of the mouth of the Nanga Koli A. F. H. HEUSCH collected in 1890 according to PANNEKOEK [1]) quartzite and greywakke(?). On the south-coast at Nanga Mbawa I found granite (read quartz-diorite) and gabbro [2]) and finally in the valley of the river Ndona quartz-diorite. None of these rocks were ever found as rock, they are consequently at least "auf tertiärer Lagerstätte". They are the last remains of rockmasses that got into the conglomerates by washing from which they got afterwards free again.

The oldest formations of Flores belong to the effusive rocks of the character of dacites, labrador-andesites and hornblende-andesites with their tufas, on which those of the limestones of the Reo-formation and those of the tuffas of the Soa formation follow. They were uncovered by subsequent elevation. Afterwards the island was over its entire length the scene of violent volcanic eruptions, from which the only partly known crater mountains proceeded. The material produced by them consists — as far as our knowledge reaches — exclusively of pyroxene-andesites belonging to the Pacific type of rocks. The younger coral limestones occurring only sporadically have only been formed after the formation of the volcanoes.

VON SCHELLE's postulation that the bottom "to the north of Mount Rokka is very rich in tin-ore" appears to have been not only vain but also very expensive.

Mathematics. — "*The theory of the combination of observations and the determination of the precision, illustrated by means of vectors.*" By Dr. M. J. VAN UVEN. (Communicated by Prof. W. KAPTEYN),"

(Communicated in the meeting of June 27, 1914).

By L. VON SCHRUTKA [3]) and C. RODRIGUEZ [4]) a method has been given of illustrating geometrically the theory of the combination of observations by the method of least squares, namely by means of vector operations. RODRIGUEZ however chooses in the case of rigorous equations of condition another way, whilst VON SCHRUTKA, who consistently

[1]) Overzicht der geographische en geologische gegevens l. c. p. 229.

[2]) l. c. p. 229.

[3]) L. VON SCHRUTKA, Eine vektoranalytische Interpretation der Formeln der Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Archiv der Mathematik und Physik. 3, Reihe, Bd. 21, (1913), p. 293.

[4]) C. RODRIGUEZ, La compensacion de los Errores desde al punto de visto geometrico. Mexico, Soc. Cient. "Antonio Alzate", vol. 33 (1913—1914), p. 57.

operates with vectors, restricts himself to two variables and one rigorous equation of condition.

It is our purpose not only to extend their method to the case of an arbitrary number (N) of variables and an equally arbitrary number (ν) of conditions, but also to derive the *weight* of the unknown quantities in the same way.

I. There are given N quantities $x, y, z, \ldots$ which are to be determined from n (approximate) equations of condition (equations of observation):

$$a_i x + b_i y + c_i z + \ldots + m_i = 0 \qquad i = 1, \ldots n.$$

These equations have the weights g_i resp., and so are equivalent to the equations

$$a_i \sqrt{g_i} \,.\, x + b_i \sqrt{g_i} \,.\, y + c_i \sqrt{g_i} \,.\, z + \ldots + m_i \sqrt{g_i} = 0 \qquad i = 1, \ldots n,$$

each of which has the weight unity.

We now introduce

$$\alpha_i = \frac{a_i \sqrt{g_i}}{\sqrt{[g_i a_i^2]}}, \quad \beta_i = \frac{b_i \sqrt{g_i}}{\sqrt{[g_i b_i^2]}}, \quad \gamma_i = \frac{c_i \sqrt{g_i}}{\sqrt{[g_i c_i^2]}}, \ldots \mu_i = \frac{m_i \sqrt{g_i}}{\sqrt{[g_i m_i^2]}};$$

$$A = x \sqrt{[g_i a_i^2]}, \quad B = y \sqrt{[g_i b_i^2]}, \quad C = z \sqrt{[g_i c_i^2]}, \ldots M = \sqrt{[g_i m_i^2]}$$

$$A_i = A\alpha_i = a_i \sqrt{g_i} \,.\, x, \quad B_i = B\beta_i = b_i \sqrt{g_i} \,.\, y, \quad C_i = C\gamma_i = c_i \sqrt{g_i} \,.\, z, \ldots$$

$$\ldots M_i = M\mu_i = m_i \sqrt{g_i},$$

[ ] denoting summation over i from 1 to n.

So the equations of observation run in the form

$$A_i + B_i + C_i + \ldots M_i = 0 \qquad i = 1, \ldots n$$

We now consider $A_i, B_i, C_i, \ldots M_i$ as the components of the vectors $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, .. $\mathfrak{M}$, resolved parallel to the rectangular coordinate axes of an n-dimensional space. Thus the tensors are $A, B, C, \ldots M$, $\alpha_i, \beta_i, \gamma_i, \ldots \mu_i$ representing the direction cosines.

The set of n equations of observation may now be condensed in the single vector-equation

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \ldots + \mathfrak{M} = 0,$$

which expresses, that the vectors $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots \mathfrak{M}$ must form a closed polygon. The coefficients $a_i, b_i, c_i, \ldots$ and the weights g_i being given, the unit vectors $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ of the vectors $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ are determinate. So the vector-equation requires that $\mathfrak{M}$ may be resolved in the N directions $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$, in other words: that $\mathfrak{M}$ lies in the N-dimensional space R_N, determined by the vectors $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ and called the space of the variables (or unknown quantities).

In consequence of the errors of observation this condition is not fulfilled. The most probable corrected value of $\mathfrak{M}$ is the projection of $\mathfrak{M}$ on the space R_N of the variables.

Denoting the projecting vector by $\mathfrak{K}$ (tensor K, direction cosines $\varkappa_i$, components K_i) we have really

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \ldots + \mathfrak{M} = \mathfrak{K}.$$

As $\mathfrak{K}$ is perpendicular to $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$, we have

$$(\mathfrak{A}, \mathfrak{K}) = 0, \qquad (\mathfrak{B}, \mathfrak{K}) = 0, \qquad (\mathfrak{C}, \mathfrak{K}) = 0, \text{ etc}$$

or

$$[\alpha_i K_i] = 0, \qquad [\beta_i K_i] = 0, \qquad [\gamma_i K_i] = 0, \text{ etc.}$$

or because

$$K_i = A_i + B_i + C_i + \ldots + M_i = \alpha_i A + \beta_i B + \gamma_i C + \ldots + M_i,$$

$$[\alpha_i^2] A + [\alpha_i \beta_i] B + [\alpha_i \gamma_i] C + \ldots + [\alpha_i M_i] = 0,$$
$$[\beta_i \alpha_i] A + [\beta_i^2] B + [\beta_i \gamma_i] C + \ldots + [\beta_i M_i] = 0,$$
$$[\gamma_i \alpha_i] A + [\gamma_i \beta_i] B + [\gamma_i^2] C + \ldots + [\gamma_i M_i] = 0,$$
$$\cdots\cdots\cdots\cdots\cdots$$

By multiplying these equations by $\sqrt{[g_i a_i^2]}$, $\sqrt{[g_i b_i^2]}$, $\sqrt{[g_i c_i^2]}$, ... resp., we obtain the "normal equations":

$$[g_i a_i^2]\, x + [g_i a_i b_i]\, y + [g_i a_i c_i]\, z + \ldots + [g_i a_i m_i] = 0,$$
$$[g_i b_i a_i]\, x + [g_i b_i^2]\, y + [g_i b_i c_i]\, z + \ldots + [g_i b_i m_i] = 0,$$
$$[g_i c_i a_i]\, x + [g_i c_i b_i]\, y + [g_i c_i^2]\, z + \ldots + [g_i c_i m_i] = 0,$$
$$\cdots\cdots\cdots\cdots\cdots$$

II. After these developments which also are given by VON SCHRUTKA and RODRIGUEZ we proceed to determine the *weights* of the variables.

For this we notice that all the quantities M_i have the weight 1, and therefore have an equal mean error ε. From this ensues, that the projection of $\mathfrak{M}$ in any direction has the same mean error ε.

We have to investigate the influence on $\mathfrak{A}$ due to the variation of $\mathfrak{M}$, if the other variables $\mathfrak{B}, \mathfrak{C}, \ldots$ do not undergo that influence.

A variation of $\mathfrak{M}$ which does not displace the foot on R_N of the projecting vector $\mathfrak{K}$, does not act upon any vector $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ So we have only to do with a variation of the projection $\mathfrak{M}'$ of $\mathfrak{M}$ on R_N. In order to leave the vectors $\mathfrak{B}, \mathfrak{C}, \ldots$ intact, the foot is to be moved in a direction $\mathfrak{s}$ perpendicular to $\mathfrak{B}, \mathfrak{C}, \ldots$, and, because it lies in R_N, also perpendicular to $\mathfrak{K}$.

Denoting by σ_i the direction cosines of $\mathfrak{s}$, we may put the equation

$$(\mathfrak{A}, \mathfrak{s}) + (\mathfrak{M}, \mathfrak{s}) = 0,$$

obtained by multiplying the equation of observation scalarly with $\mathfrak{s}$, in the form

$$A\,[\alpha_i \sigma_i] = -M_s$$

M_s designating the projection of $\mathfrak{M}$ on $\mathfrak{s}$.

As M_s has the mean error ε, the mean error ε_A of A equals

$$\varepsilon_A = \frac{\varepsilon}{[\alpha_i \sigma_i]},$$

whence

$$g_A = [\alpha_i \sigma_i]^2.$$

The vector $\mathfrak{s}$, lying in R_N, may be resolved in the directions $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ Denoting its components in these directions by $X, Y, Z, \ldots$ we find

$$\mathfrak{s} = X\mathfrak{a} + Y\mathfrak{b} + Z\mathfrak{c} + \ldots,$$

or

$$\sigma_i = X\alpha_i + Y\beta_i + Z\gamma_i + \ldots$$

Now, $\mathfrak{s}$ being perpendicular to $\mathfrak{B}, \mathfrak{C}, \ldots$, whence $[\beta_i \sigma_i] = 0, [\gamma_i \sigma_i] = 0, \ldots$, we have

$$1 = [\sigma_i^2] = X [\alpha_i \sigma_i]$$

or

$$X = \frac{1}{[\alpha_i \sigma_i]}.$$

From the equations

$$[\alpha_i \sigma_i] = \frac{1}{X}, \quad [\beta_i \sigma_i] = 0, \quad [\gamma_i \sigma_i] = 0, \ldots$$

which may also be written

$$\begin{array}{l} [\alpha_i^2]\ X + [\alpha_i \beta_i]\ Y + [\alpha_i \gamma_i]\ Z + \ldots = \frac{1}{X}, \\ [\beta_i \alpha_i]\ X + [\beta_i^2]\ Y + [\beta_i \gamma_i]\ Z + \ldots = 0, \\ [\gamma_i \alpha_i]\ X + [\gamma_i \beta_i]\ Y + [\gamma_i^2]\ Z + \ldots = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

or

$$\begin{array}{l} [\alpha_i^2]\ X^2 + [\alpha_i \beta_i]\ XY + [\alpha_i \gamma_i]\ XZ + \ldots - 1 = 0, \\ [\beta_i \alpha_i]\ X^2 + [\beta_i^2]\ XY + [\beta_i \gamma_i]\ XZ + \ldots + 0 = 0, \\ [\gamma_i \alpha_i]\ X^2 + [\gamma_i \beta_i]\ XY + [\gamma_i^2]\ XZ + \ldots + 0 = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

the first unknown quantity X^2 takes the value

$$X^2 = \frac{1}{[\alpha_i \sigma_i]^2} = \frac{\varepsilon_A^2}{\varepsilon^2} = \frac{1}{g_A}.$$

The reciprocal value of the weight of A is therefore found to be the first unknown of the "modified normal equations".

Putting further

$$X = \xi \sqrt{[g_i a_i^2]}, \quad Y = \eta \sqrt{[g_i b_i^2]}, \quad Z = \zeta \sqrt{[g_i c_i^2]}, \ldots$$

the modified normal equations pass into

$$\begin{array}{l} [g_i a_i^2]\ \xi^2 + [g_i a_i b_i]\ \xi\eta + [g_i a_i c_i]\ \xi\zeta + \ldots - 1 = 0, \\ [g_i b_i a_i]\ \xi^2 + [g_i b_i^2]\ \xi\eta + [g_i b_i c_i]\ \xi\zeta + \ldots + 0 = 0, \\ [g_i c_i a_i]\ \xi^2 + [g_i c_i b_i]\ \xi\eta + [g_i c_i^2]\ \xi\zeta + \ldots + 0 = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Now, from $A = x\sqrt{[g_i a_i^2]}$ ensues

$$\varepsilon_A = \varepsilon_x \sqrt{[g_i a_i^2]},$$

hence

$$\frac{1}{g_x} = \frac{\varepsilon_x^2}{\varepsilon^2} = \frac{1}{[g_i a_i^2]} \times \frac{\varepsilon_A^2}{\varepsilon^2} = \frac{X^2}{[g_i a_i^2]} = \mathfrak{x}^2,$$

which is the well-known theorem on the weights of the variables.

Example: 3 equations of observation with 2 variables.

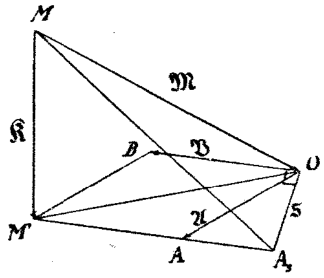

Fig. 1.

The unit-vectors $\mathfrak{a}$ and $\mathfrak{b}$ determine a plan R_2. The extremity M of $-\mathfrak{M} = OM$ is projected on this plane in the point M'. OM' is resolved parallel to $\mathfrak{a}$ and $\mathfrak{b}$ into the components $OA = \mathfrak{A}$ and $OB = \mathfrak{B}$. In the plane $R_2(\mathfrak{A}, \mathfrak{B})$ the vector $\mathfrak{s}$ is erected perpendicular to $\mathfrak{B}$. On this vector $OM = -\mathfrak{M}$ and $OA = \mathfrak{A}$ have the same projection $OA_s = M_s$. This segment M_s has the mean error ε; the variable A, i.e. the segment OA therefore has the mean error $\varepsilon_A = \dfrac{\varepsilon}{\cos AOA_s}$.

III. We now suppose that besides the n *approximate* equations of condition (equations of observation) ν *rigorous* equations of condition are given, viz.:

$$a_{n+j}x + b_{n+j}y + c_{n+j}z + \ldots + m_{n+j} = 0 \quad (j = 1, \ldots \nu).$$

For the sake of regularity in the notation, we will also provide these equations with factors g_{n+j} (which afterwards disappear from the calculation). Thus we really operate with

$$a_{n+j}\sqrt{g_{n+j}}\cdot x + b_{n+j}\sqrt{g_{n+j}}\cdot y + c_{n+j}\sqrt{g_{n+j}}\cdot z + \ldots + m_{n+j}\sqrt{g_{n+j}} = 0 (j = 1, . \nu).$$

Agreeing, that $[i]$ now means a summation over i from 1 to $n + \nu$, we may, retaining the notation used above, consider $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots, \mathfrak{M}$ as vectors in a space of $n + \nu$ dimensions.

The vector-equation

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \ldots + \mathfrak{M} = 0$$

is again not fulfilled on account of the errors of observation. The last ν component-equations $(n+1) \ldots (n+\nu)$ however hold exactly this time.

Putting again

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \ldots + \mathfrak{M} = \mathfrak{K}$$

the ν projections $K_{n+1}, \ldots K_{n+\nu}$ of $\mathfrak{K}$ must be zero, whence

$$\varkappa_{n+j} = 0 \quad (j = 1, \ldots \nu).$$

So the vector $\mathfrak{K}$ is perpendicular to the space R_ν "of condition" determined by the coordinate-axes x_{n+j} and therefore cannot generally be any longer assumed to be perpendicular to the space $R_N(\mathfrak{A}, \mathfrak{B}, \mathfrak{C},..)$ of the variables. $\mathfrak{K}$ lies in the n-dimensional space R'_n $x_{n+j} = M_{n+j}$ $(j=1,.\nu)$, which is parallel to the space R_n "of observation" determined by the axes x_h $(h=1, .. n)$.

The parallel-space R'_n cuts the space R_N of the variables in a linear space of $N+n-(n+\nu)=N-\nu$ dimensions, which we shall denote by $\varrho'_{N-\nu}$. This latter is parallel to the space $\varrho_{N-\nu}$ of intersection of the space R_n of observation with the space R_N.

We now project the extremity of $\mathfrak{M}$ lying in R'_n *in this space* on the space $\varrho'_{N-\nu}$ of intersection. The projecting vector will now be the "correction-vector" $\mathfrak{K}$.

Translating $\mathfrak{K}$ to the origin into the vector OP, OP will be perpendicular to the space $\varrho_{N-\nu}$ common to R_N and R_n.

Next we construct the normal space of $\varrho_{N-\nu}$ which passes through the origin O. This space has $n+\nu-(N-\nu)=n+2\nu-N$ dimensions. It contains the space R_ν of condition (as normal space of R_n), further the line OP, and also the normal space of $n+\nu-N$ dimensions which can be drawn from P perpendicular to R_N. This latter space therefore lies together with R_ν in a space of $n+2\nu-N$ dimensions and thus cuts R_ν in a space of $(n+\nu-N)+$ $+\nu-(n+2\nu-N)=0$ dimensions, consequently in a point. As for this point Q, it thus lies both in R_ν and in the normal space drawn from P perpendicular to R_N, from which among other things follows, that PQ makes right angles with each line of R_N, more particularly with the vectors $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ So, projecting OP and OQ on $\mathfrak{A}$, these projections are equal. The same holds for the projections on $\mathfrak{B}, \mathfrak{C}, \ldots$

Representing OQ by the vector $\mathfrak{K}'(K', \varkappa_i', K_i')$, we have, as $\mathfrak{K}'$ lies in R_ν,

$$K_h' = 0 \text{ and } \varkappa_h' = 0. \quad (h=1,...n)$$

From

$$(\mathfrak{K}.\mathfrak{A}) = (\mathfrak{K}',\mathfrak{A}),\ (\mathfrak{K},\mathfrak{B}) = (\mathfrak{K}',\mathfrak{B}),\ (\mathfrak{K},\mathfrak{C}) = (\mathfrak{K}',\mathfrak{C}), ..$$

follows

$$K[\varkappa_i \alpha_i] = K'[\varkappa_i' \alpha_i],\ K[\varkappa_i \beta_i] = K'[\varkappa_i' \beta_i],\ K[\varkappa_i \gamma_i] = K'[\varkappa_i' \gamma_i], \ldots$$

As $\varkappa_{n+j}=0$ for $j=1,\ldots\nu$, the sum $[\varkappa_i \alpha_i]$ is only to be extended from 1 to n; hence $[\varkappa_i \alpha_i] = \sum_1^n \varkappa_h \alpha_h = [\varkappa_h \alpha_h]'$; and since $\varkappa_h'=0$ for $h=1,..n$, the sum $[\varkappa_i' \alpha_i]$ is to be extended from $n+1$

to $n+\nu$, so that $[\varkappa_i'a_i] = \sum_1^\nu \varkappa_{n+j}'a_{n+j} = [\varkappa_{n+j}'a_{n+j}]''$. Here and in what follows $[_h]'$ will denote a sum over h from 1 to n, and $[_{n+j}]''$ a sum over j from 1 to ν.

We may therefore write

$$[a_h K_h]' = [a_{n+j} K_{n+j}]'',$$

or, because

$$K_h = A_h + B_h + C_h + \ldots + M_h = \alpha_h A + \beta_h B + \gamma_h C + \ldots + M_h,$$

$$[\alpha_h^2]'A + [\alpha_h\beta_h]'B + [\alpha_h\gamma_h]'C + \ldots + [\alpha_h M_h]' = [\alpha_{n+j}K_{n+j}]'',$$

$$[\beta_h\alpha_h]'A + [\beta_h^2]'B + [\beta_h\gamma_h]'C + \ldots + [\beta_h M_h]' = [\beta_{n+j}K_{n+j}]'',$$

$$[\gamma_h\alpha_h]'A + [\gamma_h\beta_h]'B + [\gamma_h^2]'C + \ldots + [\gamma_h M_h]' = [\gamma_{n+j}K_{n+j}]'',$$

. . . . , ...

Putting

$$\alpha_{n+j}=\alpha_j',\ \beta_{n+j}=\beta_j',\ \gamma_{n+j}=\gamma_j', \ldots K_{n+j}'=-Q_j,\ M_{n+j}=M_j',\ m_{n+j}=m_j',$$

we have

$$[\alpha_h^2]'A + [\alpha_h\beta_h]'B + [\alpha_h\gamma_h]'C + \ldots + [\alpha_h M_h]' + [\alpha_j'Q_j]'' = 0,$$

$$[\beta_h\alpha_h]'A + [\beta_h^2]'B + [\beta_h\gamma_h]'C + \ldots + [\beta_h M_h]' + [\beta_j'Q_j]'' = 0,$$

$$[\gamma_h\alpha_h]'A + [\gamma_h\beta_h]'B + [\gamma_h^2]'C + \ldots + [\gamma_h M_h]' + [\gamma_j'Q_j]'' = 0,$$

. . . .

Introducing

$$a_j' = a_{n+j} = \frac{\alpha_j'\sqrt{[g_i a_i^2]}}{\sqrt{g_{n+j}}},\quad b_j' = b_{n+j} = \frac{\beta_j'\sqrt{[g_i b_i^2]}}{\sqrt{g_{n+j}}},$$

$$c_j' = c_{n+j} = \frac{\gamma_j'\sqrt{[g_i c_i^2]}}{\sqrt{g_{n+j}}}, \ldots, q_j = Q_j\sqrt{g_{n+j}}$$

we obtain, after multiplying successively by $\sqrt{[g_i a_i^2]}$, $\sqrt{[g_i b_i^2]}$, $\sqrt{[g_i c_i^2]}, \ldots$

$$[g_h a_h^2]'x + [g_h a_h b_h]'y + [g_h a_h c_h]'z + \ldots + [g_h a_h m_h]' + [a_j'q_j]'' = 0,$$

$$[g_h b_h a_h]'x + [g_h b_h^2]'y + [g_h b_h c_h]'z + \ldots + [g_h b_h m_h]' + [b_j'q_j]'' = 0,$$

$$[g_h c_h a_h]'x + [g_h c_h b_h]'y + [g_h c_h^2]'z + \ldots + [g_h c_h m_h]' + [c_j'q_j]'' = 0,$$

. . . . ,

N equations, which together with the ν conditions

$$a_j'x + b_j'y + c_j'z + \ldots + m_j' = 0$$

serve to determine the N variables $x, y, z, \ldots$ and the ν auxiliary quantities q_j.

IV. In order to determine the weights of $x, y, z, \ldots$, i. e. of $A, B, C, \ldots$, we must examine the influence undergone by $\mathfrak{A}$ from a variation of $\mathfrak{M}$, the vectors $\mathfrak{B}, \mathfrak{C}, \ldots$ remaining unaltered.

A variation of $\mathfrak{M}$ only acts upon $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ when the foot of $\mathfrak{K}$ on the space $\varrho'_{N-\nu}$ of intersection moves. If the foot is fixed, $\mathfrak{K}$ may freely move in the space S, common to the normal space of $\varrho'_{N-\nu}$ (of $n+2\nu-N$ dimensions) and the space R'_n parallel to R_n. The space S obviously has $(n+2\nu-N)+n-(n+\nu)=n+\nu-N$ dimensions. A component of $\mathfrak{M}$ in this space has no effect on the vectors $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ A component of $\mathfrak{M}$ will only have any effect on $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$, when it lies in the normal space S'_N of S, which has $n+\nu-(n+\nu-N)=N$ dimensions. By translating this normal space S'_N to O, it contains both R_ν and $\varrho_{N-\nu}$ (intersection of R_N and R_n).

The variation of $\mathfrak{M}$ will exclusively influence $\mathfrak{A}$, when the component of $\mathfrak{M}$ undergoing this variation is perpendicular to $\mathfrak{B}, \mathfrak{C}, \ldots$

These considerations lead to the result that we want that direction $\mathfrak{s}$, which lies in S'_N and is perpendicular to $\mathfrak{B}, \mathfrak{C}, \ldots$ The vectors $\mathfrak{B}, \mathfrak{C}, \ldots$ determine together a space of $N-1$ dimensions. The vector $\mathfrak{s}$ must lie in the normal space (of $n+\nu-N+1$ dimensions) of the space $(\mathfrak{B}, \mathfrak{C}, \ldots)$. This normal space cuts S'_N in a space of $(n+\nu-N+1)+N-(n+\nu)=1$ dimension, hence in a straight line. So there is always one and only one line $\mathfrak{s}$ fulfilling the imposed conditions.

Since $\mathfrak{s}$ lies in S'_N, i. e. in the space joining R_ν with $\varrho_{N-\nu}$, the projection $\mathfrak{t}$ of $\mathfrak{s}$ on R_n will fall into $\varrho_{N-\nu}$.

Now we have for the direction cosines τ_i of the projection $\mathfrak{t}$ of $\mathfrak{s}$ on R_n:

$$\tau_h = \frac{\sigma_h}{\sqrt{[\sigma_h^2]'}} \; (h=1,\ldots n)\,; \quad \tau_{n+j}=0 \quad (j=1,\ldots,\nu).$$

As $\mathfrak{t}$, being a line of $\varrho_{N-\nu}$, also lies in the space R_N and therefore may be resolved in the directions $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$, we have

$$\tau_h = P\alpha_h + Q\beta_h + R\gamma_h + \ldots, \qquad (h=1,\ldots n)$$
$$\tau_{n+j} = P\alpha_{n+j} + Q\beta_{n+j} + R\gamma_{n+j} + \ldots = 0. \qquad (j=1,\ldots\nu)$$

Putting

$$P\sqrt{[\sigma_h^2]'} = P' \;, \quad Q\sqrt{[\sigma_h^2]'} = Q' \;, \quad R\sqrt{[\sigma_h^2]'} = R' ,\ldots$$

we obtain:

$$\alpha_h P' + \beta_h Q' + \gamma_h R' + \ldots = \sigma_h \,, \qquad (h=1,\ldots n)$$
$$\alpha_{n+j} P' + \beta_{n+j} Q' + \gamma_{n+j} R' + \ldots = 0 \,, \qquad (j=1,\ldots\nu)$$

and, $\mathfrak{s}$ being perpendicular to $\mathfrak{B}, \mathfrak{C}, \ldots$,

$$[\beta_i\sigma_i]=0 \,, \qquad [\gamma_i\sigma_i]=0 \;\ldots \qquad [\sigma_i^2]=1 \,.$$

In this way we have collected $n+\nu+N$ equations to determine the $n+\nu$ unknown quantities σ_i and the N unknown quantities $P', Q', R', \ldots$.

S'_N being perpendicular to $\mathfrak{R}$, $\mathfrak{s}$ is also perpendicular to $\mathfrak{R}$. By multiplying the equation

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + ... + \mathfrak{M} = \mathfrak{K}$$

scalarly by $\mathfrak{s}$, it reduces to

$$(\mathfrak{A}, \mathfrak{s}) + (\mathfrak{M}, \mathfrak{s}) = 0$$

or

$$A\,[\alpha_i \sigma_i] = -\,M_s$$

In order to determine the mean error of M_s, we remark that of all the lines through O in R_n $\mathfrak{t}$ is that which makes the smallest angle with $\mathfrak{s}$. The error of M_s therefore depends for the most part on the error in the components M_t of $\mathfrak{M}$ in the direction $\mathfrak{t}$. We may consequently write

$$m.\ e.\ of\ M_s = m.\ e.\ of\ M_t \times \cos(\mathfrak{s}, \mathfrak{t}) = \varepsilon \cos(\mathfrak{s}, \mathfrak{t})$$

or

$$\varepsilon_s = \varepsilon\,[\sigma_i \tau_i] = \varepsilon \left[\sigma_h \cdot \frac{\sigma_h}{\sqrt{[\sigma_h^2]'}} \right]' = \varepsilon \sqrt{[\sigma_h^2]'},$$

hence

$$\varepsilon_A = \frac{\varepsilon_s}{[\alpha_i \sigma_i]} = \frac{\sqrt{[\sigma_h^2]'}}{[\alpha_i \sigma_i]}\,\varepsilon\,.$$

Since

$$M_s = M_t\,[\sigma_i \tau_i] = [M_h \tau_h]' \cdot [\sigma_i \tau_i] = \left[M_h \frac{\sigma_h}{\sqrt{[\sigma_h^2]'}} \right]' \cdot \sqrt{[\sigma_h^2]'} = [M_h \sigma_h]',$$

we have

$$A = -\frac{M_s}{[\alpha_i \sigma_i]} = -\left[\frac{\sigma_h}{[\alpha_i \sigma_i]} \cdot M_h \right]'$$

or, putting

$$\frac{\sigma_h}{[\alpha_i \sigma_i]} = p_h\,,$$

$$A = -\,[p_h M_h]'.$$

$$\frac{1}{g_A} = \frac{\varepsilon_A^2}{\varepsilon^2} = \frac{[\sigma_h^2]'}{[\alpha_i \sigma_i]^2} = [p_h^2]'.$$

Introducing

$$X = \frac{P'}{[\alpha_i \sigma_i]}\,,\ Y = \frac{Q'}{[\alpha_i \sigma_i]}\,,\ Z = \frac{R'}{[\alpha_i \sigma_i]}\,, \ldots;\ p_{n+j} = \frac{\sigma_{n+j}}{[\alpha_i \sigma_i]}\ (j = 1, ..\nu)$$

we arrive at

$$\alpha_h X + \beta_h Y + \gamma_h Z + \ldots = p_h \qquad (h = 1, \ldots n)$$

$$\alpha_{n+j} X + \beta_{n+j} Y + \gamma_{n+j} Z + \ldots = 0 \qquad (j = 1, \ldots \nu)$$

$$[\alpha_i p_i] = 1\ ,\ [\beta_i p_i] = 0\ ,\ [\gamma_i p_i] = 0\ , \ldots$$

From these $n + \nu + N$ equations we can solve the $n+\nu$ unknown quantities p_i $(i = 1 \ldots n + \nu)$ and the N auxiliary quantities $X, Y, Z, \ldots$

The quantity $\dfrac{1}{g_A} = [p_h^2]'$ in question is also found as follows

$$\frac{1}{g_A}=[p_h{}^2]'=[p_h(\alpha_h X+\beta_h Y+\gamma_h Z+\ldots)]'=X[p_h\alpha_h]'+Y[p_h\beta_h]'+Z[p_h\gamma_h]'+\ldots=$$
$$=X-X[p_{n+j}\alpha_{n+j}]''-Y[p_{n+j}\beta_{n+j}]''-Z[p_{n+j}\gamma_{n+j}]''\ldots$$
$$=X-[p_{n+j}(\alpha_{n+j}X+\beta_{n+j}Y+\gamma_{n+j}Z+\ldots)]''$$
$$=X.$$

Returning to the original variables $x,y,z,\ldots$, we derive from

$$x=\frac{A}{\sqrt{[g_i a_i^2]}}$$

firstly

$$\varepsilon_x=\frac{\varepsilon_A}{\sqrt{[g_i a_i^2]}}$$

and

$$\frac{1}{g_x}=\frac{\varepsilon_x^2}{\varepsilon^2}=\frac{[p_h^2]'}{[g_i a_i^2]}.$$

Further, putting

$$p_h=k_h\frac{\sqrt{[g_i a_i^2]}}{\sqrt{g_h}},\qquad p_{n+j}=k_{n+j}\frac{\sqrt{[g_i a_i^2]}}{\sqrt{g_{n+j}}}$$
$$X=\xi\,[g_i a_i^2],\qquad Y=\eta\,[g_i b_i^2],\qquad Z=\zeta\,[g_i c_i^2],\ldots$$

the $n+\nu+N$ equations pass into

$$a_h\,\xi+b_h\,\eta+c_h\,\zeta+\ldots=\frac{k_h}{g_h},\qquad (h=1,\ldots n)$$
$$a_{n+j}\,\xi+b_{n+j}\eta+c_{n+j}\zeta+\ldots=0\,,\qquad (j=1,\ldots \nu)$$
$$[a_i k_i]=1,\qquad [b_i k_i]=0,\qquad [c_i k_i]=0,\ldots,$$

whence

$$\frac{1}{g_x}=\left[\frac{k_h^2}{g_h}\right]'=\xi.$$

Example: 2 equations of observation with 2 variables and 1 condition. The unit-vectors $\mathfrak{a}$ and $\mathfrak{b}$ determine a plane $R_N\,(N=2)$, the plane of the variables. This plane cuts the plane of observation $R_n\,(n=2)$ in the line $\varrho_{N-\nu}\,(N-\nu=1)$, which thus coincides with the line $\mathfrak{r}$. The line OP is drawn in the plane R_n perpendicular to $\varrho_{N-\nu}\,(\mathfrak{r})$. Through the extremity M of the vector $\mathfrak{M}$ a line is drawn parallel to OP; this line cuts the plane R_N of the variables in M'. The vector MM' =//= PO is the correction-vector $\mathfrak{K}$. OM' is resolved in the directions $\mathfrak{a}$ and $\mathfrak{b}$ into the components $OA=\mathfrak{A}$ and $OB=\mathfrak{B}$. The lengths of these lines represent the most probable values of the variables A and B.

The line PQ is perpendicular to the plane R_N and meets the normal R_ν (line of condition), erected in O on R_n, in the point Q. The vector OQ is called $\mathfrak{K}'$.

The space joining $\varrho_{N-\nu}$ with R_ν is here the plane S'_N. The plane

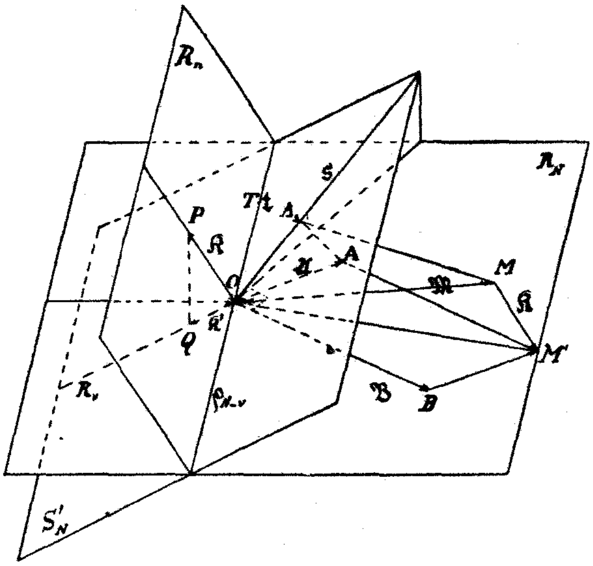

Fig. 2.

erected in O perpendicular to $\mathfrak{B}$, intersects S'_N in the line $\mathfrak{s}$, which therefore is perpendicular to $\mathfrak{B}$ and $\mathfrak{R}$. So A and M are projected on $\mathfrak{s}$ in the same point A_s.

The normal plane $A A_s M$ of $\mathfrak{s}$ cuts $\mathfrak{t}$ in a point T, the distance of which to O amounts to M_t (with mean error ε). The mean error ε_s of A_s thus has the value $\varepsilon_s = \varepsilon \cos(\mathfrak{s}, \mathfrak{t})$, and that of A the value

$$\varepsilon_A = \frac{\varepsilon_s}{\cos(\mathfrak{s}, \mathfrak{a})} = \varepsilon \frac{\cos(\mathfrak{s}, \mathfrak{t})}{\cos(\mathfrak{s}, \mathfrak{a})}.$$

V. The errors (residuals) of $M_1, M_2, \ldots M_n$ are $K_1, K_2, \ldots K_n$ resp. The sum of their squares is $[K_h^2] = K^2$.

For the case that no equations of condition are given, $\mathfrak{R}$ must be perpendicular to R_N. So $\mathfrak{R}$ may dispose of a space of $n-N$ dimensions (the normal space of R_N). Hence $\mathfrak{R}$ has $n-N$ components, all with the same mean value ε. Consequently

$$K^2 = (n-N) \times \varepsilon^2$$

hence

$$\varepsilon = \sqrt{\frac{[K_i^2]}{n-N}}.$$

In case ν conditions are imposed, $\mathfrak{R}$ may dispose of the space S of $n+\nu-N$ dimensions. Consequently $\mathfrak{R}$ now has $n+\nu-N$ components, all with a mean value ε. In this case we have therefore

$$\varepsilon = \sqrt{\frac{[K_h^2]}{n+\nu-N}}.$$