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F. M. Jaeger and Jul. Kahn: Ibid IV. "Measurements of some Aliphatic and Aromatic Ethers". (Communicated by Prof. P. van Romburgh), p. 395.
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Mathematics. — *"A triple involution of the third class."* By Professor Jan de Vries.

(Communicated in the meeting of May 30, 1914).

1. I consider the projective nets of conics represented by

$$\lambda a_x^2 + \lambda' a_x'^2 + \lambda'' a_x''^2 = 0 \text{ and } \lambda b_x^2 + \lambda' b_x'^2 + \lambda' b_x''^2 = 0 \quad . \quad (1)$$

The points of intersection of corresponding conics form a quadruple involution [1]).

On the straight line YZ, which we may represent by $x_k = \varrho y_k + \sigma z_k$, the two nets determine the pairs of points, indicated by

$$\sum_3 \lambda (\varrho^2 a_y^2 + 2\varrho\sigma a_y a_z + \sigma^2 a_z^2) = 0 \text{ and } \sum_3 \lambda (\varrho^2 b_y^2 + 2\varrho\sigma b_y b_z + \sigma^2 b_z^2) = 0.$$

These equations produce the same pair of points, as soon as the relations

$$\sum_3 \lambda a_y^2 = \tau \sum_3 \lambda b_y^2, \quad \sum_3 \lambda a_y a_z = \tau \sum_3 \lambda b_y b_z, \quad \sum_3 \lambda a_z^2 = \tau \sum_3 \lambda b_z^2.$$

are satisfied.

By elimination of $\lambda, \lambda', \lambda''$ we find from this system the relation

$$| \, a_y^2 - \tau b_y^2, \quad a_y a_z - \tau b_y b_z, \quad a_z^2 - \tau b_z^2 \, | = 0 \quad . \quad . \quad . \quad . \quad (2)$$

[1]) This involution is an intersection of the linear congruence of elliptic twisted quartics, which I have considered in my communication in vol. XIV, p. 1127 of these Proceedings.

from which it appears that YZ contains three pairs of the involution; the latter is consequently of the *third class*.

2. We shall now suppose that the two nets have a common base point A; they produce then a *triple involution* of the *third class*. We choose the base point A for vertex O_3 of a triangle of co-ordinates.

Through O_3 pass ∞^1 conics of the first net, which are touched there by the corresponding conics. For we have the conditions

$$\sum_3 \lambda a_{13} = \tau \sum_3 \lambda b_{13} \text{ and } \sum_3 \lambda a_{23} = \tau \sum_3 b_{23},$$

so that the parameters λ, λ', λ'' are connected by the relation

$$\begin{vmatrix} \sum_3 a_{13}\lambda & \sum_3 b_{13}\lambda \\ \sum_3 a_{23}\lambda & \sum_3 b_{23}\lambda \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now we find from (1)

$$\lambda = \begin{vmatrix} a_x'^2 & a_x''^2 \\ b_x'^2 & b_x''^2 \end{vmatrix} \text{ etc.}$$

If we substitute these formulae λ, λ', λ'' in (3), an equation of the eighth order will arise. The locus of the pairs X', X'' of the triple involution (X^3) associated to $O_3 \equiv A$ is therefore a curve of the *eighth order*, which we shall indicate by α^8; A is a *singular point* of *order eight*.

By (3) two projective systems with index two are separated from the two nets, which systems produce the curve α^8. Their intersections with the arbitrary straight line r, are the coincidences of the (4,4), which the two systems determine on r. If r is laid through A, the free points of intersection are connected by a (2,2); one of the 4 coincidences of this correspondence lies in A, because two homologous conics touch each other and r in A. Hence it appears that the *singular curve* α^8 has a *quintuple point* in A. This corresponds to the fact that (X^3) must be of the third class; the three pairs on a straight line r laid through A are formed by A with the three points in which r is moreover cut by α^8. The line $x = X'X''$ envelops a curve of the *fifth class*; for of the system (x) only the lines which touch α^8 in A pass through A.

3. A is not the only singular point of (X^3). The homologous conics intersecting in a point Y are determined by

$$\sum_3 \lambda a_y^2 = 0 \qquad \text{and} \qquad \sum_3 \lambda b_y^2 = 0\,.$$

If these equations are dependent, Y becomes a singular point.

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Through Y pass then two projective pencils of conics, which determine a quartic represented by

$$| a_y^2 \quad a_x^2 \quad b_x^2 | = 0 \,, \quad \ldots \ldots \quad (4)$$

or also by

$$| b_y^2 \quad b_x^2 \quad a_x^2 | = 0 \,. \quad \ldots \ldots \quad (4^*)$$

The singular points are determined by the relations

$$\left\| \begin{matrix} a_y^2 & a_y'^2 & a_y''^2 \\ b_y^2 & b_y'^2 & b_y''^2 \end{matrix} \right\| = 0 \quad \ldots \ldots \quad (5)$$

Now the curves $a_y^2\, b_y''^2 = a_y''^2\, b_y^2$ and $a_y^2\, b_y'^2 = a_y'^2\, b_y^2$ have apart from the point O_3 (which is node on both) 12 points in common. To them belong the three points, which $a_y^2 = 0$ and $b_y^2 = 0$ have in common apart from O_3; they do not lie, however, on the curve $a_y'^2\, b_y''^2 = a_y''^2\, b_y'^2$. There are therefore, besides the singular point A, *nine* more *singular* points B_k; the pairs of points, which form with B_k groups of the involution (X^3) lie on a curve β_k^4, so that B_k is a *singular point* of *order four*.

The singular curve β_k^4 is produced by two projective pencils with common base points A and B_k, it has therefore *nodes* in these two points. From (4) and (5) it appears that this curve also passes through the remaining singular points. The straight lines x, which contain the pairs X', X'' lying on β_k^4, envelop a *conic*.

As β_k^4 passes through A twice, there are in (X^3) two groups in which the pair A, B_k occurs; so B_k belongs twice to α^8. This singular curve has therefore besides its *quintuple point* A, *nine* more *nodes* B_k, is consequently of genus *two* and of class 18.

On each of the 8 tangents of α^8, passing through A, two pairs of the (X^3) coincide; from this it ensues that the straight lines s on which two pairs have coincided, envelop a curve of *class eight*, which we indicate by $(s)_8$.

4. We can now determine the order x of the locus λ of the pairs of points X', X'', which form groups of the (X^3) with the points X of a straight line l. As α^8 contains eight points of l, λ passes eight times through A; analogously it has quadruple points in B_k. The x points of intersection of λ with an other straight line l^* are vertices of triangles of involution, of which a second vertex lies on l, so that the third vertex must be a common point of λ and λ^*. As these curves, besides in two vertices of the triangle determined by the point ll^* and the x points mentioned, can only intersect moreover in the singular points, we have for the determination of x, the relation $x^3 = x + 2 + 8^2 + 9 \times 4^2$; hence $x = 15$.

The transformation (X, X'), which replaces each point by the two points, which (X^3) associates to it, transforms therefore a straight line into a *curve of order fifteen with an octuple point and nine quadruple points.*

As l contains three pairs X, X', which supply six intersections with λ^{15}, the *curve of coincidences* δ is of *order nine.* Apparently δ^9 has a *quintuple point* in A and *nodes* in B_k.

With α^8, δ^9 has $5 \times 6 + 9 \times 4 = 66$ intersections in A and B_k; the remaining *six* are coincidences of the involution of pairs lying on α^8. Analogously we find that I^2 has *four* coincidences on β_k^4.

The supports d of the coincidences envelop a curve of the *tenth class* $(d)_{10}$, which has a quintuple point in A.

5. The locus of the pairs X', X'', which are collinear with a point E, is a curve ε^8, passing twice through E where it is touched by the lines to the points E' and E'', which form a triangle of involution with E. It is clear that ε^8 will pass three times through A and twice through each point B; it is consequently of class 30.

To the 26 tangents of ε^8, passing through E, belong 10 lines d; the remaining ones are represented by 8 bitangents, which are straight lines s.

If E is brought in A, then ε^8 passes into α^8. For a point B_k ε^8 consists of β_k^4 and a curve ε_k^4, which passes through A and the points B_l and has a node in B_k. The two curves have 14 intersections in the singular points; the remaining two are points E' and E'', belonging to $E \equiv B_k$. The 6 tangents passing through B_k at ε_k are supports of coincidences, the curve $(d)_{10}$, has B_k for *node*.

The curve ε^8 has with δ^9 51 intersections in A and B_k; of the remaining common points 10 lie in the coincidences mentioned above, of which the supports d pass through E. Consequently there lie on ε^8 11 coincidences $X \equiv X'$, of which the supports do not pass through E, whereas X' and X'' are collinear with E. These 11 points belong to the curve ε_*, which contains the points X, for which the line $x = X'X''$ passes through E. The curves ε^8 and ε_* also have the points E' and E'' in common, forming a triangle of involution with E. As E is collinear with 5 pairs of the I^2 lying on α^8 and with 2 pairs of the I^2 lying on B_k, ε_* passes five times through A and twice through B_k. Consequently ε^8 and ε_* have in all $3 \times 5 + 9 \times 2 + 13 = 64$ points in common; the locus of X is therefore a curve ε_*^8.

As E is collinear with 5 pairs[1]) X', X'' of a^8, and with two pairs of β_k^2, ε_*^8 has a quintuple point in A and nodes in B_k.

If E is brought in A, ε_*^8 coincides with a^8.

For B_1 ε_*^8 consists of the curve β_1^4 and a curve $*\beta_1^4$, which passes three times through A and once through the 8 points B_k.

The intersections X of ε_*^8 with the straight line l determine 8 lines $x = X'X''$ passing through E; we conclude from this that x envelops a *curve of the eighth class* $(l)_8$, when X describes the straight line l. In confirmation of this result we observe that with the 8 intersections X of l and a^8 correspond the 8 straight lines passing through $A(X'')$ to the associated points X'.

As $(l)_8$ must be *rational*, consequently possesses 21 *bitangents*, l contains 21 pairs X, Y, for which the corresponding points X', X''; Y', Y'' are collinear.

6. An arbitrary straight line contains three pairs (X', X''), (Y', Y''), (Z', Z'') of X^3; the corresponding points X, Y, Z apparently form a group of a *new triple involution*[2]), which we shall indicate by (XYZ); it appears to be of *class* 21.

Apparently (XYZ) has singular points in A and B_k. Let x be the order of the curve α, which contains the pairs Y, Z, belonging to $X \equiv A$; let further y be the order of the corresponding curve β_k belonging to B_k.

Let the straight line l be described by a point Z, the associated pair XY will then describe a curve λ, the order of which we shall indicate by z. If attention is paid to the points of intersection of l with α and β_k, it will be seen that λ must have an x-fold point in A, a y-fold point in B_k.

In order to determine the numbers x, y, z, we may obtain three equations.

We consider in the first place the intersections of the curves λ and μ, which are determined by the straight lines l and m. To them belong the two points which form a triplet with lm, further z points Z, for which X lies on l and Y on m; the remaining intersections lie in the singular points. So we have the relation

$$z^2 = 2 + z + x^2 + 9y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Let the curve a^8 be described by Z, then the figure of order $8z$,

[1]) The curves α^8 and ε^8 have $3 \times 5 + 9 \times 2 \times 2 = 51$ intersections in the singular points; they have 3 more points in common on EA; the remaining 10 intersections form 5 points X', X'' collinear with E. From this appears anew that the curve of involution α^3 is of class 5.

[2]) This property is characteristic of the triple involutions of the *third* class.

which is described by the pair X, Y, will be the combination of twice α^8, five times α^x and twice $\beta_k{}^y$.

Hence

$$8z = 16 + 5x + 18y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If Z describes the curve $^*\beta_1{}^4$, the corresponding figure of order $4z$ consists of the curve $\beta_1{}^4$, of three times α^x, and of the 8 curves $\beta_k{}^y$ $(k \neq 1)$. Hence:

$$4z = 4 + 3x + 8y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Out of (6), (7), (8) we find by elimination of x and y,

$$z^2 - 77z + 882 = 0;$$

so z is equal to 63 or 14. The second value, however, must be rejected; for we have proved above, that (XYZ) is of the class 21, so that l has 42 points in common with λ at the least. So we find the values

$$z = 63, \qquad x = 40, \qquad y = 16.$$

For the involution (XYZ), A is a *singular point* of order 40, B_k a *singular point* of order 16.

As l and λ besides the 21 pairs already mentioned can only have coincidences in common, the *curve of coincidences* (XYZ) is of *order* 21, δ^{21}.

Apparently α^{40} has in A a 20-fold point, $\beta_k{}^{16}$ in B_k an eight-fold point; in these points δ^{21} has the tangents in common with α^{40} and $\beta_k{}^{16}$.

If X is placed in A and Y in B_k, $x = X'X'$ envelops a curve of the 5th class, $y = Y'Y''$ a conic; so there are 10 straight lines $x = y$. From this it ensues that the singular curve α^{40} has ten-fold points in B_k. In a similar way we find that the curve $\beta_k{}^{16}$ has quadruple points in B_l; it passes ten times through A, eight times through B_k.

Mathematics. — "*On the functions of* HERMITE." (Third part). By Prof. W. KAPTEYN.

(Communicated in the meeting of May 30, 1914).

12. After having written the preceding pages, we met with two important, newly published papers, on the same subject. The first by Mr. H. GALBRUN: "Sur un développement d'une fonction à variable réelle en série de polynômes" (Bull. de la Soc. math. de France T. XLI p. 24), the second by Prof. K. RUNGE ' Ueber eine besondere Art von Integralgleichungen" (Math. Ann. Bd. 75 p. 130).