Mathematics. — *"On* HERMITE'S *and* ABEL'S *polynomia."* By N. G. W. H. BEEGER. (Communicated by Prof. W. KAPTEYN).

(Communicated in the meeting of May 30, 1914).

Prof. KAPTEYN has deduced the following expansion [1]:

$$\frac{1}{\sqrt{(1-\theta^2)}} e^{\alpha^2-\frac{(\alpha-\theta x)^2}{1-\theta^2}} = \sum_0^\infty \frac{\theta^n H_n(x) H_n(\alpha)}{2^n \cdot n!} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $H_n(x)$ represent the polynomia of HERMITE. Let in this expansion $\alpha = 0$, then we find:

$$\frac{1}{\sqrt{(1-\theta^2)}} e^{-\frac{\theta^2 x^2}{1-\theta^2}} = \sum_0^\infty \frac{\theta^n H_n(x) H_n(o)}{2^n \cdot n!} \quad . \quad . \quad . \quad . \quad (2)$$

Now it holds good for the polynomia of HERMITE that:

$$H_{2n+1}(0) = 0 \qquad H_{2n}(0) = (-1)^n \frac{(2n)!}{n!} \sqrt{\pi} \quad . \quad . \quad . \quad (3)$$

On account of which the above relation passes into:

$$\frac{1}{\sqrt{(1-\theta^2)}} e^{-\frac{\theta^2 x^2}{1-\theta^2}} = \sum_0^\infty (-1)^n \frac{H_{2n}(x)}{2^{2n} \cdot n!} \theta^{2n} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For the polynomia $\varphi_n(x)$ of ABEL we know the expansion:

$$\frac{1}{1-\theta} e^{-\frac{x^2\theta}{1-\theta}} = \sum_0^\infty \varphi_n(x^2)\, \theta^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If we replace in (4) θ^2 by θ we find:

$$\frac{1}{\sqrt{(1-\theta)}} e^{-\frac{\theta x^2}{1-\theta}} = \sum_0^\infty (-1)^n \frac{H_{2n}(x)}{2^{2n} \cdot n!} \theta^n$$

If we multiply the first member of this relation by $\frac{1}{\sqrt{(1-\theta)}}$ and the second member by $\sum_0^\infty \frac{(2n)!}{2^{2n} \cdot (n!)^2} \theta^n$, the first member becomes equal to the first member of (5). By equalizing the coefficient of θ^n in the two second members, we find the following relation between the polynomia of ABEL and those of HERMITE:

$$\varphi_n(x^2) = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^{n-k} \frac{(2k)!}{(k!)^2 (n-k)!} H_{2n-2k}(x) \quad . \quad . \quad . \quad (6)$$

If we multiply both members of (6) by

$$H_{2n-2}(x)\, e^{-x^2}\, dx$$

[1] These Proceedings. Vol. XVI, p. 1198 (22).

and integrate between $-\infty$ and $+\infty$, then we find by application of the well-known integrals:

$$\int_{-\infty}^{+\infty} H_m(x)\, H_n(x)\, e^{-x^2}\, dx = 0 \qquad m \neq n$$

$$\int_{-\infty}^{+\infty} H_m^2(x)\, e^{-x^2}\, dx = 2^m \,.\, m!\, \sqrt{\pi}:$$

$$\int_{-\infty}^{+\infty} \varphi_n(x^2)\, H_{2n-2i}(x)\, e^{-x^2}\, dx = (-1)^n \frac{(2n-2i)!(2i)!}{(i!)^2(n-i)!2^{2i}} \sqrt{n} \,. \quad (7)$$

Prof. KAPTEYN deduces the following representation by means of an integral for HERMITE's polynomia [1]):

$$H_n(x) = \frac{e^{x^2}}{\sqrt{\pi}} \int_0^\infty e^{-\frac{u^2}{4}} u^n \cos\left(xu - \frac{n\pi}{2}\right) du.$$

If we substitute this expression in (6):

$$\varphi_n(x^2) = \frac{e^{x^2}}{2^{2n}\sqrt{\pi}} \int_0^\infty e^{-\frac{u^2}{4}} \sum_{k=0}^{n} (-1)^{n-k} \frac{(2k)!}{(k!)^2(n-k)!} u^{2n-2k} \cos\left(xu - (n-k)\,\pi\right) du$$

or, if we work out the cosine

$$\varphi_n(x^2) = \frac{e^{x^2}}{2^{2n}\sqrt{\pi}} \int_0^\infty e^{-\frac{u^2}{4}} \cos xu\, du \sum_{k=0}^{n} \frac{(2k)!}{(k!)^2(n-k)!} u^{2(n-k)} \quad . \quad . \quad (8)$$

Now is

$$(2k)! = \int_0^\infty e^{-y} y^{2k} dy$$

consequently

$$\left.\begin{aligned} &\sum_0^n \frac{(2k)!}{(k!)^2(n-k)!} u^{2(n-k)} = \sum_0^n \frac{u^{2(n-k)}}{(k!)^2(n-k)!} \int_0^\infty e^{-y} y^{2k} dy = \\ &= u^{2n} \int_0^\infty e^{-y} dy \sum_0^n \frac{1}{(k!)^2(n-k)!} \left(\frac{y}{u}\right)^{2k} = \frac{u^{2n}}{n!} \int_0^\infty e^{-y} dy \sum_0^n \frac{1}{k!} \binom{n}{k} \left(\frac{y}{u}\right)^{2k} \end{aligned}\right\} \quad (9)$$

For ABEL's polynomia we have:

$$\varphi_n\left(-\frac{y^2}{u^2}\right) = \sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} \left(\frac{y}{u}\right)^{2k}$$

1) l. c. p. 1194 (9).

so that we can write for (9)

$$\frac{u^{2n}}{n!}\int_0^\infty e^{-y}\varphi_n\left(-\frac{y^2}{u^2}\right)du$$

Substituted in (8) we get the double integral

$$\varphi_n(x^2)=\frac{e^{x^2}}{n!\,2^{2n}\sqrt{\pi}}\int_0^\infty e^{-\frac{u^2}{4}}u^{2n+1}\cos xu\,du\int_0^\infty e^{-ut}\varphi_n(-t^2)dt$$

if we introduce $y=ut$.

By substitution of $u=2y$ it passes into

$$\varphi_n(x^2)=\frac{4e^{x^2}}{n!\sqrt{\pi}}\int_0^\infty e^{-y^2}y^{2n+1}\cos 2xy\,dy\int_0^\infty e^{-2yt}\varphi_n(-t^2)dt\ .\ \ .\quad (10)$$

Now we make use of the relation [1]) also deduced by Prof. KAPTEYN

$$\int_0^\infty e^{-t}\frac{t^n}{(1+t)^{n+1}}dt=\int_0^\infty e^{-t}\frac{\varphi_n(t)}{1+t}dt\ .\ .\ .\ .\ .\quad (11)$$

In (10) we substitute $x=\sqrt{t}$ and then multiply both members by

$$\frac{1}{1+t}e^{-t}dt$$

and integrate between 0 and ∞, then we get by making use of (11).

$$\int_0^\infty e^{-t}\frac{t^n}{(1+t)^{n+1}}dt=\frac{4}{n!\sqrt{\pi}}\int_0^\infty e^{-y^2}y^{2n+1}dy\int_0^\infty e^{-2yu}\varphi_n(-u^2)du\int_0^\infty\frac{\cos 2y\sqrt{t}}{1+t}dt.$$

According to a well-known integral in the theory of the integral-logarithm, is [2])

$$\int_0^\infty\frac{\cos 2y\sqrt{t}}{1+t}dt=2\int_0^\infty\frac{x\cos 2yx}{1+x^2}dx=-e^{-2y}li_1(e^{2y})-e^{2y}li(e^{-2y})$$

consequently

$$\left.\begin{aligned}\int_0^\infty e^{-t}\frac{t^n}{(1+t)^{n+1}}dt=-\frac{4}{n!\sqrt{\pi}}\int_0^\infty e^{-y^2}y^{2n+1}\{e^{-2y}li_1(e^{2y})+e^{2y}li(e^{-2y})\}dy\ .\\ .\int_0^\infty e^{-2yu}\varphi_n(-u^2)du\end{aligned}\right\}.(12)$$

By summation from $n=0$ to $n=\infty$ we find:

1) l. c. XV, p 1250 (14).

2) See for instance "Theorie des Integrallogarithmus Dr. NIELSEN page 24.

$$\int\limits_0^{\vartheta} e^{-t}\,dt = -\frac{4}{\sqrt{\pi}}\int\limits^{\infty} e^{-y^2}\{e^{-2y}li_1(e^{2y})+e^{2y}li(e^{-2y})\}\,dy\int\limits_0^{\infty} e^{-2yu}du\sum_0^{\infty}\frac{y^{2n+1}}{n!}\varphi_n(-u^2)$$

Now is [1]):

$$\sum_0^{\infty}\frac{x^m}{m!}\varphi_m(a) = e^{a}J_0(2\sqrt{ax})$$

in which J_0 represents the function of BESSEL of order zero. From this it ensues consequently that

$$1 = -\frac{4}{\sqrt{\pi}}\int\limits_0^{\infty} y\{e^{-2y}li_1(e^{2y})+e^{2y}li(e^{-2y})\}\,dy\int\limits_0^{\infty} e^{-2yu}J_0(2iuy)\,du.$$

As is known,

$$J_0(2iuy) = \sum_{u=0}^{\infty}\frac{u^{2n}y^{2n}}{(n!)^2}$$

so

$$\int\limits_0^{\infty} e^{-2uy}J_0(2iuy)\,du = \sum_{n=0}^{\infty}\frac{y^{2n}}{(n!)^2}\int\limits_0^{\infty} e^{-2uy}u^{2n}du = \sum_{n=0}^{\infty}\frac{1}{y(n!)^2 2^{2n+1}}\int\limits_0^{\infty} e^{-z}z^{2n}\,dz =$$

$$= \sum_{n=0}^{\infty}\frac{(2n!)}{y(n!)^2 2^{2n+1}}.$$

Introducing this we have

$$1 = -\frac{4}{\sqrt{\pi}}\int\limits_0^{\infty}[e^{-2y}li_1(e^{2y}) + e^{2y}li(e^{-2y})]\,dy\,.\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2 2^{2n+1}}$$

or

$$\frac{1}{\sum\limits_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{2n+1}}} = -\frac{4}{\sqrt{\pi}}\int\limits_0^{\infty}[e^{-2y}li_1(e^{2y}) + e^{2y}li(e^{-2y})]\,dy \quad . \quad (13)$$

According to an integral used before, is

$$\int\limits_0^{\infty}[e^{-2y}li_1(e^{2y})+e^{2y}li(e^{-2y})]dy = -2\int\limits_0^{\infty} t\cos t\,dt\int\limits_0^{\infty}\frac{dy}{4y^2+t^2} = -\lim_{y=\infty}\int\limits_0^{\infty}\cos t\,\mathrm{arctg}\frac{2y}{t}\,dt\,.$$

Formula (13) may also be written as follows:

$$\frac{1}{\sum\limits_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{2n+1}}} = \frac{4}{\sqrt{\pi}}\lim_{y=\infty}\int\limits_0^{\infty}\cos t\,\mathrm{arctg}\frac{2y}{t}\,dt \quad . \quad . \quad (14)$$

By multiplying formula (11) by $\frac{x^n}{n!}$ and by summation from $n=0$ to $n=\infty$ we find:

[1]) These Proceedings XV, p 1246 (9).

$$\int_0^\infty \frac{e^{-t}}{1+t}\,dt \sum_0^\infty \frac{x^n}{n!}\left(\frac{t}{1+t}\right)^n = \int_0^\infty \frac{e^{-t}}{1+t}\,dt \sum_0^\infty \frac{x^n}{n!}\varphi_n(t) = \int_0^\infty \frac{e^{-t+x}}{1+t} J_0(2\sqrt{xt})$$

or

$$\int_0^\infty \frac{e^{-t^2+(x-1)t}}{1+t}\,dt = e^x \int_0^\infty \frac{e^{-t}}{1+t} J_0(2\sqrt{xt})\,dt \quad . \quad . \quad . \quad (15)$$

In order to deduce some more relations from formula (11) we set to work as follows. In Dr. NIJLAND's dissertation[1]) the following relation is deduced for ABEL's polynomia:

$$\varphi_n'(x) = -\sum_{k=0}^{n-1} \varphi_k(x)\,.$$

By summation of formula (11) from $n=0$ to $n-1$:

$$\int_0^\infty e^{-t} \sum_0^{n-1} \frac{t^n}{(1+t)^{n+1}}\,dt = \int_0^\infty \frac{e^{-t}}{1+t} \sum_0^{n-1} \varphi_n(t)\,dt$$

or

$$\int_0^\infty e^{-t} \left\{ \frac{t^n}{(t+1)^n} - 1 \right\} dt = \int_0^\infty \frac{e^{-t}}{1+t} \varphi_n'(t)\,dt$$

or

$$-1 + \int_0^\infty e^{-t} \frac{t^n}{(t+1)^n}\,dt = \int_0^\infty \frac{e^{-t}}{1+t} \varphi_n'(t)\,dt \quad . \quad . \quad . \quad . \quad (16)$$

We integrate the second member partially:

$$\int_0^\infty \frac{e^{-t}}{1+t} \varphi_n'(t)\,dt = \left| \varphi_n(t) \frac{e^{-t}}{1+t} \right|_0^\infty + \int_0^\infty \frac{(1+t)+1}{(1+t)^2} e^{-t} \varphi_n(t)\,dt =$$

$$= -1 + \int_0^\infty \frac{e^{-t}}{1+t} \varphi_n(t)\,dt + \int_0^\infty \frac{e^{-t}}{(1+t)^2} \varphi_n(t)\,dt\,.$$

Formula (16) passes into:

$$\int_0^\infty e^{-t} \frac{t^n}{(1+t)^n}\,dt = \int_0^\infty \frac{e^{-t}}{1+t} \varphi_n(t)\,dt + \int_0^\infty \frac{e^{-t}}{(1+t)^2} \varphi_n(t)\,dt$$

or by application of (11):

1) Over een bijzondere soort van geheele functiën. Utrecht. 1896 p. 19.

$$\int_0^\infty e^{-t}\frac{t^n}{(1+t)^n}\,dt = \int_0^\infty e^{-t}\frac{t^n}{(1+t)^{n+1}}\,dt + \int_0^\infty \frac{e^{-t}}{(1+t)^2}\,\varphi_n(t)\,dt.$$

The first integral of the second member we convey to the first member, and we find:

$$\int_0^\infty e^{-t}\frac{t^{n+1}}{(1+t)^{n+1}}\,dt = \int_0^\infty \frac{e^{-t}}{(1+t)^2}\,\varphi_n(t)\,dt \quad . \; . \; . \; . \; . \quad (17)$$

If we apply the same process to this, and again to the result, etc., we find at last after m-fold appliance:

$$\int_0^\infty e^{-t}\frac{t^{n+m}}{(1+t)^{n+1}}\,dt = m!\int_0^\infty \frac{e^{-t}}{(1+t)^{m+1}}\,\varphi_n(t)\,dt \quad . \; . \; . \; . \quad (18)$$

We can render this formula still more general by summation from $n=0$ to $n=\infty$ after division by $(-1)^m\, m!$; we get:

$$\int_0^\infty e^{-2t}\frac{t^n}{(1+t)^{n+1}}\,dt = \int_0^\infty \frac{e^{-t}}{t+2}\,\varphi_n(t)\,dt. \quad . \; . \; . \; . \; . \quad (19)$$

We apply the process explained above to this again and by summation again after division by $(-1)^m\, m!$ etc. we finally find:

$$\int_0^\infty e^{-kt}\frac{t^{n+m}}{(1+t)^{n+1}}\,dt = m!\int_0^\infty e^{-t}\frac{\varphi_n(t)}{(t+k)^{m+1}}\,dt. \quad . \; . \; . \; . \quad (20)$$

in which k and m represent positive integers.

Of course a formula analogous to (12) may be deduced from this

$$\left.\begin{array}{l} \displaystyle\int_0^\infty e^{-k^2u}\frac{u^n}{(1+u)^{n+1}}\,du = \\ \displaystyle -\frac{4}{n!\sqrt{\pi}}\int_0^\infty e^{-y^2}y^{2n+1}\left[e^{-2yk}li_1(e^{2yk})+e^{2yk}li(e^{-2yk})\right]dy\int_0^\infty e^{-2ty}\varphi_n(-t^2)\,dt. \end{array}\right\}(21)$$

By summation, formula (13) is, however, found again.

The formulae (4) and (5) may also be used in order to express the polynomia H_{2n} in φ's. For this purpose we multiply the two members of (5) by

$$\sqrt{(1-\theta)} = 1 - \frac{1}{2}\,\theta - \frac{1}{2!}\cdot\frac{1}{2^2}\,\theta^2 - \frac{1}{3!}\cdot\frac{1.3}{2^3}\,\theta^3 - \frac{1}{4!}\,\frac{1.3.5}{2^4}\,\theta^4 - \ldots$$

By equalizing the coefficient of θ^n in the second member of the

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equation, thus obtained, to the coefficient of θ^n in the second member of (4) we find:

$$H_{2n}(x)=(-1)^n 2^{2n}n!\left\{\varphi_n(x^2)-\frac{1}{2}\varphi_{n-1}(x^2)-\sum_{k=2}^{n}\frac{1}{k!}\cdot\frac{1.3.5...(2k-3)}{2^k}\varphi_{n-k}(x^2)\right\} \quad (22)$$

By means of this expression an integral may be deduced.

For if we multiply both members by

$$e^{-x}\varphi_m(x)\,dx$$

after replacing x^2 by x and if we then integrate between 0 and ∞, we find, using the following well-known formulae[1]:

$$\int_0^\infty e^{-x}\varphi_m(x)\,\varphi_n(x)\,dx=0 \quad m\neq n$$

$$\int_0^\infty e^{-x}\varphi_m^2(x)\,dx=1 \quad :$$

$$\int_0^\infty e^{-x}\varphi_m(x)\,H_{2n}(\sqrt{x})\,dx=(-1)^{n+1}\,2^{2n}\,.\,n!\,\frac{1}{(n-m)!}\cdot\frac{1.3.5\,..(2n-2m-3)}{2^{n-m}}$$

or after some reduction:

$$\int_0^\infty e^{-x^2}\varphi_m(x^2)H_{2n}(x)x\,dx=(-1)^{n+1}2^{n+m-1}\frac{n!}{(n-m)!}\,1.3.5..(2n-2m-3) \quad (23)$$

$$m<n-1\,.$$

In the same way we find

$$\int_0^\infty e^{-x^2}\varphi_n(x^2)\,H_{2n}(x)\,x\,dx=(-1)^n\,.\,2^{2n-1}\,n! \quad . \quad . \quad (24)$$

and

$$\int_0^\infty e^{-x^2}\varphi_{n-1}(x^2)\,H_{2n}(x)\,x\,dx=(-1)^{n-1}\,.\,2^{2n-2}\,.\,n! \quad . \quad (25)$$

If we write formula (22) in this form:

$$H_{2n}(x)=(-1)^n 2^{2n}\,.\,n!\left\{\varphi_n(x^2)-\tfrac{1}{2}\varphi_{n-1}(x^2)-\frac{1}{2\sqrt{\pi}}\sum_{k=2}^{n}\frac{\Gamma(k-\frac{1}{2})}{k!}\varphi_{n-k}(x^2)\right\}$$

we know that[2])

$$\varphi_{n-k}(x^2)=\frac{e^{x^2}}{(n-k)!}\int_0^\infty e^{-\alpha}\alpha^{n-k}J_0(2x\sqrt{\alpha})d\alpha.$$

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1) Dr NIJLAND's Dissertation page 11.

2) These Proceedings XV. p. 1247.

This we substitute·

$$H_{2n}(x)=(-1)^n\,.\,2^{2n}\{n!\varphi_n(x^2)-\tfrac{1}{2}n!\varphi_{n-1}(x^2)-\frac{1}{2\sqrt{\pi}}e^{x^2}\int_0^\infty e^{-\alpha}J_0(2x\sqrt{\alpha})d\alpha.$$

$$.\sum_{k=2}^{n}\frac{n!}{k!(n-k)!}\Gamma(k-\tfrac{1}{2})\alpha^{n-k}.$$

We further introduce:

$$\Gamma(k-\tfrac{1}{2})=\int_0^\infty e^{-t}t^{k-\frac{3}{2}}\,dt$$

and

$$\tfrac{1}{2}\,n!\,\varphi_{n-1}(x^2)=\frac{e^{x^2}}{2\sqrt{\pi}}\int_0^\infty e^{-\alpha}J_0\,(2x\sqrt{\alpha})d\alpha\,.\,n\alpha^{n-1}\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt.$$

We find then:

$$H_{2n}(x)=(-1)^n2^{2n}\Bigg\{n!\varphi_n(x^2)-\frac{e^{x^2}}{2\sqrt{\pi}}\int_0^\infty e^{-\alpha}J_0(2x\sqrt{\alpha})d\alpha\int_0^\infty e^{-t}\,t^{-\frac{3}{2}}\sum_{k=1}^{n}\binom{n}{k}\alpha^{n-k}t^k$$

or after some reduction

$$\left.\begin{aligned}&(-1)^{n-1}\frac{1}{2^{2n}}H_{2n}(x)+n!\,\varphi_n(x^2)=\\&=\frac{e^{x^2}}{2\sqrt{\pi}}\int_0^\infty e^{-\alpha}J_0\,(2x\sqrt{\alpha})\,d\alpha\int_0^\infty e^{-t}t^{-\frac{3}{2}}\Big\{(\alpha+t)^n-\alpha^n\Big\}\,dt\end{aligned}\right\}\quad(26)$$

For $x=0$, the following identities arise from the formulae (6) and (22):

$$2^{2n}=\sum_{k=0}^{n}\frac{(2k)!(2n-2k)!}{k!^2(n-k)!^2}\quad\text{or}\quad\frac{n!^2}{(2n)!}2^{2n}=\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{\binom{2n}{2k}}\quad.\quad.\quad(27)$$

and

$$\frac{(2n)!}{n!}=2^{2n}\,.\,n!\left\{\frac{1}{2}-\sum_{k=2}^{n}\frac{1}{k!}\frac{1.3\,5\,..(2k-3)}{2^k}\right\}$$

or

$$\frac{1}{\dfrac{n!^2}{(2n)!}2^{2n}}=\frac{1}{2}-\sum_{k=2}^{n}\frac{1}{k!}\frac{1.3.5\ldots(2k-3)}{2^k}\quad.\quad.\quad.\quad.\quad(28)$$

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