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Physics. — "*On the theory of the string galvanometer of* EINTHOVEN."
By Dr. L. S. ORNSTEIN. (Communicated by Prof. H. A. LORENTZ.)

(Communicated in the meeting of September 26, 1914).

§ 1. Mr. A. C. CREHORE has developed some considerations in the Phil. Mag. of Aug. 1914 [1]), on the motion of the string galvanometer, which cause me to make some remarks on this subject.

For a string, immersed in a magnetic field H, and carrying a current of the strength J, the differential equation for the elongation in the motion of the string is

$$\frac{\partial^2 y}{\partial t^2} + \varkappa \frac{\partial y}{\partial t} = a^2 \frac{\partial^2 y}{\partial x^2} + \frac{HJ}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $\varkappa$ is the constant damping factor, $a^2 = \frac{T_1}{\varrho}$, T_1 is the tension and ϱ is the density. The direction of the stretched string has been chosen as the x-axis. For $x = 0$ and $x = l$ the string is fixed, so $y = 0$. In deducing the equation the ponderomotive force is supposed to be continually parellel to the elongation y, which is only approximately true, since the force is at every moment perpendicular to the elements of the string (perpendicular to J and H); but if y may be taken small, then the equation (1) is valid. The approximation causes a parabola to be found for the state of equilibrium with constant H and J, instead of the arc of a circle, as it ought to be; however, the parabola is identical with a circle to the degree of approximation used.

Dr. CREHORE now observes, that the equation (1) may be treated after the method of normal coördinates by putting

$$y = \Sigma \varphi_s \sin \frac{s\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Besides the equation 1, he deduces a set of equations, the "circuit equations", which give a second relation between φ_s and J (from (1) there originates in the well-known way an equation for every coordinate φ_s). The obtained solutions will be independent, when the circuit equation is true, and again their sum is a solution of the problem. However, from the deduction of the circuit equation it cannot well be seen whether this is the case, since not entirely exact energetic considerations underlie this deduction. Now supposing the string to be linked in a circuit with resistance R, and self-induction L, the circuit-equation may be easily found by applying MAXWELL's

[1]) Theory of the String Galvanometer of EINTHOVEN. Phil. Mag. Vol. 28, 1914, p. 207.

induction-equation. For in consequence of the motion of the string in the magnetic field the number of lines of force passing through the circuit changes to an amount proportional to

$$\int_0^l \frac{\partial y}{\partial t}\, dx$$

Expressed in the units used by Dr. CREHORE, the induction-equation now takes the form:

$$E = RJ + L\frac{dJ}{dt} + H\int_0^l \frac{\partial y}{\partial t}\, dx \, . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where E is an external electromotive force acting on the circuit.

§ 2. The problem of finding the vibrations governed by the equations (1), (3) and the condition $y = 0$ for $x = 0$ and $x = l$, can be easily solved. First, let E be 0, and so the question of free (damped) vibrations may be put. Suppose that

$$y = \varphi\, e^{i\omega t} \qquad J = I\, e^{i\omega t}$$

where φ is a function of x and I is a constant. Then the equations change into

$$-\,\omega^2 \varphi + i\omega\varkappa\varphi - a^2 \frac{\partial^2 \varphi}{\partial x^2} = \frac{HI}{\varrho}$$

$$0 = RI + Li\omega I + iH\omega \int_0^l \varphi\, dx\, .$$

Hence

$$(\omega^2 - i\omega\varkappa)\,\varphi + a^2 \frac{\partial^2 \varphi}{\partial x^2} = \frac{H^2 i\omega}{\varrho(R + Li\omega)} \int_0^l \varphi\, dx\, .$$

Putting $\omega^2 - i\omega\varkappa$ in the first member n^2 and $\dfrac{H^2 i\omega}{\varrho(R + Li\omega)} = p$ we have

$$n^2 \varphi + a^2 \frac{\partial^2 \varphi}{\partial x^2} = p \int_0^l \varphi\, dx\, .$$

This equation may be satisfied by

$$\varphi = A \cos \frac{n}{a} x + B \sin \frac{n}{a} x + C$$

provided that

$$n^2 C = p\left(\frac{a}{n} A \sin\frac{nl}{a} + \frac{a}{n} B\left(1 - \cos\frac{nl}{a}\right) + C\,l\right). \quad . \qquad (4)$$

whereas, because of the boundary conditions, we must have

$$A + C = 0$$

$$A \cos\frac{nl}{a} + B \sin\frac{nl}{a} + C = 0.$$

This gives for the frequency the transcendental equation

$$n^2 = p\left(-\frac{a}{n}\sin\frac{nl}{a} - \frac{a}{n}\,\frac{\left(1-\cos\frac{nl}{a}\right)^2}{\sin\frac{nl}{a}} + l\right)$$

or

$$n^2 \sin\frac{nl}{a} = p\left(l \sin\frac{nl}{a} - 2\frac{a}{n}\left(1 - \cos\frac{nl}{a}\right)\right).$$

From this it appears immediately that we must have

$$\sin\frac{nl}{2a} = 0. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \qquad (5)$$

or

$$n^2 \cos\frac{nl}{2a} = p\left(l \cos\frac{nl}{2a} - \frac{2a}{n}\sin\frac{nl}{2a}\right) \quad . \ . \ . \ . \ . \ . \qquad (6)$$

(5) can be satisfied by

$$\frac{nl}{2a} = k\pi \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \qquad (7)$$

or, hence

$$\omega^2 - i\omega\varkappa = \left(\frac{2k\pi a}{l}\right)^2.$$

As is immediately to be seen, these are the damped vibrations of even order, which the string can perform in the absence of the current. It is evident that the presence of current and field have no influence on the vibrations of even order. If the resistance is infinitely great, the constant p in the equation (6) is zero. In this case the equations can be satisfied by $n = 0$, or $\omega = 0$, i. e. the string is at rest; and further by

$$\cos\frac{nl}{2a} = 0.$$

Hence

$$\frac{nl}{2a} = (2k+1)\frac{\pi}{2} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \qquad (8)$$

or

$$\omega^2 - i\omega\varkappa = \left(\frac{(2k+1)\,\pi a}{l}\right)^2.$$

The frequencies arrived at are those of odd order, altered by current and field. For large values of R an approximate value of n can easily be expressed in the form $n_\infty + \frac{q}{R}$. From (6) follows

$$\bar{n}_s = n_s + \frac{H^2 \omega_s}{\varrho R} \frac{4a^2}{n_s^3\, l} i$$

s being an odd number, l being taken zero, while for ω and n_s their values for $R = \infty$ must be put. Taking $\varkappa = 0$, i.e. neglecting the air-damping in comparison with the electrical damping, we find

$$\omega_s = \frac{\pi\, sa}{l} + \frac{4H^2 il}{R\varrho\, s^2 \pi^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In the solution, therefore, there is a damping factor of the form

$$e^{-\frac{4H^2 l}{R\rho s^2 \pi^2} t}.$$

The influence of the damping is the less, the greater the value of s is. This is directly evident, for if s is great, the string vibrates in a great number of parts with opposite motion. The electromotive force generated by those parts therefore is annulled.

In case R is small, the roots of the equation (6) are those of the transcendental equation

$$l \cos\frac{nl}{2a} - \frac{2a}{n} \sin\frac{nl}{2a} = 0$$

or

$$\frac{2a}{nl}\, tg\, \frac{nl}{2a} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)^{1)}$$

The quantity $\frac{nl}{2a}$ approaches to odd multiples of $\frac{\pi}{2}$. For small values of R an approximate form $n_0 + aR$ can be easily indicated. Taking again $L = 0$ and $\varkappa = 0$, we find

$$\bar{n}_s = n_s + \frac{2a\, R\varrho}{l^2 H^2} i.$$

where n_s is an arbitrary root of (10). In case the resistance is small, all vibrations suffer the same damping.

For φ we find

$$\varphi = \frac{\sin\frac{nl}{2} - \sin\frac{n(l-x)}{a} - \sin\frac{nx}{a}}{\sin\frac{nl}{a}}.$$

[1]) Compare for instance RIEMANN-WEBER, Partielle-Differential Gleichungen, II, p. 129.

hence for y

$$y = e^{i\omega t} \frac{\sin\frac{nl}{a} - \sin\frac{n(l-x)}{a} - \sin\frac{nx}{a}}{\sin\frac{nl}{a}} \quad . \quad . \quad . \quad . \quad (11)$$

The real and imaginary part of this expression satisfy the equations and the boundary conditions. A sum of solutions for different values of ω satisfies the equation. If y and $\frac{dy}{dt}$ are given for $t = 0$, we can with the aid of the given functions find the solution. The found proper functions are not orthogonal, but by an appropriate linear substitution orthogonal functions can be obtained. If y is known, I can be calculated from (3).

§ 3. It is useful to work out the problem. Using the assumption (2) of CREHORE, we obtain for φ_s the following set of equations (taking k and l zero):

$$\ddot{\varphi}_s + n_s^2 \varphi_s = \frac{4HJ}{s\pi\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and

$$RJ + \frac{2lH}{\pi} \Sigma \frac{\dot{\varphi}_s}{s} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$n_s = \frac{s\pi a}{l}.$$

Here s is an odd number; for even values the second member of (12) is zero, and the even vibrations are therefore unchanged.

Now putting

$$\varphi_s = a_s e^{i\omega t} \quad , \quad J = I e^{i\omega t},$$

and

$$a_s = \frac{4HI}{s\pi\varrho} \frac{1}{n_s^2 - \omega^2},$$

we find

$$RI + \frac{2l H^2 i\omega}{\pi} \Sigma \frac{a_s}{s} = 0.$$

The frequency-equation therefore is

$$R + \frac{8l H^2 i\omega}{\varrho\pi_2} \Sigma \frac{1}{s^2 (n_s^2 - \omega^2)} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

This frequency-equation has the same roots as equation (6), which if $\varkappa$ and l have been taken 0, takes the form

$$\omega \cos\frac{\omega l}{2a} = \frac{iH^2 l}{R\varrho}\left(\cos\frac{\omega l}{2a} - \frac{2a}{l\omega}\sin\frac{\omega l}{2a}\right).$$

The identity of these frequency-equations can be easily shown. Put $\frac{8l\,H^2}{R\varrho}\,i = k$, then (6$a$) takes the form

$$1 - k\,.\,\frac{8}{\pi^2}\,\Sigma\,\frac{1}{s^2} + k\,.\,\frac{8}{\pi^2}\,\Sigma\,\frac{n_s^2\omega}{s^2\,(n_s^2 - \omega^2)} = 0$$

The sum of inverse squares of odd numbers is $\frac{\pi^2}{8}$. Further, $n_s^2 = \frac{\pi^2 a^2}{c^2}$, therefore the first member amounts to

$$1 - \frac{k}{\omega} + 8k\frac{a^2}{l^2\omega}\,\Sigma\,\frac{1}{n_s^2 - \omega^2} = 0.$$

For $tg\,z$ we have

$$tg\,z = -\sum_1^\infty \frac{2z}{z^2 - \left(\frac{s\pi}{2}\right)^2}$$

where s is again an odd number, therefore we obtain

$$1 - \frac{k}{\omega} + \frac{2a}{\omega^2 l}\,k\,tg\,\frac{\omega l}{2a} = 0 \quad \ldots\ldots\ldots \quad (14)$$

The equation (6) takes the form

$$\omega\cos\frac{\omega l}{2a}\left(1 - \frac{k}{\omega} + \frac{2a}{l\omega^2}\,k\,tg\,\frac{\omega l}{2a}\right) = 0 \quad \ldots\ldots \quad (15)$$

The equations (14) and (15) have the same roots, for the vectors ω and $\cos\frac{\omega l}{2a}$ do not contribute roots to (15).

Having found the roots of (14), we can determine y. Each root yields a FOURIER series. In the case that ($R = \infty$), $\sin\frac{s\pi x}{e}$ must be combined with one frequency only. For our case we have

$$y = \Sigma_s\,\Sigma_\varkappa\,\frac{A_\varkappa}{s\,(n_s^2 - \omega_\varkappa^2)}\,.\,e^{i\omega_\varkappa t}\,\sin\frac{s\pi x}{l}$$

$$\Sigma_s\frac{1}{s}\sin\frac{s\pi x}{l}\,\Sigma_R\,\frac{A_\varkappa}{n_s^2 - \omega_\varkappa^2}\,e^{i\omega_\varkappa t} \quad \ldots\ldots\ldots \quad (16)$$

The FOURIER series which is the vector of $A\,e^{i\omega_\varkappa t}$ must be equal to the function which in § 2 appears as the vector of the same exponential. This can be shown by direct development. It is apparent that by a given frequency all the original normal coordinates are set into motion. For very great and very small values of R, the constants A in the expression (16) can easily be determined.

We can also use (9) and (11). Let us write (11) in the form

$$y = e^{i\omega t}\left(1 - \frac{\cos\left[\frac{nl}{2a} - \frac{xn}{a}\right]}{\cos\frac{nl}{2a}}\right)$$

and let us introduce the value of n from (9), we then find

$$y = e^{n_s i t - \frac{4H^2 l}{R\varrho s^2\pi^2} t}\left(1 - \cos\frac{n_s x}{a} - i\frac{R}{\sigma_s}\sin\frac{n_s x}{a}\right)$$

where $\sigma_s = \dfrac{2H^2l^2}{R\varrho s^2\pi^2}$. Separating the real and imaginary parts, we find

$$y = \Sigma_s\, e^{-\frac{4H^2 l}{R\varrho s^2\pi^2 t}}\left\{\left(\frac{\sigma_s}{R}\left(1 - \cos\frac{n_s x}{a}\right)\cos n_s t + \sin\frac{n_s x}{a}\sin n_s t\right)A_s\right.$$
$$\left. + \left(\frac{\sigma_s}{R}\left(1 - \cos\frac{n_s x}{a}\right)\sin n_s t - \sin\frac{n_s x}{a}\cos n_s t\right)B_s\right.$$

For the time $t = 0$

$$y_0 = \Sigma_s\left\{\frac{\sigma_s}{R}\left(1 - \cos\frac{n_s x}{a}\right)A_s - \sin\frac{n_s x}{a}B_s\right\}$$

and

$$\left(\frac{dy}{dt}\right)_0 = \Sigma_s n_s\left\{\sin\frac{n_s x}{a}A_s + \frac{\sigma_s}{R}\left(1 - \cos\frac{n_s x}{a}\right)B_s + \frac{4H^2 l}{R\varrho s^2 n_s}B_s\right\}.$$

Putting $\int_0^l y_0 \sin\frac{s\pi x}{l}\,dx = a_s$ and $\int_0^l \frac{y\, s\pi x}{l}\,dx = b_s$ we have

$$b_s = \Sigma_s \frac{2\sigma_s}{R}A_s - \frac{l}{2}B_s$$

$$a_s = n_s\frac{l}{2}A_s + \frac{2n_s}{R}\Sigma\,\sigma_s B_s + \frac{4H^2 l}{R\varrho s^2 n_s}B_s\,.$$

For $R = \infty$ we get $b_s = -\frac{l}{2}B_s$, $a_s = \frac{n_s l}{2}A_s$. Therefore $B_s = -\frac{l}{2}b_s$ and $A_s = \frac{2}{n_s l}a_s$. Putting

$$A_s = \frac{2}{l n_s}a_s + \frac{\alpha_s}{R}$$

$$B_s = -\frac{2}{l}b_s + \frac{\beta_s}{R}$$

we have

$$0 = \frac{4}{Rl} \Sigma \frac{\delta_s a_s}{n_s} - \frac{l}{2} \frac{\beta_s}{R}$$

$$0 = \frac{a_s}{R} - \frac{4}{lR} \Sigma n_s \delta_s b_s - \frac{8H^2 b_s}{R\varrho s^2 n_s}.$$

These series are convergent, if the conditions for the ordinary FOURIER series are fulfilled. We can therefore calculate α_s and β_s with the help of the given formulae.

§ 4. In the case E is a given function of the time, our equation can also easily be solved.

a. First if E is constant, we have

$$\frac{\partial^2 y}{\partial t^2} + \varkappa \frac{\partial y}{\partial t} = a^2 \frac{\partial^2 y}{\partial x^2} + \frac{HJ}{\varrho}$$

$$E = RJ + H \int_0^l y \, dx.$$

The current J and y can be divided into two parts, the one depending on t, the other not; we indicate those parts by the indices 1 and 2. For the first part we have

$$0 = a^2 \frac{\partial^2 y_1}{\partial x^2} + \frac{HJ_1}{\varrho}$$

$$E = RJ_1$$

therefore

$$a^2 \frac{\partial^2 y_1}{\partial x^2} = -\frac{EH}{R\varrho},$$

from which y_1 can be determined if we take into account that y_1 vanishes for $x = 0$ and $x = l$. The determining of the second part leads to the problem treated in § 3. The solution can be used in order to fulfill given initial conditions. If an initial value of J is given, then y must fulfill at $t = 0$ a condition following from (3).

b. Further, we can consider the case $E = \mathrm{E} \cos pt$.

Putting $L = 0$, we can try the solution

$$y = \varphi \cos (pt + \beta)$$

$$J = I \cos (pt + \beta)$$

where φ is a function of x. The first equation gives

$$-p^2 \varphi - a^2 \frac{\partial^2 \varphi}{\partial x^2} = HI.$$

This equation can be solved by

$$\varphi = A \cos \frac{p}{a} x + B \sin \frac{p}{a} x + C$$

or according to the above

$$\varphi = -\frac{HI}{p^2}\left(-\cos \frac{p}{a} x - \frac{\left(1 - \cos \frac{pl}{a}\right)}{\sin \frac{pl}{a}} \sin \frac{p}{a} x + 1\right) \quad . \tag{17}$$

Introducing this result into the second equation, we obtain

$$\mathrm{E} \cos pt = RI \cos(pt + \beta) + \frac{H^2 I}{\varrho p} \sin(pt + \beta)$$

$$\left(-\frac{a}{p}\sin\frac{pl}{a} - \frac{a}{p}\frac{\left(1\text{-}\cos\frac{pl}{a}\right)^2}{\sin\frac{pl}{a}} + l\right). \tag{18}$$

Now take

$$\frac{H^2 I}{\varrho p R}\left(-\frac{a}{p}\sin\frac{pl}{a} - \frac{\left(1 - \cos\frac{pl}{a}\right)^2}{\sin\frac{pl}{r}} + l\right) = tg\, \alpha$$

then we find

$$\mathrm{E} \cos pt = I \sqrt{R^2 + \frac{H^4}{\varrho^2 p^2}\left(-\frac{a}{p}\sin\frac{p}{a} l - \frac{a}{p}\frac{\left(1 - \cos\frac{p}{a} l\right)^2}{\sin\frac{p}{a} l} + l\right)^2} \cos(pt + \beta - \alpha)$$

From this we find for the retardation of phase, $\beta = \alpha$; and for the amplitude of I

$$I = \frac{E}{r}$$

where r represents the square root in the second member. The current I being found in this way, y can be determined from (17).

When L does not vanish, we can suppose y and I to depend on e^{ipt}; and finally taking the real part, and following the above method we find the values of y and I.

If we express y by (2), the solution can also easily be found. We then have

$$\varphi_s = \frac{4HJ}{s\,\pi(n_s^2 - p^2)}.$$

Substituting this into the second equation of § 3 (where zero has been replaced by $E \cos pt$) we find

$$E \cos pt = RI \cos (pt + \beta) - \frac{\pi^2}{p 8 H^2 I} l \sin (pt + \beta) \Sigma \frac{1}{s^2} \frac{1}{(n_s^2 - p^2)}$$

from which I can be found. The sum in the second member can be put in a way analogous to that of § 3, into a form identical with (18). Our result does not agree with that of CREHORE (compare p. 214). In our solution the retardation of phase is the same for all vibrations, which is not the case in CREHORE's paper.

It may be observed that in our problem we have to do with a system of an infinite number of variables in which a dissipation-function couples the variables; for eliminating J from (12) and (13), we obtain

$$\ddot{\varphi}_s + n_s^2 \varphi = - \frac{8H^2}{s\pi\varrho R} \Sigma \frac{\dot{\varphi}_s}{s}.$$

The dissipation F takes the form

$$F = \frac{8H^2}{\pi^2 \varrho R} \left(\Sigma \frac{\dot{\varphi}_s}{s} \right)^2.$$

Groningen, Sept. 1914.

Physics. — *"Accidental deviations of density and opalescence at the critical point of a single substance."* By Dr. L. S. ORNSTEIN and F. ZERNIKE. (Communicated by Prof. H. A. LORENTZ.)

(Communicated in the meeting of September 26, 1914).

1. The accidental deviations for a single substance as well as for mixtures have been treated by SMOLUCHOWSKI [1]) and EINSTEIN [2]) with the aid of BOLTZMANN's principle; by ORNSTEIN [3]) with the aid of statistical mechanics. It appears as if the considerations used and the results obtained remain valid in the critical point. SMOLUCHOWSKI has applied the formula found for the probability of a deviation to the critical point itself, and has found for the average deviation of density

$$\bar{\delta} = \frac{1.13}{\sqrt[4]{v}}.$$

He has used this formula to express in terms of the mean density

[1]) M. SMOLUCHOWSKI, Theorie Cinétique de l'opalescence. Bull. Crac 1907 p. 1057. Ann. der Phys. Bd. 25, 1908, p. 205. Phil. Mag. 1912. On opalescence of gases in the critical state. W. H. KEESOM, Ann. der Phys. 1911 p. 591.

[2]) A. EINSTEIN. Ann. der Phys. Bd. 33, 1910, p. 1276.

[3]) ORNSTEIN, These Proc., **15**, p. 54 (1912).