but this line bears ∞^1 pairs Y', Y'' and the corresponding points Y of B_1B_2 are all associated to X. Apart from these three lines the line l is transformed by means of the birational correspondence (X, Y) into a curve of order 35, λ^{35}. It cuts l in 10 pairs X, Y (§ 4) and in 15 coincidences $X = Y$. There is consequently a *curve of coincidences* of order *fifteen*. The figure of order 22 found above consists of the three lines b_k and a curve λ^{10}, for to the conic (b_3, b_{12}) corresponds the tangent b_3 of $(x)_6$.

8. We shall now determine the *fundamental curves* which are associated to the fundamental points A, B_k, C_h. The curves of involution $(x)_3$ belonging to β_1^5 and β_2^5 (§ 2) have 9 tangents in common, there are consequently 9 lines, for which X lies in B_1 and Y in B_2. Therefore the fundamental curve of B_1 has nonuple points in B_2 and B_3. No other point Y of the line B_2B_3 can correspond to a point X lying in B_1; the said curve is therefore of *order* 18. It has a nonuple point in B too and passes three times through each of the points A and C_h; for through T or M_h passes *one* line, bearing a pair X', X'' of β_1^6 and a pair Y', Y'' of α^3 or γ_h^2; through which then $B_1 = X$ corresponds to a point Y lying in A or C_h.

The fundamental curve of A is apparently identical with the curve ε_{t}^{6} (§ 5) belonging to the point T; we shall indicate it by α^6. As α^3 has two pairs in common with τ^3 (§ 6) A is a node of α^6. That α^6 passes through the points C_h and has triple points in B_k ensues from the consideration of the lines TM_h and of the tangents out of T to the $(x)_3$ belonging to B_k.

It appears analogously that the fundamental curve of C_1 has triple points in B_k and a node in C_1; it passes through A and the remaining points C_h and is of *order six*. This curve is at the same time the ε_{ω}^{6} belonging to M_1.

We can now prove once more that the birational correspondence is of *order* 35. To the intersection X of two lines l, corresponds the point Y, which the two curves λ, apart from the fundamental points, have in common. As appears from what was mentioned above λ passes 18 times through B_k and 6 times through A and C_h; from $1 + 3 \times 18^2 + 7 \times 6^2 = 1225 = 35^2$ it appears now that λ is a curve of order 35.

Physics. — "*On the manner in which the susceptibility of paramagnetic substances depends on the density.*" By Dr. W. H. KEESOM Supplement N°. 36*c* to the Communications from the Physical Laboratory at Leiden. Communicated by Prof. H. KAMERLINGH ONNES.

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§ 1. *Introduction.* In Suppl. N°. 32*a* (Oct. '13) an expression was developed for the molecular rotatory energy in a system of freely rotating molecules as a function of the temperature. This expression was introduced into the theories of LANGEVIN and WEISS, on the supposition that, when the equipartition laws are deviated from, the statistics of the molecules under the action of an exterior directing field, in this case a magnetic field, is determined by the value u_1 of the rotatory energy in the same way as for equipartition it is by kT. It then appeared that different experimental results can be represented very satisfactorily in that way [1]).

[1]) The expressions developed in the above-mentioned paper appear to be also suitable to give a quantitative representation (as far as observations are available) of the decrease of the temperature of the CURIE-point by the addition of a diamagnetic metal to a ferromagnetic one, with which it forms mixed crystals, on the supposition that the diamagnetic metal exerts no other influence than that the mutual action of the ferromagnetic molecules is lessened in consequence of the