Calculated for $C_5H_8SO_2 : 132$.

Hence, the crystallized compound is formed from one mol. of isoprene and one mol. of $SO_2$.

The substance is soluble in water. The aqueous solution has a neutral reaction.

If a solution of the compound in carbon tetrachloride or ether is shaken with a solution of bromine in the same solvent, the colour of the bromine is not discharged; bromine water, however, is gradually decolourised. With dilute alkaline potassium permanganate a reduction sets in at once.

As to the structure of this compound I do not as yet venture to pronounce an opinion. In connexion with THIELE's theory the occurrence of a compound of the formula $CH_2 - C[CH_3] = CH - CH_2$ (the two terminal $CH_2$ groups linked by $SO_2$) would not be improbable.

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**Geophysics.** — "*The treatment of frequencies of directed quantities*". By Dr. J. P. VAN DER STOK.

(Communicated in the meeting of June 27, 1914).

**1.** The frequency-curves of barometric heights, atmospheric temperatures and other meteorological quantities assume different and peculiar forms, which can be considered as climatological characteristics and, as the number of available data increases, it is desirable to subject these curves to such a treatment that these characteristic peculiarities are represented by climatological constants.

If we choose for this purpose the development in series-form, the first question is, what treatment is to be chosen for each special case, in conformity with the distinctive features of the quantities under consideration and the limits between which they are comprised. The purpose of this investigation is to inquire, what form is to be chosen for frequencies of wind-velocities independent of direction, and of direction without regard to velocity. Furthermore, to state in how far the observed series of quantities may be regarded as normal- or standard-values, and the problem may be stated also in this way: what is the best form for frequencies of directed quantities assuming the form of linear quantities, and further, how to integrate the expression

$$\left.\begin{array}{l}\frac{h\,h'}{\pi}\,e^{-f(R\,.\,\theta)}\,RdRd\theta \\ f(R\,.\,\theta) = h^2\,[R\sin(\theta-\beta) - a]^2 + h'^2\,[R\cos(\theta-\beta) - b]^2\end{array}\right\} \quad . \quad . \quad (1)$$

i.e. the standard-value of directed quantities, on the one hand with respect to $\theta$ between the limits $2\pi$ and zero, on the other hand with respect to $R$ between the limits $\infty$ and zero.

Both problems were treated in previous communications [1] [2]), but it may appear from the following that now a more principal, and therefore more complete, solution can be obtained than seemed possible a few years hence.

**2.** If we wish to develop a function of one variable in an infinite series of polynomia

$$F(x) = \sum_{n=0}^{n=\infty} A_n\,U_n$$

$$U_n = x^n + a_1 x^{n-1} + a_2\,x^{n-2} \ldots a_n,$$

the quantities $a$ can be determined so that — as in the FOURIER-series — for the assumed limits, $\alpha$ and $\beta$

$$\int U_n\,U_m\,dx = 0$$

for all values of $m$ different from $n$.

The constants $A_n$ are then given by the equation:

$$A_n \int U_n^2\,dx = \int F(x)\,U_n\,dx.$$

The values of the constants $a$ are determined by the $n$ equations:

$$\int U_n\,dx = 0,\ \int U_n\,x\,dx = 0 \ldots \int U_n\,x^{n-1}\,dx = 0 \quad . \quad . \quad . \quad (2)$$

every integral being taken between the assumed limits.

By partial integration we have:

1) The treatment of wind-observations. Proc. Sci. Kon. Akad. v. Wet. IX, (684—699).

2) On the Analysis of Frequency-curves according to a general method. Proc. Sci. K. Akad. Wet. X, (799–817).

$$\int_0^x U_n\,dx = \varphi_1$$

$$\int_0^x U_n\,x\,dx = x\,\varphi_1 - \varphi_2 \qquad \varphi_2 = \int_0^x \varphi_1\,dx$$

$$\int_0^x U_n\,x^2\,dx = x^2\,\varphi_1 - 2x\,\varphi_2 + \varphi_3 \qquad \varphi_3 = \int_0^x \varphi_2\,dx\text{, etc.}$$

By (2) it follows from these equations that the imposed conditions are fulfilled when, in the development

$$\int_0^x U_n\,x^n\,dx = x^n\,\varphi_1 - n\,x^{n-1}\,\varphi_2 \ldots (-1)^{n-1}\,n(n-1)\quad .\,2\varphi_n\,(-1)^n n!\,\varphi_{n+1} \quad (3)$$

$\varphi_n$ be given such a value that this function, as also its $(n-1)$ first differential-quotients, become zero for $x=\beta$ and $x=\alpha$ and that then

$$U_n = \frac{d^n\varphi_n}{dx^n} \quad\text{and}\quad \int_\beta^\alpha U_n^2\,dx = (-1)^n\,n!\int_\beta^\alpha \varphi_n\,dx \quad . \quad . \quad . \quad . \quad (4)$$

This simple method of determining the terms of the required series was indicated in 1833 by MURPHY as a new method of coming to zonal harmonics; in THOMSON and TAIT's "Natural Philosophy" it is mentioned in article 782.

The method, however, is by no means restricted to the calculation of zonal harmonics but can easily be generalized and applied to other circumstances than those mentioned above.

Instead of a complete polynomium we can also consider separately even and uneven polynomia; polynomia multiplied by an exponential factor as $e^{-x^2}$ or $e^{-x}$ may be used, and instead of $dx$ we can take $xdx$ (plane) or $x^2dx$ (space) as the element of integration, whereas for $x$ also quantities of another kind, e.g. $\sin\,\alpha$, may be substituted.

3. If the limits are $+1$ and $-1$, it is rational to put:

$$\varphi_n = C\,(x^2-1)^n \qquad U_n = C\,\frac{d^n}{dx^n}\,(x^2-1)^n$$

$C$ being an arbitrary constant.

Putting

$$C = \frac{n!}{(2n)!}$$

$U_n$ becomes

$$U_n = x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\,4.(2n-1)(2n-3)} x^{n-4} - \ldots \text{ etc.} \quad . \; (5)$$

the well known form (but for a constant factor) of the zonal harmonic function and, according to (4):

$$\int_{-1}^{+1} U^2{}_n dx = \frac{n!n!}{(2n)!} \int_{-1}^{+1} (x^2 - 1)^n \, dx = \frac{2^{n+1}(n!)^4}{(2n+1)!(2n)!}$$

Putting $C = \frac{1}{2^n n!}$, we find, if by $P_n$ the commonly used form of zonal harmonics is denoted,

$$P_n = \frac{(2n)!}{2^n.n!n!} U_n$$

from which

$$\int_{-1}^{+1} P_n^2 \, dx = \frac{2}{2n+1}.$$

If the limits are $+\infty$ and $-\infty$ it is rational to choose for $\varphi_n$:

$$\varphi_n = C e^{-x^2} \qquad U_n = C \frac{d^n}{dx^n} e^{-x^2}.$$

Putting

$$C = \frac{(-1)^n}{2^n},$$

$U_n$ assumes the form:

$$\left. \begin{aligned} U_n = U_n' e^{x^2} = e^{-x^2} \Bigg[ x^n - \frac{n(n-1)}{2^2.1!} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^4.2!} x^{n-4} \\ \ldots\ldots (-1)^{\frac{n-1}{2}} \frac{n!}{2^{n-1} \frac{n-1}{2}!} x \quad (n \text{ uneven}) \\ \ldots\ldots (-1)^n \frac{n!}{2^n \frac{n}{2}!} \quad (n \text{ even}) \Bigg] \end{aligned} \right\} (6)$$

or

$$U_n' = \frac{(-1)^n}{2^n} \left( \frac{d}{dx} - 2x \right)^n$$

and, by (4):

$$\int_{-\infty}^{+\infty} U_n^2\, dx = \frac{n!}{2^n}\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \frac{n!}{2^n}\sqrt{\pi}\,.$$

The series (6), proposed by BRUNS [1]) and CHARLIER [2]), is in mathematics known as HERMITE's function and might, if applied to analysis of frequencies, be called the $\varphi_n$ function, as proposed by BRUNS.

It is the most appropriate form for quantities as atmospheric and watertemperatures, barometric heights etc., moving between uncertain limits, and also for wind-observations if generalized for application to functions of two variables.

In either of the cases considered above the terms of even and uneven power are separated automatically because

$$\int_{-1}^{+1} x^{2n+1}\, dx = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} x^{2n+1}\, e^{-x^2}\, dx = 0\,.$$

If, however, the limits are 1 and 0 or $\infty$ and 0, then such a separation does not take place and we must either maintain the complete polynomium or consider both cases separately.

4. Considering the even polynomia separately for the limits 1 and 0, every polynomium $U_{2n}$ contains only $n$ constants and the development (5) takes the form:

$$\left.\begin{aligned}\int_0^x U_{2n}\, x^{2n}\, dx = x^{2n}\,\varphi_1 - 2n\, x^{2n-2}\,\varphi_2 + 2^2\, n\,(n-1)\, x^{2n-4}\,\varphi_3 - \\ (-1)^{n-1}\, 2^{n-1}\, n\,(n-1)\ldots 2\, x^2\,\varphi_n\, (-1)^n\, 2^n\,.\, n!\,\varphi_{n+1}\end{aligned}\right\} \quad (7)$$

where

$$\varphi_1 = \int_0^x U_{2n}\, dx \quad , \quad \varphi_2 = \int_0^x \varphi_1\, x dx \ldots \varphi_{n+1} = \int_0^x \varphi_n\, x dx$$

$$\int_0^x U^2_{2n}\, dx = (-1)^n\, 2^n\,.\, n! \int_0^x \varphi_n\, x dx\,.$$

Putting

$$\Delta = \frac{1}{x}\frac{d}{dx}$$

1) Wahrscheinlichkeitsrechnung und Kollektivmasslehre. 1906.

2) Researches into the theory of Probability. (Comm. from the Astron. Observ. Lund.). 1906.

we find

$$U_{2n} = x \, \Delta^n \, \varphi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

whereas for $\varphi_n$, as the simplest expression, we must take:

$$\varphi_n = C x^{2n-1} (x^2 - 1)^n.$$

Assigning to $C$ the value:

$$\frac{1}{(4n-1)(4n-3)\ldots(2n+1)}$$

the zonal harmonic function, as given in (5), is again found also for the limits 1 and 0.

In the case of uneven polynomia

$$U_{2n+1} = C \, \Delta^n \, x^{2n+1} (x^2 - 1)^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which for

$$C = \frac{1}{(4n+1)(4n-1)\ldots(2n+3)}$$

again leads to the expression (5).

Giving $C$ the value $\frac{1}{2^n n!}$, we obtain from (8) as well as from (9) the zonal harmonic function in the form as commonly used.

No more as for the limits 1 and 0, the development (7) for the limits $\infty$ and 0 leads to new expressions; we have to put

$$\varphi_n = C \, x^{2n+1} \, e^{-x^2}$$

for even as well as for uneven functions, and by the formulae

$$\left.\begin{aligned} U_{2n} &= \frac{(-1)^n}{(2n+1)2^n} \, e^{-x^2} \, \frac{d}{dx} (\Delta - 2)^n \, x^{2n+1} \\ U_{2n+1} &= \frac{(-1)^n}{2^n} \, e^{-x^2} \, (\Delta - 2)^n \, x^{2n+1} \end{aligned}\right\} \quad . \quad . \quad (10)$$

we find the same expression as in § 3 for $\varphi_n$ of formula (6), but by an abridged calculation.

5. The problem, which form of development is the fittest for frequencies of a quantity which assumes the form of a function of one variable, moving between the limits 1 and 0 or $\infty$ and 0, but, as a matter of fact, must be considered as a function of two variables, is not solved satisfactorily in § 5, at least if we are not satisfied by a merely formal representation.

A graphical representation of such a function is given by the distribution of points in a plane about a given origin, the element of integration is then, not $dx$, but $2\pi R dR$ and the question must be put as follows: to find a polynomium such that

$$\int U_n \, U_m \, R dR = 0$$

for all values of $m$ different from $n$.

The development by partial integration then becomes:

$$\left.\begin{array}{l}\int U_{2n} R^{2n+1} dx = R^{2n} \varphi_1 - 2n R^{2n-2} \varphi_2 + 2^2 n (n-1) R^{2n-4} \varphi_3 \\ (-1)^{n-1} 2^{n-1} n (n-1) \ldots 2R^2 (-1)^n 2^n \, . \, n! \, \varphi_{n+1}\end{array}\right\} \quad (11)$$

where

$$\varphi_1 = \int U_{2n} R \, dR, \quad \varphi_2 = \int \varphi_1 R \, dR \text{ etc.}$$

and

$$\int U^2{}_{2n} R \, dR = (-1)^n \, n! \int \varphi_n R dR.$$

If the limits are 1 and 0, then we have to put:

$$\varphi_n = C R^{2n} (R^2 - 1)^n$$

so that

$$U_{2n} = C \Delta^n R^{2n} (R^2 - 1)^n.$$

Putting $C = \frac{1}{2^n}$ we find for the polynomium:

$$U_{2n} = \frac{(2n)!}{n!} R^{2n} - {}^nC_1 \frac{(2n-1)!}{(n-1)!} R^{2n-2} + {}^nC_2 \frac{(2n-2)!}{(n-2)!} R^{2n-4} \ldots \text{ etc.} \quad (12)$$

where ${}^nC_p$ denotes the $p^{\text{th}}$ binomium-coefficient of the $n^{\text{th}}$ power, further:

$$\int_0^1 U^2{}_{2n} R \, dR = 2^n (2n)! \int_0^1 \varphi_n R \, dR = (2n)! \int_0^1 R^{2n+1} (R^2 - 1)^n dR = \frac{1}{2} \frac{n! n!}{2n+1}.$$

This new function may be considered as a zonal harmonic generalized for the case of directed quantities and might be applied e.g. to the distribution of hits on a target.

The analogy of (12) with the zonal harmonic function becomes conspicuous if the latter (5), by multiplication by $\frac{(2n-1)!}{(n-1)!}$, be given the form:

$$U_n = \frac{(2n-1)!}{(n-1)!} x^n - \frac{C_1}{2} \cdot \frac{(2n-2)!}{(n-2)!} x^{n-2} + \frac{C_2}{2} \cdot \frac{(2n-4)!}{(n-4)!} x^{n-4} - \text{ etc.}$$

The expression (12) satisfies the differential equation:

$$R(1-R^2) \frac{d^2 U_{2n}}{dR^2} + (1 - 3R^2) \frac{dU_{2n}}{dR} + 4n(n+1) R U_{2n} = 0.$$

For uneven polynomia $\varphi_n$ has to be given the same value as (9) and then again the common zonal harmonic would result. As, however, the quantities under consideration are essentially positive, uneven functions can be left out of consideration.

If the limits are $\infty$ and 0, then the same reasoning holds; it is then rational to put:

$$\varphi_n = C R^{2n} e^{-R^2}$$

$$U_{2n} = C \, \Delta^n R^{2n} e^{-R^2} \qquad U'_{2n} = C \, (\Delta - 2)^n R^{2n}.$$

Putting

$$C = \frac{(-1)^n}{2^n}$$

the polynomium assumes the form:

$$U'_{2n} = R^{2n} - n^2 R^{2n-2} + \frac{n^2(n-1)^2}{2!} R^{2n-4} - \ldots (-1)^n n! \quad . \quad (13a)$$

and

$$\int_0^\infty U^2_{2n} \, RdR = 2^n \, . \, n! \int_0^\infty \varphi_n \, RdR = \frac{n!n!}{2}.$$

In analogy with (12) the polynomium, by putting

$$C = \frac{(-1)^n}{2^n n!},$$

may be written also:

$$U'_{2n} = \frac{R^{2n}}{n!} - {}^nC_1 \frac{R^{2n-2}}{(n-1)!} + {}^nC_2 \frac{R^{2n-4}}{(n-2)!} \ldots (-1)^n \quad . \quad . \quad (13b)$$

This new function (13) seems to be the proper form of development in the case of directed quantities as wind-velocities, disregarding direction; it satisfies the diff. equations:

$$R \frac{d^2 U'_{2n}}{dR^2} - (2R^2 - 1) \frac{dU'_{2n}}{dR} + 4n \, RU'_{2n} = 0$$

$$R \frac{d^2 U_{2n}}{dR^2} + (2R^2 + 1) \frac{dU_{2n}}{dR} + 4\,(n+1)\, RU_{2n} = 0.$$

In applying this development, a simplification may be obtained by a change of scale-value: writing $HR$ for $R$ and putting

$$H^2 = \frac{1}{M^2},$$

the second term with the coefficient $A_2$ will disappear as

$$U'_2 = (R^2 - 1).$$

Here $M^2$ denotes the moment of the second order of the given frequency-series.

6. In the same manner as in § 5 in the case of a directed quantity in a plane, the development appropriate for quantities in space may be found, e. g. for distances of stars, disregarding direction.

The element of integration is then $4\pi R^2 dR$, and the development (11) holds good if in the left member $R^{2n+1}$ is written instead of $R^{2n}$ and, at the same time for $\varphi_1$

$$\varphi_1 = \int_0^\infty U_{2n} R^2 dR$$

so that

$$U_{2n} = \frac{C}{R} \Delta^n \varphi_n \text{ and } \varphi_n = CR^{2n+1} e^{-R^2}.$$

Putting

$$C = \frac{(-1)^n}{2^n}$$

$U'_{2n}$ becomes:

$$\left.\begin{array}{l} U'_{2n} = e^{R^2} U_{2n} = R^{2n} - \overset{n}{C}_1 \dfrac{2n+1}{2} R^{2n-2} + \overset{n}{C}_2 \dfrac{(2n+1)(2n-1)}{2^2} R^{2n-4} \\ \qquad \ldots (-1)^n \dfrac{(2n+1)!}{2^{2n} n!} \end{array}\right\} \quad . \quad (14)$$

and

$$\int_0^\infty U'^2_{2n} R^2 dR = (-1)^n \cdot 2^n \cdot n! \int_0^\infty \varphi_n R dR = (-1)^n \, 2^{n-1} \, n! n!$$

In applying this development a simplification may be obtained by writing $HR$ for $R$ and putting:

$$H^2 = \frac{3}{2M^2},$$

then $A_2 = 0$, because

$$U'_2 = R^2 - {}^3/_2.$$

7. Although we may expect *a priori* that the FOURIER-series is the most appropriate form of development for frequencies of directions (disregarding velocity), it seems desirable in connection with the foregoing to show that, following the same method, we, in fact, come to this result.

If

$$U = \sin^{2n} \alpha + a_1 \sin^{2n-2} \alpha + \ldots a_n,$$

then we may distinguish four different types of functions, namely:

$F_1 = U \qquad F_2 = U \sin \alpha \cos \alpha \qquad F_3 = U \cos \alpha$ en $F_4 = U \sin \alpha$

For $F_1$ the development holds good:

$$\int U \sin^{2n} \alpha d\alpha = \varphi_1 \sin^{2n} \alpha - 2n \varphi_2 \sin^{2n-2} \alpha + \ldots$$
$$(-1)^{n-1} . 2^{n-2} . n(n-1) \ldots 2 . \varphi_n (-1)^n 2^n n! \varphi_{n+1}$$

where

$$\varphi_1 = \int U d\alpha \qquad \varphi_2 = \int \varphi_1 \sin \alpha \cos \alpha \, d\alpha \quad \text{etc.}$$

Therefore, putting

$$\Delta = \frac{1}{\sin \alpha \cos \alpha} \frac{d}{d\alpha}$$

$$\varphi_n = C \sin^{2n-1} \alpha \cos^{2n-1} \alpha \quad \text{and} \quad C = \frac{2^n n!}{(2n)!}$$

we find for the limits $\frac{\pi}{2}$ and $0$:

$$F_1 = \frac{2^n . n!}{(2n)!} \sin \alpha \cos \alpha \, \Delta^n \varphi_n = \cos 2n\alpha.$$

In the same manner:

$$F_2 = \frac{2^n . n!}{(2n-1)!} \Delta^{n-1} \sin^{2n-1} \alpha \cos^{2n-1} \alpha = \sin 2n\alpha$$

$$F_3 = \frac{2^n . n!}{(2n)!} \sin \alpha \, \Delta^n \sin^{2n-1} \alpha \cos^{2n+1} \alpha = \cos (2n+1) \alpha$$

$$F_4 = \frac{2^n . n!}{(2n)!} \cos \alpha \, \Delta^n \sin^{2n+1} \alpha \cos^{2n-1} \alpha = \sin (2n+1) \alpha .$$

8. The solution of the second problem, as formulated in § 1, can be simplified by putting $\theta - \beta = \varphi$ in form. (1), i.e. by counting the angular values not, as usual, from the North-direction, but from $N\beta E$; this has, of course, no influence on the sums of the velocities.

It is, however, unfeasible to apply a similar correction for the components $a$ and $b$ of the resulting wind, and the problem to be solved comes to the development in series-form of the expression:

$$\frac{hh'}{\pi} e^{-h^2(x-a)^2 - h'^2(y-b)^2} \qquad \begin{array}{l} R \cos \theta = y \\ R \sin \theta = x . \end{array}$$

It appears from the first of the communications cited in § 1 that, in following the usual method of developing, difficulties are experienced which practically are unsurmountable. In the second communication however, it was shown that the development (6) may be extended to the case of two variables $x$ and $y$, and that such a function can be developed in a series of polynomia of the form:

$$F(x,y)=e^{-x^2-y^2}[A_{00}U_0+A_{1.0}U_1+A_{0.1}V_1+A_{2.0}U_2+A_{1.1}U_1V_1+A_{0.2}V_2+\text{etc.}] \quad (15)$$

where $V$ represents the same function of $(y)$ as $U$ of $(x)$ in form. (6).

The coefficients $A$ are then determined by the expression:

$$A_{n.m} = \varepsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\, U_n V_m\, dx dy = \varepsilon\, S_{n.m}$$

$$\varepsilon^{-1} = \frac{n!\,m!}{2^{m+n}}\pi \, . \quad (16)$$

Substituting again for $x$ and $y$, $R \sin \theta$ and $R \cos \theta$, then, by integration with respect to $\theta$, all uneven polynomia vanish and, because

$$\int_0^{2\pi} \begin{matrix} \sin^{2n} a \\ \cos^{2n} a \end{matrix}\, da = \frac{(2n)!}{2^{2n}.n!n!},$$

we find

$$\left.\begin{aligned}\int_0^{2\pi} U_{2n} V_{2m}\, d\theta = \frac{2\pi}{2^{2(n+m)}} \frac{(2n)!(2m)!}{n!m!} e^{-H^2R^2} \Bigg[ \frac{(HR)^{2(m+n)}}{(m+n)!} - \\ - {}^{m+n}C_1 \frac{(HR)^{2(m+n-1)}}{(m+n-1)!} + {}^{m+n}C_2 \frac{(HR)^{2(m+n-2)}}{(m+n-2)!} - \text{etc.} \Bigg]\end{aligned}\right\} \quad (17)$$

i.e. the same expression as $13^b$, found in a different way.

As to the determination of the $A$ coefficients, it is expedient to consider first the case that $a$ and $b$ are equal to zero.

It is then easily found that

$$S_{2n.0} = \frac{(2n)!}{2^n.n!}(M^2H^2 - {}^1/_2)^n = \frac{(2n)!}{2^n.n!}P^n$$

and similarly for the $V$ function

$$S_{0.2n} = \frac{(2m)!}{2^m.m!}(M'^2H^2 - {}^1/_2)^n = \frac{(2m)!}{2^m.m!}Q^m$$

$$M^2 = \frac{1}{2h^2} \qquad M'^2 = \frac{1}{2h'^2}.$$

The arbitrary constant $H$ now can be given such a value that $P$ or $Q = 0$; putting $P = 0$, then $H = h$, and in the development only the $V$ functions remain.

If $a$ and $b$ are different from zero, then it appears that (for $P=0$)

$$V \left\{ \begin{aligned} S_2 &= Q + h^2b^2 \\ S_4 &= 3\,Q^2 + 6\,h^2b^2Q + h^4b^4 \\ S_6 &= 15\,Q^3 + 45\,h^2b^2Q^2 + 15\,h^4b^4Q + h^6b^6 \end{aligned}\right.$$

or, generally:

$$S_{2n\,.\,2m} = h^{2n}\, a^{2n} \frac{(2m)!}{m!\, 2^m} \left[ Q^m + {}^mC_1 \frac{h^2 b^2 Q^{m-1}}{1} + \right.$$
$$\left. + {}^mC_2 \frac{h^4 b^4 Q^{m-2}}{1\,.\,3} + \ldots \frac{h^{2m} b^{2m}}{1\,.\,3 \ldots (2m-1)} \right] \quad . \; (18)$$

Although, therefore, in this case the $U$ functions do not altogether vanish, still the form remains the same as in ($13^a$) and ($13^b$) because, as appears from (17), the polynomium has the same value for all terms where $n + m$ has the same value so that e.g. the terms with

$$A_{4.0} \qquad A_{2.2} \quad \text{and} \quad A_{0.4}$$

can be taken together.

In order to investigate in how far a given collection of wind-observations may be considered as a collection of two independent quantities depending on chance, we have, therefore, in the first place to calculate the constants $a, b, \beta$, $h$ and $h'$ from the set of observations.

In the second place the development ($13b$) has to be applied to the frequency-series of the wind-velocities, thereby taking for $H$ either $h$ or $h'$ so that the term $U_2$ remains.

A comparison between the $A$ constants calculated in this way with those determined according to (18) then gives an answer to the question.

9. By writing in (15) $hR \sin\theta$ and $hR \cos\theta$ for $x$ and $y$, multiplying by $RdR$ and integrating with respect to $R$ between the limits $\infty$ and zero, we obtain a development representing the frequencies of the directions independent of velocity.

The even terms $U_{2n}$ and $V_{2m}$, or the product $V_{2n}\, U_{2m}$ then give rise to a series of terms of the type $F_1$ (§ 7) all of which have the factor $\cos 2n\alpha$ in common.

The even terms $U_{2n+1}\, V_{2m+1}$, produced by the product of two uneven terms have $\sin\alpha \cos\alpha$ as a common factor and give rise to terms with $\sin 2n\alpha$, according to the functions $F_2$ in § 7.

The uneven terms, analogous to $F_3$ and $F_4$, assume a simpler form, namely:

$$U_{2n+1} = K \sin\alpha \cos^{2n}\alpha \quad \text{and} \quad V_{2n+1} = K \cos\alpha \sin^{2n}\alpha$$

and therefore give rise to terms with $\sin(2n+1)\,\alpha$ and $\cos(2n+1)\,\alpha$, whereas all non-periodic terms vanish, except in the first term with $A_0$.

A comparison with the Fourier-series thus produced and calculated on the base of the five wind-constants with the Fourier-series as directly deduced from the observations of direction-frequencies, then again gives an answer to the question.

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