Mathematics. — *"Combination of observations with and without conditions and determination of the weights of the unknown quantities, derived from mechanical principles.* By Prof. M. J. VAN UVEN. (Communicated by Prof. JAN DE VRIES).

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The theory of the combination of observations by the method of least squares has already been the object of numerous geometrical and mechanical illustrations. In the geometrical representations the leading part is usually played by vectors (L. VON SCHRUTKA [1]), C. RODRIGUEZ [2])); the mechanical ones are taken partly from the theory of the "pedal barycentre" (Y. VILLARCEAU [3]), M. D'OCAGNE [4])), partly from the theory of elasticity (S. FINSTERWALDER [5]) R. D'EMILIO [6]), S. WELLISCH, PANTOFLIČEK [7]), F. J. W. WHIPPLE [8]), M. WESTERGAARD [9]), G. ALBENGA [10])).

In the following paper we will try to develop a mechanical analogy of the solution of the equations furnished by observation, supposing that no conditions are added, as well as for the case that besides the *approximate* equations of condition (called by us:

[1]) L. VON SCHRUTKA. Eine vectoranalytische Interpretation der Formeln der Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Archiv der Math. u. Physik, 3. Reihe Bd. 21 (1913), p. 293.

[2]) C. RODRIGUEZ. La compensacion de los Errores desde el punto de vista geometrico Mexico, Soc. Cientif. "Antonio Alzate", vol. 33 (1913—1914), p. 57.

[3]) Y. VILLARCEAU. Transformations de l'astronomie nautique. Comptes Rendus, 1876 I, 531.

[4]) M. D'OCAGNE. Sur la détermination géométrique du point le plus probable donné par un système de droites non convergentes. Comptes Rendus, 1892 I, p. 1415. Journal de l'Ecole Polytechn Cah. 63 (1893), p. 1.

[5]) S. FINSTERWALDER Bemerkungen zur Analogie zwischen Aufgaben der Ausgleichungsrechnung und solchen der Statik. Sitzungsber. der K. B. Akad. d. Wissensch. zu München, Bd. 33 (1903), p. 683.

[6]) R. D'EMILIO. Illustrazioni geometriche e meccaniche del principio dei minimi quadrati. Atti d. R. Instituto Veneto di scienze, lettre ed arti, T. 62 (1902—1903), p. 363.

[7]) S. WELLISCH. Fehlerausgleichung nach der Theorie des Gleichgewichts elastischer Systeme. PANTOFLIČEK. Fehlerausgleichung nach dem Prinzipe der kleinsten Deformationsarbeit. Oesterr. Wochenschrift f. d. off. Baudienst, 1908, p. 428.

[8]) F. J. W. WHIPPLE. Prof. Bryan's mean rate of increase. A mechanical illustration. The mathematical Gazette, vol. 3 (1905), p. 173.

[9]) M. WESTERGAARD. Statisk Fejludjaevning. Nyt Tidsskrift for Matematik, B, T. 21 (1910), pp. 1 and 25.

[10]) G. ALBENGA. Compensazione grafica con la figura di errore (Punti determinati per intersezione). Atti d. R. Accad. d. Sc. di Torino, T. 47 (1912), p. 377.

"equations of observation") also *rigorous* equations of condition are given.

Moreover, in either of these cases also the weights of the unknown quantities will be derived from mechanical considerations.

The method here developed is founded on the statics of a point acted upon by elastic forces and is in principle closely related to the procedure of the last-mentioned mathematicians.

To obtain general results, we will operate with an arbitrary number (N) of unknown quantities or variables, which are considered as coordinates in N-dimensional space. In order to render the results more palpable, we shall, at the end, recapitulate them for the case of two variables.

I. To determine the N unknown quantities

$$x,\ y,\ z, \ldots \ (N)$$

the n (approximate) equations of condition or equations of observation

$$a_i x + b_i y + c_i z + \ldots + m_i = 0 \qquad (i = 1, \ldots n),$$

are given, with the weights g_i resp.

In the sums, frequently occurring in the sequel, we will denote by Σ a summation over the coordinates $x, y, z, \ldots$ or over the corresponding quantities (for inst. their coefficients $a_i, b_i, c_i, \ldots$) and by [ ] a summation over the n equations of observation, thus over i from 1 to n.

Putting accordingly

$$a_i^2 + b_i^2 + c_i^2 + \ldots = \Sigma a_i^2$$

and introducing

$$\alpha_i = \frac{a_i}{\sqrt{\Sigma a_i^2}},\ \beta_i = \frac{b_i}{\sqrt{\Sigma a_i^2}},\ \gamma_i = \frac{c_i}{\sqrt{\Sigma a_i^2}}, \ldots \mu_i = \frac{m_i}{\sqrt{\Sigma a_i^2}},$$

we may write the equations of observation in the following form

$$V_i \equiv \alpha_i x + \beta_i y + \gamma_i z + \ldots + \mu_i = 0 \qquad (i = 1, \ldots n)$$

or

$$V_i \equiv \Sigma \alpha_i x + \mu_i = 0 \qquad (i = 1, \ldots n).$$

These equations have resp. the weights

$$p_i = g_i \Sigma a_i^2.$$

The equations $V_i = 0$ represent ($N-1$)-dimensional linear spaces; their normals have the direction cosines $(\alpha_i, \beta_i, \gamma_i, \ldots)$ resp.

In consequence of the errors of observation, the approximate equations $V_i = 0$ are incompatible; in other words: the n linear spaces $V_i = 0$ do not meet in the same point. By substituting the coordinates $x, y, z, \ldots$ of an arbitrary point P in the expressions

V_i, the latter obtain the values v_i, representing the distances of the point P to the spaces $V_i = 0$.

The distance from $V_i = 0$ to P is to be considered as a vector $\mathfrak{v}_i$ with tensor v_i and direction cosines $\alpha_i, \beta_i, \gamma_i, \ldots$

We now imagine a force $\mathfrak{F}_i$ acting upon P (in N-dimensional space) in the direction of the normal $\mathfrak{v}_i$ (from P to $V_i = 0$) and the magnitude of which is proportional to the distance v_i and a factor p_i characteristic of the space V_i. (The space $V_i = 0$, for instance, may be considered as the position of equilibrium of a space $V_i = v_i$ passing through P by elastic flexion.)

So the space V_i acts upon P with the force

$$\mathfrak{F}_i = -p_i \mathfrak{v}_i.$$

All the spaces $V_i (i = 1, \ldots n)$ combined consequently exert on P a resultant force, amounting to

$$\mathfrak{F} = [\mathfrak{F}_i] = -[p_i \mathfrak{v}_i].$$

This resultant force depends on the position of the point P. Hence we have in N-dimensional space a vector-field $\mathfrak{F}$. determined by the above equation.

Now the question to be answered, is: at which point P are these forces $\mathfrak{F}_i$ in equilibrium? For this point P we have

$$\mathfrak{F} = 0$$

or

$$[p_i \mathfrak{v}_i] = 0.$$

The "components" of this vector-equation in the directions of the axes are

$$[p_i v_i \alpha_i] = 0, \quad [p_i v_i \beta_i] = 0, \quad [p_i v_i \gamma_i] = 0, \ldots \quad .$$

Substituting for v_i the expression $V_i = \Sigma\, \alpha_i x + \mu_i$, we obtain

$$[p_i \alpha_i^2]\, x + [p_i \alpha_i \beta_i]\, y + [p_i \alpha_i \gamma_i]\, z + \ldots + [p_i \alpha_i \mu_i] = 0,$$
$$[p_i \beta_i \alpha_i]\, x + [p_i \beta_i^2]\, y + [p_i \beta_i \gamma_i]\, z + \ldots + [p_i \beta_i \mu_i] = 0,$$
$$[p_i \gamma_i \alpha_i]\, x + [p_i \gamma_i \beta_i]\, y + [p_i \gamma_i^2]\, z + \ldots + [p_i \gamma_i \mu_i] = 0,$$

. . . .

or by

$$\alpha_i = \frac{a_i}{\sqrt{\Sigma a_i^2}},\ \beta_i = \frac{b_i}{\sqrt{\Sigma a_i^2}},\ \gamma_i = \frac{c_i}{\sqrt{\Sigma a_i^2}},\ \ldots\ \mu_i = \frac{m_i}{\sqrt{\Sigma a_i^2}},\ p_i = g_i\, \Sigma a_i^2,$$

$$[g_i a_i^2]\, x + [g_i a_i b_i]\, y + [g_i a_i c_i]\, z + \ldots + [g_i a_i m_i] = 0,$$
$$[g_i b_i a_i]\, x + [g_i b_i^2]\, y + [g_i b_i c_i]\, z + \ldots + [g_i b_i m_i] = 0,$$
$$[g_i c_i a_i]\, x + [g_i c_i b_i]\, y + [g_i c_i^2]\, z + \ldots + [g_i c_i m_i] = 0,$$

. . . .

In this way the "normal equations" are found.

The force $\mathfrak{F}_i = -p_i v_i$ has the potential

$$U_i = \tfrac{1}{2} p_i v_i^2 = \tfrac{1}{2} p_i V_i^2 ;$$

for

$$(F_i)_x = -\frac{\partial U_i}{\partial x} = -p_i V_i \frac{\partial V_i}{\partial x} = -p_i v_i a_i \text{ etc.}$$

The whole potential therefore amounts to

$$U = [U_i] = \tfrac{1}{2} [p_i V_i^2] .$$

As the equation $V_i \equiv \Sigma a_i x + \mu_i = 0$ has the weight p_i, the mean error of weight 1 is determined by

$$\varepsilon^2 = \frac{[p_i v_i^2]}{n-N},$$

hence

$$\varepsilon^2 = \frac{2U}{n-N}.$$

At the point P satisfying the normal equations the potential and consequently also ε^2 is a minimum. The "weight" of the distance v_i was p_i. This weight may be determined a posteriori, if we know the influence of the space V_i alone acting upon any point. We then have but to divide the amount F_i of the force $\mathfrak{F}_i$ by v_i.

II. In order to find the weights of the unknown quantities, we now remove the origin by translation to the point P, which satisfies the normal equations.

Calling the minimum potential U_0, denoting the new coordinates by $x', y', z', \ldots$ and introducing

$$V_i' \equiv a_i x' + \beta_i y' + \gamma_i z' + \ldots = \Sigma a_i x',$$

we obtain

$$[p_i V_i'^2] = 2(U - U_0) = 2U'.$$

So U' is the difference of potential existing between a point $(x', y', z', \ldots)$ and the *minimum point* P.

The equation $[p_i V_i'^2] = 2U'$ represents a quadratic $(N-1)$-dimensional space Ω, closed (ellipsoidal) and having P as centre. This space is an equipotential space and at the same time the locus of the points of equal ε. We shall call these spaces Ω briefly *hyper-ellipsoids*. The hyperellipsoids Ω are homothetic round P as centre of similitude.

Introducing the principal axes as axes of the coordinates $X, Y, Z, \ldots$, we obtain for Ω an equation of the form

$$AX^2 + BY^2 + CZ^2 + \ldots = 2U'.$$

The components of $\mathfrak{F}$ in the directions of the principal axes are found to be

$$F_X = -\frac{\partial U'}{\partial X} = -AX\,,\; F_Y = -\frac{\partial U'}{\partial Y} = -BY\,,\; F_Z = -\frac{\partial U'}{\partial Z} = -CZ, \text{ etc.}$$

We may therefore attribute these components to attractive forces of the spaces $X=0$, $Y=0$, $Z=0, \ldots$ (principal diametral spaces), which are perpendicular to these spaces and proportional to the "principal weights" $A, B, C, \ldots$

For a point on the principal axis of X holds

$$F_X = -AX \;,\; F_Y = 0 \;,\; F_Z = 0, \text{ etc.}$$

Consequently the principal weight A may be determined by dividing the force at a point of the principal axis of X by the distance X of that point to the principal diametral space $X=0$. To determine the weight of another direction $\mathfrak{l}$, only those points are required, at which the direction of the force coincides with the direction $\mathfrak{l}$, i. e. the points the normals of which to the hyperellipsoids Ω have the direction $\mathfrak{l}$. When dividing the amount of the force existing at such a point Q by the distance of the tangent space of Q to the centre P, the quotient found is equal to the weight of the given direction.

So, in order to determine the weight g_x of the direction of the original x'-axis (or of the x-axis), we only have to turn back to the coordinate system $x', y', z', \ldots$, relatively to which the equipotential spaces have the equation

$$[p_i V_i'^2] = 2U'.$$

For a point $Q(x', y', z', \ldots)$ at which the normal to the equipotential space, passing through Q, is parallel to the x'-axis (or to the x-axis), we have

$$F_{x'} = -g_x x' \;,\; F_{y'} = 0 \;,\; F_{z'} = 0, \text{ etc.}$$

or

$$\frac{\partial U'}{\partial x'} = g_x x' \;,\; \frac{\partial U'}{\partial y'} = 0 \;,\; \frac{\partial U'}{\partial z'} = 0, \text{ etc.}$$

hence

$$[p_i \alpha_i V_i'] = g_x x' \;,\; [p_i \beta_i V_i'] = 0 \;,\; [p_i \gamma_i V_i'] = 0, \text{ etc.}$$

or

$$[p_i \alpha_i^2]\, x' + [p_i \alpha_i \beta_i]\, y' + [p_i \alpha_i \gamma_i]\, z' + \ldots = g_x x',$$
$$[p_i \beta_i \alpha_i]\, x' + [p_i \beta_i^2]\, y' + [p_i \beta_i \gamma_i]\, z' + \ldots = 0,$$
$$[p_i \gamma_i \alpha_i]\, x' + [p_i \gamma_i \beta_i]\, y' + [p_i \gamma_i^2]\, z' + \ldots = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

or

$$[p_i \alpha_i^2]\frac{1}{g_x} + [p_i \alpha_i \beta_i]\frac{y'}{g_x x'} + [p_i \alpha_i \gamma_i]\frac{z'}{g_x x'} + \ldots - 1 = 0,$$

$$[p_i \beta_i \alpha_i]\frac{1}{g_x} + [p_i \beta_i^2]\frac{y'}{g_x x'} + [p_i \beta_i \gamma_i]\frac{z'}{g_x x'} + \ldots + 0 = 0,$$

$$[p_\iota \gamma_\iota \alpha_\iota]\frac{1}{g_x} + [p_\iota \gamma_\iota \beta_\iota]\frac{y'}{g_x x'} + [p_\iota \gamma_\iota^2]\frac{z'}{g_x x'} + \ldots + 0 = 0,$$

. . . .

or

$$[g_\iota a_\iota^2]\frac{1}{g_x} + [g_\iota a_\iota b_\iota]\frac{y'}{g_x x'} + [g_\iota a_\iota c_\iota]\frac{z'}{g_x x'} + \ldots - 1 = 0,$$

$$[g_\iota b_\iota a_\iota]\frac{1}{g_x} + [g_\iota b_\iota^2]\frac{y'}{g_x x'} + [g_\iota b_\iota c_\iota]\frac{z'}{g_x x'} + \ldots + 0 = 0,$$

$$[g_\iota c_\iota a_\iota]\frac{1}{g_x} + [g_\iota c_\iota b_\iota]\frac{y'}{g_x x'} + [g_\iota c_\iota^2]\frac{z'}{g_x x'} + \ldots + 0 = 0,$$

. . . .

So $\frac{1}{g_x}$ is apparently found as the first unknown quantity in the "modified" normal equations, modified in this way, that the constant terms are replaced by $-1,0,0,\ldots$ resp.

Considering U (c.q. U_ι) as an $(N+1)^{\text{th}}$ coordinate perpendicular to the N-dimensional space $(x, y, z, \ldots)$, the equation

$$p_\iota V_\iota^2 = 2U_\iota$$

represents a quadratic space of N dimensions, built up of ∞ $(N-1)$-dimensional linear generator-spaces, all parallel to $(V_\iota = 0,\ U = 0)$, the intersections of which with the planes perpendicular to $(V_\iota = 0,\ U = 0)$ are congruent parabolae. The parameter of these congruent parabolae is $\frac{1}{p_\iota}$.

The quadratic space $p_\iota V_\iota^2 = 2U_\iota$ will briefly be called a *parabolic cylindric space with parameter* $\frac{1}{p_\iota}$.

The equation

$$[p_\iota V_\iota^2] = 2\,U$$

represents a quadratic space Ψ of N dimensions, the centre of which is at $U = \infty$, and the intersections of which with the N-dimensional spaces $U = const.$ are hyperellipsoids Ω. Thus Ψ is the extension of the elliptic paraboloid.

The point T of Ψ with minimum U (U_0), and hence closest to $U = 0$, which is called the *summit* of Ψ, is projected on $U = 0$ in the point P, satisfying the normal equations.

By displacing the system of coordinate axes $(x, y, z, \ldots, U)$ (by translation) from O to T, Ψ obtains the equation

$$[p_\iota V_\iota'^2] = 2\,U' = 2\,(U - U_0).$$

By constructing the enveloping cylindric space, the vertex of which coincides with the set of points of the space $x = 0$ at infinity, thus the tangent cylindric space, the generator-spaces of which are parallel to the x-axis, we find for this cylindric space the equation

$$g_x x'^2 = 2\ U'.$$

Its parameter is $\frac{1}{g_x}$, or the reciprocal value of the weight of the direction x.

III. We now suppose, that the variables $x, y, z, \ldots$ must at the same time satisfy the following ν *rigorous* equations of condition

$$\Phi_j\,(x, y, z, \ldots) = 0 \qquad (j = 1, \ldots \nu)$$

Then the point P is constrained to the common $(N-\nu)$-dimensional space Φ of intersection of the $\nu\,(N-1)$-dimensional spaces Φ_j.

Now the point P, subjected to the elastic forces $\mathfrak{F}_i$, is in equilibrium, when the resultant $\mathfrak{F} = [\mathfrak{F}_i]$ is perpendicular to Φ.

Let the normal at P to Φ_j have the direction cosines

$$\alpha_j' = \frac{\frac{\partial \Phi_j}{\partial x}}{\sqrt{\Sigma\left(\frac{\partial \Phi_j}{\partial x}\right)^2}}\,,\quad \beta_j' = \frac{\frac{\partial \Phi_j}{\partial y}}{\sqrt{\Sigma\left(\frac{\partial \Phi_j}{\partial x}\right)^2}}\,,\quad \gamma_j' = \frac{\frac{\partial \Phi_j}{\partial z}}{\sqrt{\Sigma\left(\frac{\partial \Phi_j}{\partial x}\right)^2}}\,,\ \text{etc.}$$

The normals at P to the spaces Φ_j form a linear ν-dimensional space. In this space $\mathfrak{F}$ must lie, which means $\mathfrak{F}$ can be resolved in the directions of these normals, the unit-vectors of which will be denoted by $\mathfrak{w}_j$.

So we have

$$\mathfrak{F} = [q_j\,\mathfrak{w}_j]'$$

where $[\ ]'$ signifies the summation over j from 1 to ν.

The components of this vector-equation are

$$[p_i v_i \alpha_i] + [q_j\,\alpha_j']' = 0,\ [p_i v_i \beta_i] + [q_j\,\beta_j']' = 0,\ [p_i v_i \gamma_i] + [q_j\,\gamma_j']' = 0,\ \text{etc.}$$

or

$$[p_i \alpha_i^2]\,x + [p_i \alpha_i \beta_i]\,y + [p_i \alpha_i \gamma_i]\,z + \ldots + [p_i \alpha_i \mu_i] + [q_j\,\alpha_j']' = 0,$$
$$[p_i \beta_i \alpha_i]\,x + [p_i \beta_i^2]\,y + [p_i \beta_i \gamma_i]\,z + \ldots + [p_i \beta_i \mu_i] + [q_j\,\beta_j']' = 0,$$
$$[p_i \gamma_i \alpha_i]\,x + [p_i \gamma_i \beta_i]\,y + [p_i \gamma_i^2]\,z + \ldots + [p_i \gamma_i \mu_i] + [q_j\,\gamma_j']' = 0,$$

. . . .

Putting

$$q_j' = q_j \sqrt{\Sigma\left(\frac{\partial \Phi_j}{\partial x}\right)^2}, \quad (j = 1, \ldots \nu)$$

we may write the above equations in the form

$$[g_i a_i^2]\,x + [g_i a_i b_i]\,y + [g_i a_i c_i]\,z + \ldots + [g_i a_i m_i] + [q_j'\,\frac{\partial \Phi_j}{\partial x}\,]' = 0,$$

$$[g_i b_i a_i]\,x + [g_i b_i^2]\,y + [g_i b_i c_i]\,z + \ldots + [g_i b_i m_i] + [q_j' \frac{\partial \Phi_j}{\partial y}]' = 0,$$

$$[g_i c_i a_i]\,x + [g_i c_i b_i]\,y + [g_i c_i^2]\,z + \ldots + [g_i c_i m_i] + [q_j' \frac{\partial \Phi_j}{\partial z}]' = 0,$$

. . . .

These N equations serve, together with the ν conditions $\Phi_j = 0$, to determine the N variables $x, y, z, \ldots$ and the ν auxiliary quantities q_j'.

Now the solution of the problem is not represented by the centre of the hyperellipsoids Ω, but by the point, in which the intersection space Φ (space of conditions) is *touched* by an individual of the set of the hyperellipsoids Ω.

The analytical treatment of the problem is simplified by taking the coordinates so small, that in the expressions Φ_j homogeneous linear forms suffice. The geometrical meaning of this is that a new origin $O'(x_0, y_0, z_0, \ldots)$ is chosen in the space of conditions Φ near the probable position of the required point. So the spaces Φ_j are replaced by their tangent spaces R_j, and the space of conditions by its tangent space R of $N-\nu$ dimensions, intersection of the tangent spaces R_j.

Denoting the coordinates obtained by translation to O' by $\xi, \eta, \zeta, \ldots$, so that $x = x_0 + \xi, \ldots$ and putting

$$\alpha_i x_0 + \beta_i y_0 + \gamma_i z_0 + \ldots \mu_i = \overline{\mu}_i \;,\; a_i x_0 + b_i y_0 + c_i z_0 + \ldots + m_i = \overline{m}_i,$$

we find

$$2U = [p_i(\alpha_i x + \beta_i y + \gamma_i z \ldots + \mu_i)^2] = [p_i(\alpha_i \xi + \beta_i \eta + \gamma_i \zeta + \ldots + \overline{\mu}_i)^2]$$

or, putting

$$\alpha_i \xi + \beta_i \eta + \gamma_i \zeta + \ldots \overline{\mu}_i = \overline{V}_i,$$

$$2U = [p_i \overline{V}_i^2].$$

The equations $\Phi_j(x, y, z, \ldots) = 0$ may now be written:

$$\Phi_j(x_0, y_0, z_0, \ldots) + \left(\frac{\partial \Phi_j}{\partial x}\xi + \frac{\partial \Phi_j}{\partial y}\eta + \frac{\partial \Phi_j}{\partial z}\zeta + \ldots\right) + \ldots = 0$$

or, since O' is assumed in $\Phi_j = 0$, and higher powers of $\xi, \eta, \zeta, \ldots$ are to be neglected,

$$\frac{\partial \Phi_j}{\partial x}\xi + \frac{\partial \Phi_j}{\partial y}\eta + \frac{\partial \Phi_j}{\partial z}\zeta + \ldots = 0 \quad (j = 1, \ldots \nu)$$

or

$$W_j \equiv \alpha_j' \xi + \beta_j' \eta + \gamma_j' \zeta + \ldots = \Sigma\, \alpha_j' \xi = 0. \quad (j = 1, \ldots \nu).$$

So the normal equations appear in the following form

$$[g_i a_i^2]\,\xi + [g_i a_i b_i]\,\eta + [g_i a_i c_i]\,\zeta + \ldots + [g_i a_i \overline{m}_i] + [q_j \alpha_j']' = 0,$$

$$[g_i b_i a_i]\,\xi + [g_i b_i^2]\,\eta + [g_i b_i c_i]\,\zeta + \ldots + [g_i b_i \overline{m}_i] + [q_j \beta_j']' = 0,$$

$$[g_i c_i a_i]\,\xi + [g_i c_i b_i]\,\eta + [g_i c_i^2]\,\zeta + \ldots + [g_i c_i \overline{m_i}] + [q_j \gamma_j']' = 0,$$

. . . .

IV. To determine the weights of the directions $x, y, z, \ldots$, we again begin by shifting the origin (by translation) from O' to the point P, satisfying the normal equations and $W_j = 0$.

Calling U_0 the potential in P, $U - U_0 = U'$ the difference of potential relatively to P, $\xi', \eta', \zeta', \ldots$ the coordinates with respect to P, and putting finally

$$\alpha_i \xi' + \beta_i \eta' + \gamma_i \zeta' + \ldots = \overline{V}_i', \quad \alpha_j' \xi' + \beta_j' \eta' + \gamma_j' \zeta' + \ldots = W_j'$$

we find

$$2U' = [p_i \overline{V}_i'^2] - 2\,[q_j W_j']'.$$

This equation represents the set of equipotential spaces Ω. $U' = 0$ furnishes the hyperellipsoid Ω_0 touching Φ (or R) in P.

Now those points must be found, at which the force can only be resolved into an (inactive) component perpendicular to R and a component parallel to the x-axis.

For such a point we have

$$F_{\xi'} = -\frac{\partial U'}{\partial \xi'} = [r_j \alpha_j']' - g_\xi \xi',$$

$$F_{\eta'} = -\frac{\partial U'}{\partial \eta'} = [r_j \beta_j']' + 0,$$

$$F_{\zeta'} = -\frac{\partial U'}{\partial \zeta'} = [r_j \gamma_j']' + 0,$$

. . . .

or

$$[p_i \overline{V}_i' \alpha_i] - [q_j \alpha_j']' = -[r_j \alpha_j']' + g_\xi \xi',$$

$$[p_i \overline{V}_i' \beta_i] - [q_j \beta_j']' = -[r_j \beta_j']',$$

$$[p_i \overline{V}_i' \gamma_i] - [q_j \gamma_j']' = -[r_j \gamma_j']',$$

. . . .

or putting

$$r_j - q_j = s_j,$$

$$[p_i \alpha_i \overline{V}_i'] + [s_j \alpha_j']' = g_\xi \xi', \; [p_i \beta_i \overline{V}_i'] + [s_j \beta_j']' = 0, \; [p_i \gamma_i \overline{V}_i'] + [s_j \gamma_j']' = 0, \text{ etc.}$$

whence

$$[p_i \alpha_i^2]\,\xi' + [p_i \alpha_i \beta_i]\,\eta' + [p_i \alpha_i \gamma_i]\,\zeta' + \ldots + [s_j \alpha_j']' = g_\xi \xi',$$

$$[p_i \beta_i \alpha_i]\,\xi' + [p_i \beta_i^2]\,\eta' + [p_i \beta_i \gamma_i]\,\zeta' + \ldots + [s_j \beta_j']' = 0,$$

$$[p_i \gamma_i \alpha_i]\,\xi' + [p_i \gamma_i \beta_i]\,\eta' + [p_i \gamma_i^2]\,\zeta' + \ldots + [s_j \gamma_j']' = 0,$$

. . . .

or

$$[g_i a_i^2]\frac{1}{g_\xi} + [g_i a_i b_i]\frac{\eta'}{g_\xi\xi'} + [g_i a_i c_i]\frac{\zeta'}{g_\xi\xi'} + \ldots + \left[\frac{s_j}{g_\xi\xi'}\cdot\alpha_j'\right] - 1 = 0,$$
$$[g_i b_i a_i]\frac{1}{g_\xi} + [g_i b_i^2]\frac{\eta'}{g_\xi\xi'} + [g_i b_i c_i]\frac{\zeta'}{g_\xi\xi'} + \ldots + \left[\frac{s_j}{g_\xi\xi'}\,\beta_j'\right] + 0 = 0,$$
$$[g_i c_i a_i]\frac{1}{g_\xi} + [g_i c_i b_i]\frac{\eta'}{g_\xi\xi'} + [g_i c_i^2]\frac{\zeta'}{g_\xi\xi'} + \ldots + \left[\frac{s_j}{g_\xi\xi'}\cdot\gamma_j'\right] + 0 = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

the conditions

$$\alpha_j'\frac{1}{g_\xi} + \beta_j'\frac{\eta'}{g_\xi\xi'} + \gamma_j'\frac{\zeta'}{g_\xi\xi'} + \ldots = 0 \qquad (j = 1\ldots\nu).$$

also being satisfied.

From the above $N+\nu$ equations with the N unknown quantities $\frac{1}{g_\xi}, \frac{\eta'}{g_\xi\xi'}, \frac{\zeta'}{g_\xi\xi'}, \ldots$ and the ν unknown quantities $\frac{s_j}{g_\xi\xi'}, \frac{1}{g_\xi}$ can be solved.

The method of solution of HANSEN is found again by introducing

$$\frac{g_i\,\Sigma a_i\xi'}{g_\xi\xi'} = k_i \quad ; \quad \frac{s_j}{g_\xi\xi'} = k_j' \, ;$$
$$\frac{\xi'}{g_\xi\xi'} = \frac{1}{g_\xi} = A \quad , \quad \frac{\eta'}{g_\xi\xi'} = B \quad , \quad \frac{\zeta'}{g_\xi\xi'} = C, \ldots,$$

whence

$$\frac{k_i}{g_i} = \frac{\Sigma a_i\,\xi'}{g_\xi\xi'} = \Sigma a_i A \qquad (i = 1, \ldots n)$$

Then the modified normal equations furnish

$$[g_i a_i^2]\,A + [g_i a_i b_i]\,B + [g_i a_i c_i]\,C + \ldots + [k_j'\alpha_j']' = 1,$$
$$[g_i b_i a_i]\,A + [g_i b_i^2]\,B + [g_i b_i c_i]\,C + \ldots + [k_j'\beta_j']' = 0,$$
$$[g_i c_i a_i]\,A + [g_i c_i b_i]\,B + [g_i c_i^2]\,C + \ldots + [k_j'\gamma_j']' = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

or

$$[g_i a_i(\Sigma a_i A)] + [k_j'\alpha_j']' = 1, \quad [g_i b_i(\Sigma a_i A)] + [k_j'\beta_j']' = 0,$$
$$[g_i c_i(\Sigma a_i A)] + [k_j'\gamma_j']' = 0, \text{ etc.}$$

or

$[k_i a_i] + [k_j'\alpha_j']' = 1, \quad [k_i b_i] + [k_j'\beta_j']' = 0, \quad [k_i c_i] + [k_j'\gamma_j']' = 0,$ etc.,

and the (rigorous) equations of condition run

$$\Sigma\,\alpha_j'\,A = 0 \qquad (j = 1, \ldots \nu).$$

From the set of equations

$\Sigma a_i A = \frac{k_i}{g_i} \qquad (i = 1, \ldots n)$

$\Sigma \alpha_j' A = 0 \qquad (j = 1, \ldots \nu)$

$[k_i a_i]+[k_j'\alpha_j']'=1$, $[k_i b_i]+[k_j'\beta_j']'=0$, $[k_i c_i]+[k_j'\gamma_j']'=0$, etc. ($N$ in number)

the N variables $A, B, C, \ldots$, the n unknown quantities k_i and the ν auxiliary quantities k_j' can now be solved.

The weight of x is thus defined by

$$g_x = g_\xi = \frac{1}{A}.$$

It may also be found by the following calculation

$$\left[\frac{k_i^2}{g_i}\right] = [\Sigma k_i a_i A] = \Sigma A [k_i a_i] = A [k_i a_i] + B [k_i b_i] + C [k_i c_i] + \ldots$$

$$= A - A [k_j' \alpha_j']' - B [k_j' \beta_j']' - C [k_j' \gamma_j']' - \ldots$$

$$= A - [k_j' \Sigma \alpha_j' A]' = A = \frac{1}{g_\xi},$$

so that g_x is also determined by

$$g_x = \frac{1}{\left[\frac{k_i^2}{g_i}\right]}.$$

By considering the quantity U as $(N+1)^{\text{th}}$ coordinate perpendicular to the N-dimensional space $(x, y, z, \ldots)$, the equation

$$[p_i \overline{V}_i'^2] - 2 [q_j W_j']' = 2 U'$$

represents the quadratic space Ψ. The origin of the coordinates $\xi', \eta', \zeta', .. U'$ now lies at the point S, the projection of which on $U' = - U_0$ $(U = 0)$ is the required point. Now this point S is *not* the summit of Ψ.

The linear space of conditions R of $N - \nu$ dimensions is now joined to the point $U' = \infty$ by an $(N - \nu + 1)$-dimensional space R_1, which passes through S and intersects the quadratic space Ψ in a quadratic space Ψ_1 having the same character as Ψ, in that it also has its centre in $U' = \infty$, but is of fewer dimensions, viz. $N + (N - \nu + 1) - (N + 1) = N - \nu$. The quadratic space Ψ_1 has its summit in S.

We now have to determine the points Q in Ψ_1, at which the $((\nu+1)$-dimensional$)$ spaces of normals are parallel to the x-axis. In such a point Q Ψ_1 is also enveloped by a parabolic cylindric space, the generator-spaces of which are parallel to the x-axis, and which therefore has an equation of the form

$$g_x \xi'^2 = 2 U'.$$

Its parameter is $\frac{1}{g_x}$.

In other words: $\frac{1}{g_x}$ is the parameter of the parabolic cylindric space, which has its generator-spaces parallel to the x-axis and envelops the quadratic space Ψ_1.

V. We conclude this paper with a short summary of the results for the case of *two* variables x and y.

The equations of observation are represented by the straight lines

$$V_i \equiv \alpha_i x + \beta_i y + \mu_i = 0 \quad (\text{weight } p_i) \quad (i = 1, \dots n).$$

The point $P(x, y)$ is subjected to the force

$$\mathfrak{F} = [\mathfrak{F}_i] = -[p_i \mathfrak{v}_i]$$

in which $\mathfrak{v}_i$ represents, in amount and direction, the distance of the line $V_i = 0$ to the point P.

The point P remains at rest, if its coordinates satisfy the equations

$$[p_i \alpha_i^2] x + [p_i \alpha_i \beta_i] y + [p_i \alpha_i \mu_i] = 0,$$
$$[p_i \beta_i \alpha_i] x + [p_i \beta_i^2] y + [p_i \beta_i \mu_i] = 0.$$

Denoting here the potential U by z, we obtain

$$[p_i (\alpha_i x + \beta_i y + \mu_i)^2] = 2z.$$

This equation represents an elliptic paraboloid Ψ, being the sum-surface of the parabolic cylinders

$$p_i (\alpha_i x + \beta_i y + \mu_i)^2 = 2z_i,$$

which have the plan $z = 0$ as summit-tangent-plane along the generator $\alpha_i x + \beta_i y + \mu_i = 0$, $z = 0$, and which are obtained by translating the parabola

$$v_i^2 = \frac{2}{p_i} z_i,$$

lying in the normal plane of $V_i \equiv \alpha_i x + \beta_i y + \mu_i = 0$, perpendicularly to $V_i = 0$. The parameter of this parabola is $\frac{1}{p_i}$.

The summit T of the elliptic paraboloid Ψ ($[p_i V_i^2] = 2z$) is projected on $z = 0$ into the point P, satisfying the normal equations.

By constructing the tangent cylinder, the vertex of which lies upon the x-axis at infinity, we obtain a parabolic cylinder, the perpendicular transverse section of which has a parameter equal to the reciprocal value of the weight g_x of the variable x.

There being only two variables, only one (rigorous) equation of condition $\Phi(x, y) = 0$ may be added; $\Phi(x, y) = 0$ represents the curve to which the point P is constrained.

We now have to determine that particular ellipse of the homothetic set $[p_i V_i^2] = \text{const.}$, which touches the curve Φ The point of contact is the point P required.

In Φ, near the probable position of P, the new origin O' is taken. We have thus only to operate with linear functions of the coordinates So we really replace Φ by its tangent R at P.

The elliptic paraboloid Ψ is cut by the vertical of P in the point S The vertical plane R_1, which intersects $z = 0$ along R, pierces the paraboloid Ψ along the parabola Ψ_1, having S as summit.

We now construct the cylinder having its vertex at the point at infinity of the x-axis and having the parabola Ψ_1 as directrix (i.e. enveloping the parabola Ψ_1) The parameter (of the perpendicular transverse section) of this cylinder is the reciprocal value of the weight g_x of the variable x.

The equipotential lines in $z=0$ are the homothetic ellipses $[p_i V_i^2] = \text{const}$. Such an ellipse is the locus of the points of equal ε.

When the (rigorous) equation of condition is: $x = \text{const}$. the parabola Ψ_1 is parallel to the plane $x=0$ The tangent cylinder is then infinitely narrow; its parameter is 0, the weight of x is infinite.

Chemistry. — "*Equilibria in ternary systems.* XVI By Prof. F. A. H SCHREINEMAKERS.

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Now we shall consider the case that the vapour contains two components

We assume that of the components A, B, and C only the component B is exceedingly little volatile, so that practically we may say that the vapour consists only of A and C. This is for instance the case when B is a salt, which is not volatile, and when A and C are solvents, as water, alcohol, etc

Theoretically the vapour consists only of $A+B+C$, herein the quantity of B is however exceedingly small in comparison with the quantity of A and C. so that the vapour consists practically totally of A and C.

When, however, we consider complexes in the immediate vicinity of the point B, the relations become otherwise. The solid or liquid substance has viz. always a vapour-pressure, although this is sometimes immeasurably small, therefore, a vapour exists however, which consists only of B, without A and C. When we now take a liquid or a complex in the immediate vicinity of point B, the quantity of B in the vapour is then still also large and is not to be neglected in comparison with that of A and C.

Consequently, when we consider equilibria, not situated in the vicinity of point B, then we may assume that the vapour consists only of A and C, when these equilibria are situated, however, in the immediate vicinity of point B, we must also take into consideration the volatility of B and we must consider the vapour as ternary.

When we consider only the occurrence of liquid and gas, then, as we have formerly seen, three regions may occur, viz. the gas-region, the liquid-region and the region L—G. This last region is