Citation:

Kapteyn, W., On the functions of Hermite. (third part), in:
KNAW, Proceedings, 17 I, 1914, Amsterdam, 1914, pp. 139-150

which is described by the pair X, Y, will be the combination of twice α^8, five times α^x and twice β_k^y.

Hence

$$8z = 16 + 5x + 18y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If Z describes the curve $^*\beta_1^4$, the corresponding figure of order $4z$ consists of the curve β_1^4, of three times α^x, and of the 8 curves β_k^y $(k \neq 1)$. Hence:

$$4z = 4 + 3x + 8y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Out of (6), (7), (8) we find by elimination of x and y,

$$z^2 - 77z + 882 = 0\,;$$

so z is equal to 63 or 14. The second value, however, must be rejected; for we have proved above, that (XYZ) is of the class 21, so that l has 42 points in common with λ at the least. So we find the values

$$z = 63, \qquad x = 40, \qquad y = 16.$$

For the involution (XYZ), A is a *singular point* of order 40, B_k a *singular point* of order 16.

As l and λ besides the 21 pairs already mentioned can only have coincidences in common, the *curve of coincidences* (XYZ) is of *order* 21, δ^{21}.

Apparently α^{40} has in A a 20-fold point, β_k^{16} in B_k an eight-fold point; in these points δ^{21} has the tangents in common with α^{40} and β_k^{16}.

If X is placed in A and Y in B_k, $x = X'X'$ envelops a curve of the 5th class, $y = Y'Y''$ a conic; so there are 10 straight lines $x = y$. From this it ensues that the singular curve α^{40} has ten-fold points in B_k. In a similar way we find that the curve β_k^{16} has quadruple points in B_l; it passes ten times through A, eight times through B_k.

Mathematics. — "*On the functions of* HERMITE." (Third part). By Prof. W. KAPTEYN.

(Communicated in the meeting of May 30, 1914).

12. After having written the preceding pages, we met with two important, newly published papers, on the same subject. The first by Mr. H. GALBRUN: "Sur un développement d'une fonction à variable réelle en série de polynômes" (Bull. de la Soc. math. de France T. XLI p. 24), the second by Prof. K. RUNGE ' Ueber eine besondere Art von Integralgleichungen" (Math. Ann. Bd. 75 p. 130).

In this section we will give their principal results though not altogether after their methods, and make some additional remarks.

13. Mr. Galbrun considers the question of the expansion of a function between the limits a and b, in a series

$$f(x) = A_0 H_0(x) + A_1 H_1(x) + \ldots$$

where

$$A_n = \frac{1}{2^n n!} \frac{1}{\sqrt{\pi}} \int_a^b e^{-\alpha^2} f(\alpha) H_n(\alpha) d\alpha.$$

He finds that this expansion is possible when $f(x)$ satisfies the conditions of Dirichlet between the limits a and b. This agrees with our result in Art. 7, the only difference being that our limits were $-\infty$ and $+\infty$. This difference however is not essential, for considering a function which has the value zero for all values $a > x > b$ Art. 7 gives immediately the expansion of Mr. Galbrun.

His proof rests on two interesting relations which may be easily deduced from the formulae in the first part of this paper.

The first relation

$$\sum_0^n \frac{H_p(x) H_p(\alpha)}{2^p p!} = \frac{1}{2^{n+1} n!} \frac{H_{n+1}(x) H_n(\alpha) - H_n(x) H_{n+1}(\alpha)}{x - \alpha} \quad (29)$$

may be established in this way.

According to (5) we have

$$2x H_n(x) = H_{n+1}(x) + 2n H_{n-1}(x)$$
$$2\alpha H_n(\alpha) = H_{n+1}(\alpha) + 2n H_{n-1}(\alpha) \qquad (n > 0)$$

Multiplying these equations by $H_n(a)$ and $H_n(x)$ we find by subtracting

$$2(x - \alpha) H_n(x) H_n(\alpha) = H_{n+1}(x) H_n(\alpha) - H_n(x) H_{n-1}(\alpha)$$
$$- 2n [H_n(x) H_{n-1}(\alpha) - H_{n-1}(x) H_n(\alpha)].$$

Hence, putting for n successively $0, 1, 2, \ldots n$, we get

1	$2(x-\alpha)H_0(x)H_0(\alpha) = H_1(x)H_0(\alpha) - H_0(x)H_1(\alpha)$
$\frac{1}{2 . 1!}$	$2(x-\alpha)H_1(x)H_1(\alpha) = H_2(x)H_1(\alpha) - H_1(x)H_2(\alpha) - 2[H_1(x)H_0(\alpha) - H_0(x)H_1(\alpha)]$
$\frac{1}{2^2 . 2!}$	$2(x-\alpha)H_2(x)H_2(\alpha) = H_3(x)H_2(\alpha) - H_2(x)H_3(\alpha) - 4[H_2(x)H_1(\alpha) - H_1(x)H_2(\alpha)]$
	. . . .
$\frac{1}{2^n . n!}$	$2(x-\alpha)H_n(x)H_n(\alpha) = H_{n+1}(x)H_n(\alpha) - H_n(x)H_{n+1}(\alpha) - 2n[H_n(x)H_{n-1}(\alpha) - H_{n-1}(x)H_n(\alpha)]$.

Multiplying these relations with the different factors written on the left, the addition of these products immediately gives the formula in question.

The second relation

$$\sum_1^\infty \frac{H_n(x)H_{n-1}(x)}{2^n \cdot n!} = e^{x^2}\int_0^x e^{-x^2}\,dx. \quad . \quad . \quad . \quad . \quad (30)$$

may be obtained by introducing (9) into the first member.

Thus we get

$$\sum_1^\infty \frac{H_n(x)H_{n-1}(x)}{2^n \cdot n!} =$$

$$= -\frac{e^{2x^2}}{\pi}\sum_1^\infty \frac{1}{2^n \cdot n!}\int_0^\infty e^{-\frac{u^2}{4}} u^n \cos\left(xu - \frac{n\pi}{2}\right)du \int_0^\infty e^{-v^2}v^{n-1}\sin\left(xv - \frac{n\pi}{2}\right)dv$$

where

$$\frac{1}{v}\sum_1^\infty \frac{u^n v^n}{2^n \cdot n!}\cos\left(xu - \frac{n\pi}{2}\right)\sin\left(xv - \frac{n\pi}{2}\right) =$$

$$= \frac{\cos xu \sin xv}{v}\sum_1^\infty \frac{u^{2k}v^{2k}}{2^{2k}.(2k)!} - \frac{\sin xu \cos xv}{v}\sum_1^\infty \frac{u^{2k+1}v^{2k+1}}{2^{2k+1}(2k+1)!}$$

$$= \frac{\cos xu \sin xv}{v}\left(\frac{e^{\frac{uv}{2}} + e^{-\frac{uv}{2}}}{2} - 1\right) - \frac{\sin xu \cos xv}{v}\left(\frac{e^{\frac{uv}{2}} - e^{-\frac{uv}{2}}}{2}\right).$$

Substituting this value, it is evident, according to the formulae of Art. 6, that all the terms of this sum vanish except only the term corresponding to —1.

Hence

$$\sum_1^\infty \frac{H_n(x)H_{n-1}(x)}{2^n \cdot n!} = \frac{e^{2x^2}}{\pi}\int_0^\infty\int_0^\infty e^{-\frac{u^2+v^2}{4}}\frac{\cos xu \sin xv}{v}\,du\,dv,$$

and because

$$\int_0^\infty e^{-\frac{u^2}{4}}\cos xu\,du = \sqrt{\pi}\,e^{-x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$\sum_1^\infty \frac{H_n(x)H_{n-1}(x)}{2^n \cdot n!} = \frac{1}{\sqrt{\pi}}e^{x^2}\int_0^\infty e^{-\frac{v^2}{4}}\frac{\sin xv}{v}\,dv\,.$$

If now we multiply the equation (a) by dx and integrate between 0 and x, we have

$$\int_0^\infty e^{-\frac{u^2}{4}} \frac{\sin xu}{u}\, du = \sqrt{\pi}\int_0^x e^{-x^2}\, dx,$$

thus finally

$$\sum_1^\infty \frac{H_n(x) H_{n-1}(x)}{2^n \cdot n!} = e^{x^2}\int_0^x e^{-x^2}\, dx.$$

14. Prof. Runge gives the solution of the integral equation

$$f(u) = \int_{-\infty}^{\infty} K(x)\, \varphi(u+x)\, dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where $f(u)$ and $K(x)$ are given functions and $\varphi(x)$ is required, by means of Hermite's functions.

He assumes

$$K(x) = e^{-x^2}[a_0 H_0(x) + a_1 H_1(x) + a_2 H_2(x) + \ldots]$$
$$\varphi(x) = e^{-x^2}[b_0 H_0(r) + b_1 H_1(x) + b_2 H_2(x) + \ldots]$$

which gives

$$f(u) = \Sigma\, a_m b_n \int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, e^{-(u+x)^2} H_n(u+x)\, dx$$

or, after some reductions

$$f(u) = \frac{\sqrt{\pi}}{\sqrt{2}} \sum_0^\infty (-1)^m a_m b_n\, e^{-\frac{u^2}{2}} \frac{H_{m+n}\left(\frac{u}{\sqrt{2}}\right)}{(\sqrt{2})^{m+n}}.$$

If now, the given function $f(u)$ is expanded in this form

$$f(u) = \frac{\sqrt{\pi}}{\sqrt{2}} e^{-\frac{u^2}{2}} \left[c_0 + c_1 \frac{H_1\left(\frac{u}{\sqrt{2}}\right)}{\sqrt{2}} + c_2 \frac{H_2\left(\frac{u}{\sqrt{2}}\right)}{(\sqrt{2})^2} + \ldots \right]$$

we have from (31)

$$c_0 = a_0 b_0, \quad c_1 = a_0 b_1 - a_1 b_0, \quad c_2 = a_0 b_2 - a_1 b_1 + a_2 b_0, \ldots$$

and it is evident that from these relations the coefficients b may be determined. If $f(u)$ and $\varphi(x)$ were the given functions, the same relations would be sufficient to determine the function $K(x)$.

15. The preceding reduction rests on the formula

$$H_n\left(\frac{x+y}{\sqrt{2}}\right) = \frac{1}{(\sqrt{2})^n}[H_n(x) + C_1^n H_{n-1}(x) H_1(y) + C_2^n H_{n-2}(x) H_2(y) + \ldots$$
$$+ C_n^n H_n(y)] \quad (32)$$

where C_i^h are the binomial coefficients. This relation may be obtained in the following way.

According to Art. 8 II we have

$$e^{-2hz-h^2} = 1 - h\,H_1(z) + \frac{h^2}{2!}H_2(z) - \frac{h^3}{3!}H_3(z) + \dots \quad , \quad (p)$$

and, expanding by TAYLOR's theorem

$$F(x+k,\, y+k)$$

where

$$F(x,y) = e^{-x^2-y^2}$$

$$e^{-(x+k)^2-(y+k)^2}.e^{x^2+y^2} = e^{x^2+y^2}\Big[e^{-x^2-y^2} + k\Big\{e^{-y^2}\frac{d}{dx}\Big(e^{-x^2}\Big) + e^{-x^2}\frac{d}{dy}\Big(e^{-y^2}\Big)\Big\}$$

$$+\frac{k^2}{2!}\Big\{e^{-y^2}\frac{d^2}{dx^2}\Big(e^{-x^2}\Big) + 2\frac{d}{dx}\Big(e^{-x^2}\Big)\frac{d}{dy}\Big(e^{-y^2}\Big) + e^{-x^2}\frac{d^2}{dx^2}\Big(e^{-y^2}\Big)\Big\} + .\Big]$$

which may be written

$$e^{-2kx-2ky-2k^2} = 1 - k[H_1(x)+H_1(y)] + \frac{k^2}{2!}[H_2(x)+2H_1(x)H_1(y)+H_2(y)] + \dots (q)$$

Putting now $z = \frac{x+y}{\sqrt{2}}$ in (p) and $k = \frac{h}{\sqrt{2}}$ in (q) we get

$$e^{-(x+y)h\sqrt{2}-h^2} = 1 - h\,H_1\left(\frac{x+y}{\sqrt{2}}\right) + \frac{h^2}{2!}H_2\left(\frac{x+y}{\sqrt{2}}\right) - \dots$$

$$e^{-(x+y)h\sqrt{2}-h^2} = 1 - \frac{h}{\sqrt{2}}[H_1(x) + H_1(y)] +$$

$$+ \frac{h^2}{(\sqrt{2})^2 2!}[H_2(x) + 2\,H_1(x)\,H_1(y) + H_2(y)] + \dots$$

Comparing the coefficients of $\frac{h^n}{n!}$ in the second members we obtain the required relation (31).

Proceeding to the reduction of the integral

$$M = \int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, e^{-(u+x)^2} H_n(u+x)\, dx$$

we put, according to (2)

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m}(e^{-x^2})$$

then

$$M = (-1)^m \int_{-\infty}^{\infty} \frac{d^m}{dx^m}(e^{-x^2})\,.\,e^{-(u+x)^2} H_n(u+x)\, dx.$$

Now, integrating by parts we have generally

$$\int U \frac{d^m V}{dx^m} dx = (-1)^m \int V \frac{d^m U}{dx^m} dx +$$
$$+ \left[U \frac{d^{m-1} V}{dx^m} - \frac{dU}{dx} \frac{d^{m-2} V}{dx^{m-2}} + \ldots + (-1)^{m-1} \frac{d^{m-1} U}{dx^{m-1}} V \right]$$

thus, assuming

$$U = e^{-(u+x)^2} H_n (u+x), \quad V = e^{-x^2}$$

and introducing the limits $-\infty$ and ∞

$$M = \int_{-\infty}^{\infty} e^{-x^2} \frac{d^m}{dx^m} [e^{-(u+x)^2} H_n (u+x)] \, dx$$

$$= (1)^n \int_{-\infty}^{\infty} e^{-x^2} \frac{d^{m+n}}{dx^{m+n}} (e^{-(u+x)^2}) \, dx$$

$$= (-1)^m \int_{-\infty}^{\infty} e^{-x^2-(u+x)^2} H_{m+n} (u+x) \, dx$$

or, adopting

$$u = v\sqrt{2} \qquad x = \frac{\xi - v}{\sqrt{2}}$$

$$M = \frac{(-1)^m}{\sqrt{2}} e^{-v^2} \int_{-\infty}^{\infty} e^{-\xi^2} H_{m+n} \left(\frac{\xi + v}{\sqrt{2}} \right) d\xi.$$

Applying now the relation (32), it is evident that the integral reduces to the first term, thus

$$M = \frac{(-1)^m}{\sqrt{2}} e^{-v^2} \frac{H_{m+n}(v)}{(\sqrt{2})^{m+n}}$$

or finally

$$M = \frac{(-1)^m}{\sqrt{2}} e^{-\frac{u^2}{2}} \frac{H_{m+n}\left(\frac{u}{\sqrt{2}}\right)}{(\sqrt{2})^{m+n}}.$$

16. We will now compare the preceding solution of the integral-equation (31) with the formal solution given by Prof. K. SCHWARZSCHILD Astr. Nachr. Bd. 185 N°. 4422).

Putting

$$t = e^{-u}, \quad s = e^{-x}$$

the equation

$$\int_0^{\infty} A(t\,.\,s)\, F(s)\, ds = B(t)$$

takes the form

$$\int_{-\infty}^{\infty} A\left(e^{-(u+x)}\right) F\left(e^{-x}\right) e^{-x}\, dx = B\left(e^{-u}\right)$$

or, assuming

$$e^{-x} F\left(e^{-x}\right) = K(x)$$
$$A\left(e^{-(u+x)}\right) = \varphi(u+x)$$
$$B\left(e^{-u}\right) = f(u)$$

$$\int_{-\infty}^{\infty} K(x)\, \varphi(u+x)\, dx = f(u).$$

Now SCHWARZSCHILD multiplies this equation by $e^{-i\lambda u}\, du$ and integrates between the limits $-\infty$ and $+\infty$, thus

$$\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du = \int_{-\infty}^{\infty} K(x)\, dx \int_{-\infty}^{\infty} \varphi(u+x)\, e^{-i\lambda u}\, du$$
$$= \int_{-\infty}^{\infty} K(x)\, e^{i\lambda x}\, dx \int_{-\infty}^{\infty} \varphi(v)\, e^{-i\lambda v}\, dv$$

and puts

$$f(u) = \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda u}\, du \quad \text{thus} \quad F(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

$$K(x) = \int_{-\infty}^{\infty} L(\lambda)\, e^{i\lambda x}\, dx \quad ,, \quad L(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(x)\, e^{-i\lambda x}\, dx$$

$$\varphi(v) = \int_{-\infty}^{\infty} \Phi(\lambda)\, e^{i\lambda v}\, dv \quad ,, \quad \Phi(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(v)\, e^{-i\lambda v}\, dv$$

therefore

$$F(\lambda) = 2\pi L(-\lambda)\, \Phi(\lambda)$$

or

$$L(\lambda) = \frac{1}{2\pi}\, \frac{F(-\lambda)}{\Phi(-\lambda)}$$

Multiplying again by $e^{i\lambda x}\, d\lambda$ and integrating between $-\infty$ and $+\infty$ this relation, he obtains

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(-\lambda)}{\Phi(-\lambda)}\, e^{i\lambda x}\, d\lambda.$$

If now we compare this result with the preceding, we have

$$F(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

or

$$F(\lambda) = \frac{1}{2\pi} \frac{\sqrt{\pi}}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-i\lambda u}\, e^{-\frac{u^2}{2}} \left(c_0 + c_1 \frac{H_1\left(\frac{u}{\sqrt{2}}\right)}{\sqrt{2}} + c_2 \frac{H_2\left(\frac{u}{\sqrt{2}}\right)}{(\sqrt{2})^2} + \cdot \right) du.$$

The general term in the series of the second member being

$$P_n = \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} H_n\left(\frac{u}{\sqrt{2}}\right) e^{-i\lambda u}\, du = \sqrt{2} \int_{-\infty}^{\infty} e^{-v^2} H_n(v)\, e^{-i\lambda v\sqrt{2}}\, dv$$

it is obvious that for $n = 2k$ the imaginary part and for $n = 2k+1$ the real part of this integral vanishes. Thus for $n = 2k$

$$P_{2k} = \sqrt{2} \int_{-\infty}^{\infty} e^{-v^2} H_{2k}(v) \cos(\lambda v \sqrt{2})\, dv\,,$$

where according to Art. 8 II

$$\cos(\lambda v \sqrt{2}) = e^{-\frac{\lambda^2}{2}} \sum_{0}^{\infty} (-1)^p \frac{\lambda^{2p}}{2^p (2p)!} H_{2p}(v)$$

thus

$$P_{2k} = (-1)^k\, 2^{\frac{2k+1}{2}} \sqrt{\pi}\, e^{-\frac{\lambda^2}{2}} \lambda^{2k}.$$

In the same way, we get

$$P_{2k+1} = -i(-1)^k\, 2^{k+1} \sqrt{\pi}\, e^{-\frac{\lambda^2}{2}} \lambda^{2k+1}$$

and therefore

$$F(\lambda) = \frac{e^{-\frac{\lambda^2}{2}}}{2\sqrt{2\pi}} \left[\Sigma \frac{c_{2k}}{2^k} P_{2k} - i\, \Sigma \frac{c_{2k+1}}{2^{\frac{2k+1}{2}}} P_{2k+1} \right]$$

$$= \frac{1}{2} e^{-\frac{\lambda^2}{2}} \left[\Sigma (-1)^k c_{2k} \lambda^{2k} - i\, \Sigma (-1)^k c_{2k+1} \lambda^{2k+1} \right].$$

In the same manner we find

$$\Phi(\lambda) = \frac{1}{2\sqrt{\pi}} e^{-\frac{\lambda^2}{4}} \left[\Sigma (-1)^k b_{2k} \lambda^{2k} - i\, \Sigma (-1)^k b_{2k+1} \lambda^{2k+1} \right]$$

and finally

$$K(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\lambda^2}{4}} \frac{\Sigma(-1)^k c_{2k}\lambda^{2k} + i\Sigma(-1)^k c_{2k+1}\lambda^{2k+1}}{\Sigma(-1)^k b_{2k}\lambda^{2k} + i\Sigma(-1)^k b_{2k+1}\lambda^{2k+1}} e^{i\lambda x}\, dx.$$

If now the conditions

$$c_0 = a_0 b_0 \qquad c_1 = a_0 b_1 - a_1 b_0 \qquad c_2 = a_0 b_2 - a_1 b_1 + a_2 b_0, ..$$

are satisfied, $K(x)$ must be reducible to

$$e^{-x^2}[a_0 H_0(x) + a_1 H_1(x) + a_2 H_2(x) + \ldots]$$

It is easy to show, that this is the case; for if the conditions are satisfied we have

$$\frac{\Sigma(-1)^k c_{2k}\lambda^{2k} + i\Sigma(-1)^k c_{2k+1}\lambda^{2k+1}}{\Sigma(-1)^k b_{2k}\lambda^{2k} + i\Sigma(-1)^k b_{2k+1}\lambda^{2k+1}} = a_0 - a_2\lambda^2 + a_4\lambda^4 \ldots - i(a_1\lambda - a_3\lambda^3 + \ldots)$$

thus

$$K(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\lambda^2}{4}} [\Sigma -1)^k a_{2k}\lambda^{2k} - i\Sigma(-1)^k a_{2k+1}\lambda^{2k+1}](\cos\lambda x + i \sin\lambda x) d\lambda$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\lambda^2}{4}} [\cos\lambda x\, \Sigma(-1)^k a_{2k}\lambda^{2k} + \sin\lambda x\, \Sigma(-1)^k a_{2k+1}\lambda^{2k+1}]\, d\lambda.$$

or, introducing (9)

$$H_{2k}(x)\, e^{-x^2} = \frac{(-1)^k}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\lambda^2}{4}} \lambda^{2k} \cos\lambda x\, d\lambda$$

$$H_{2k+1}(x)\, e^{-x^2} = \frac{(-1)^k}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\lambda^2}{4}} \lambda^{2k+1} \sin\lambda x\, d\lambda$$

$$K(x) = e^{-x^2}[a_0 H_0(x) + a_1 H_0(x) + a_2 H_2(x) + \ldots].$$

17. From the relation (32) another important result may be deduced. For multiplying by $e^{-y^2} dy$ and integrating between $-\infty$ and ∞, this relation gives

$$\int_{-\infty}^{\infty} e^{-y^2} H_n\left(\frac{x+y}{\sqrt{2}}\right) dy = \frac{\sqrt{\pi}}{(\sqrt{2})^n} H_n(x).$$

or, putting

$$x + y = a\sqrt{2}$$

$$\int_{-\infty}^{\infty} e^{-(x-a\sqrt{2})^2} H_n(a) da = \frac{\sqrt{\pi}}{(\sqrt{2})^{n+1}} H_n(x).$$

Therefore, assuming

$$\varphi_n(x) = \frac{1}{2^{\frac{n}{2}}\sqrt{n!}\sqrt[4]{\pi}} e^{-\frac{x^2}{2}} H_n(x)$$

we obtain

$$\varphi_n(x) = \frac{(\sqrt{2})^{n+1}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{3}{2}\left(x^2 - \frac{4}{3}\alpha x\sqrt{2} + \alpha^2\right)} \varphi_n(\alpha) d\alpha$$

thus, in the same way as in Art. 9

$$\lambda_n = (\sqrt{2})^{n+1}, \quad K(x,\alpha) = \frac{1}{\sqrt{\pi}} e^{-\frac{3}{2}\left(x^2 - \frac{4}{3}\alpha x\sqrt{2} + \alpha^2\right)}.$$

Here the value of the function $K(x \,.\, \alpha)$ is finite for x and $\alpha \pm \infty$. In the same manner as in Art. 9, therefore

$$K(x,\alpha) = \sum_0^\infty \frac{\varphi_n(x)\varphi_n)\alpha)}{\lambda_n}$$

or

$$e^{-(x^2 - 2x\alpha\sqrt{2} + \alpha^2)} = \sum_1^\infty \frac{H_n(x)\, H_n(\alpha)}{2^{\frac{3n+1}{2}} n!}$$

which may be verified by (9).

18. Now, according to the theory of the integral equations the determinant $D(\lambda)$ of the kernel $K(x,\alpha)$ must vanish for the values $\lambda = (\sqrt{2})^{n+1}$ $(n = 0, 1, 2 \ldots)$.

To examine this, we write $D(\lambda)$ in the form which is given by Plemelj [1])

$$\frac{D'(\lambda)}{D(\lambda)} = -(a_1 + a_2\lambda + a_3\lambda^2 + \ldots)$$

where

$$a_1 = \int_{-\infty}^{\infty} K(x,x)dx,\ a_2 = \int_{-\infty}^{\infty} K_1(x.x)dx,\ a_3 = \int_{-\infty}^{\infty} K_2(x\ x)dx, \ldots$$

$$K_n(x,\alpha) = \int_{-\infty}^{\infty} K(x,y)\, K_{n-1}(y,\alpha)\, dy \quad (n = 1,2.3 \ldots)$$

and

$$K_0(x.\alpha) = K(x.\alpha)$$

From $K(xy)$, which may be written

$$K(xy) = Ae^{-hx^2 + 2kxy - ly^2}$$

[1]) Monatshefte f. Math. und Phys. 1904 p 121.

the functions $K_n(xy)$ which have the same form

$$K_n(xy) = A_n e^{-h_n x^2 + 2k_n xy - l_n y^2},$$

may be easily deduced, for

$$K_n(x.\alpha) = AA_{n-1} \int_{-\infty}^{\infty} e^{-(l+h_{n-1})y^2 + 2(kx + k_{n-1}\alpha)y - (hx^2 + l_{n-1}\alpha^2)} dy$$

and

$$\int_{-\infty}^{\infty} e^{-fy^2+2gy-h}\, dy = e^{\frac{g^2-fh}{f}} \int_{-\infty}^{\infty} e^{-f\left(y-\frac{g}{f}\right)^2} dy$$

$$= e^{\frac{g^2-fh}{f}} \frac{\sqrt{\pi}}{\sqrt{f}}.$$

Hence

$$A_n\, e^{-h_n x^2 + 2k_n x\alpha - l_n \alpha^2} =$$

$$= \frac{A\, A_{n-1}}{\sqrt{h+h_{n-1}}} \sqrt{\pi}\; e^{-\left(h - \frac{k^2}{l+h_{n-1}}\right)x^2 + 2\frac{k k_{n-1}}{l+h_{n-1}} x\alpha - \left(l_{n-1} - \frac{k_{n-1}^2}{l+h_{n-1}}\right)\alpha^3}$$

which gives

$$A_n = \frac{A\, A_{n-1}}{\sqrt{h+h_{n-1}}} \sqrt{\pi},\ h_n = h - \frac{k^2}{l+h_{n-1}},\ k_n = \frac{k\, k_{n-1}}{l+h_{n-1}},\ l_n = l_{n-1} - \frac{k_{n-1}}{l+h_{n-1}}.$$

Now, we know

$$A = \frac{1}{\sqrt{\pi}},\ h = \frac{3}{2},\ k = \sqrt{2}, l = \frac{3}{2}$$

thus

$$A_1 = \frac{1}{\sqrt{3\pi}},\ h_1 = \frac{5}{6},\ k_1 = \frac{2}{3},\ l_1 = \frac{5}{6}$$

$$A_2 = \frac{1}{\sqrt{7\pi}},\ h_2 = \frac{9}{14},\ k_2 = \frac{2\sqrt{2}}{7},\ l_2 = \frac{9}{14}$$

$$A_3 = \frac{1}{\sqrt{15\pi}},\ h_3 = \frac{17}{30},\ k_3 = \frac{4}{15},\ l_3 = \frac{17}{30}$$

and

$$A_n = \frac{1}{\sqrt{(2^{n+1}-1)\pi}},\ h_n = \frac{2^{n+1}+1}{2(2^{n+1}-1)} = l_n,\ k_n = \frac{2^{\frac{n+1}{2}}}{2^{n+1}-1}.$$

This gives

$$a_{n+1} = \int_{-\infty}^{\infty} K_n(xx)\, dx = \frac{1}{\sqrt{(2^{n+1}-1)\pi}} \int_{-\infty}^{\infty} e^{-\frac{2^{\frac{n+1}{2}}-1}{2^{\frac{n+1}{2}}+1}x^2} dx = \frac{1}{2^{\frac{n+1}{2}}-1}.$$

Constructing now, according to WEIERSTRASS, an integral function $f(\lambda)$, with the assigned zeros

$$\lambda = \sqrt{2}, \quad \lambda = (\sqrt{2})^2, \quad \lambda = (\sqrt{2})^3 \ldots$$

we obtain

$$\frac{f(\lambda)}{f(0)} = e^{G(\lambda)} \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{(\sqrt{2})^{n+1}}\right)$$

or, assuming $f(0) = 1$, $G(\lambda) = 0$, $\frac{1}{\sqrt{2}} = r$

$$f(\lambda) = \prod_{n=0}^{\infty} (1 - \lambda r^{n+1}).$$

Thus

$$-\frac{f'(\lambda)}{f(\lambda)} = \frac{r}{1-r\lambda} + \frac{r^2}{1-r^2\lambda} + \frac{r^3}{1-r^3\lambda} + \ldots$$

and expanding the fractions of the second member

$$-\frac{f'(\lambda)}{f(\lambda)} = \sum_1^{\infty} r^p + \lambda \sum_1^{\infty} r^{2p} + \lambda^2 \sum_1^{\infty} r^{3p} + \ldots$$

Comparing this with

$$-\frac{D'(\lambda)}{D(\lambda)} = a_1 + a_2 \lambda + a_3 \lambda^2 + \ldots$$

we see that $f(\lambda) = D(\lambda)$, for $f(0) = D(0) = 1$ and

$$a_{n+1} = \sum_{p=1}^{\infty} r^{(n+1)p} = \frac{r^{n+1}}{1-r^{n+1}} = \frac{1}{2^{\frac{n+1}{2}}}.$$

Mathematics. — "*The theory of* BRAVAIS (*on errors in space*) *for polydimensional space, with applications to correlation.*" (Continuation). By Prof. M J. VAN UVEN. (Communicated by Prof. J. C. KAPTEYN.) [1])

(Communicated in the meeting of April 24, 1914).

In the theory of correlation the mean values of the products $x_j x_k$ are to be considered; denoting these by η_{jk}, we have

[1]) The list of authors who have treated upon the same subject, may be supplemented with CH. M. SCHOLS Theorie des erreurs dans le plan et l'espace. Annales de l'Ecole Polytechnique de Delft, t II (1886) p. 123.