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Chemistry. — Note on our paper: "*The Allotropy of Lead I*" by Prof. ERNST COHEN and W. D. HELDERMAN.

In our first communication on the Allotropy of Lead [1]) we stated that we resumed our investigations on this subject after having received a letter from Mr. HANS HELLER in Leipsic which showed us the way in which fresh experiments had to be directed. In this letter Mr. HELLER kindly invited us to continue these investigations.

As Mr. HELLER writes in a letter dated Jan. 21st 1915: "Gewünscht hätte ich freilich, dass der Ort, an dem ich meine Versuche machte, das hiesige Chemische Laboratorium, in der Veröffentlichung genannt worden wäre", we comply with pleasure with his request by publishing the above statement.

Utrecht, January 1915. VAN 'T HOFF-*Laboratory*.

Mathematics. — "*Characteristic numbers for a triply infinite system of algebraic plane curves*" By Prof. JAN DE VRIES.

(Communicated in the meeting of Dec. 30, 1914).

1. The curves of order n, c^n, of a triply infinite system Γ (*complex*) cut a straight line l in the groups of an involution I_n^3 of the third rank. The latter possesses $4(n-3)$ groups with a quadruple element; l is consequently four-point tangent, t_4, for $4(n-3)$ curves c^n. Any point P is base-point of a net N belonging to Γ, hence point of undulation, R_4, for *six* curves c^n. [2]) If l rotates round P, the points R_4 describe a *curve* $(R_4)_P$ *of order* $(4n-6)$, *with sixfold point* P.

The tangent t_4 cuts c^n moreover in $(n-4)$ points S; the locus $(S)_P$ has apart from P, $4(n-3)(n-4)$ points in common with l. The tangents t_4 of a net envelop a curve of class $6n(n-3)$ [3]); as P is sixfold point on the curve (R_4) belonging to the net determined by P, P will lie on $6n(n-3)-24$ or $6(n-4)(n+1)$ tangents t_4 of which the point of contact R_4 lies outside P. So P is a $6(n-4)(n+1)$-fold point on $(S)_P$, and the order of this curve is $4(n-3)(n-4)+6(n+1)(n-4)$ or $2(n-4)(5n-3)$.

Let us now consider the correspondence which is determined in

1) Proceedings **17**, 822 (1914).

2) Cf. p. 937 of my paper: "*Characteristic numbers for nets of algebraic curves*". (Proceedings Vol. XVII, p. 935). For the sake of brevity this paper will be quoted by N.

3) N. p. 936.

a pencil of rays with vertex M by the pairs of associated points R_4 and S. Any ray MR_4 contains $(4n-6)$ points R_4, consequently determines $(4n-6)(n-4)$ points S; any ray MS contains $(n-4)(10n-6)$ points S, produces therefore as many rays MR_4. To the $(n-4)(14n-12)$ coincidences, the ray MP belongs $4(n-3)(n-4)$ times; for on MP lie $4(n-3)$ points R_4, hence $4(n-3)(n-4)$ points S. The remaining coincidences arise from the coinciding of a point R_4 with one of the corresponding points S. This takes place in the point of contact R_5 of a c^n with a five-point tangent t_5. From this it ensues that the *five-point tangents of* Γ *envelop a curve of class* $10n(n-4)$.

We further consider the symmetrical correspondence between the rays of (M), containing two intersections S, S' belonging to the same point of contact R_4. Its characteristic number is apparently $(10n-6)(n-4)(n-5)$. On MP lie $4(n-3)(n-4)(n-5)$ pairs S, S'; as many coincidences are represented by MP. The remaining coincidences arise from the coinciding of a point S' with a point S. hence arise from lines $t_{4,2}$, which have with a c^n in a point R_4 a four-point contact, and in a point R_2 a two-point contact. *The tangents* $t_{4,2}$, *envelop therefore a curve of class* $16n(n-4)(n-5)$.

2. Any point of the arbitrary straight line a is, as basepoint of a net belonging to Γ, point of contact R_4 for six curves c^n. The sextuples in this way coupled to a form a system $[c^n]$, of which the index is equal to the order of the locus of the points of undulation R_4 on the curves of the net set apart out of Γ by an arbitrary point P, consequently equal to $3(6n-11)$ [1]. The tangents t_4, of which the points of contact R_4 lie on a, form a system $[t_4]$ with index $(4n-6)$ for through P pass $(4n-6)$ straight lines t_4, having their point of contact R_4 on a (§ 1). Two projective systems $[c^r]$ and $[c^s]$ with indices ϱ and σ produce a curve of order $(r\sigma+\varrho s)$. If to each c^n of the above system the line t_4 is associated, which touches it on a, a figure arises of order $3(6n-11)+n(4n-6)$. The latter consists of the straight line a counted 24 times, and the locus of the points S, which each t_4 has moreover in common with the corresponding c^n. This curve $(S)_a$ is therefore of order $(4n^2+12n-57)$.

For $n=4$, $(S)_a$ is therefore of order 55. In a complex of curves c^4 occur therefore 55 figures consisting of a c^3 and a straight line c^1. If all c^4 pass through 11 fixed points the straight lines c^1 are apparently the sides of the complete polygon determined by the base-points.

To the intersections of $(S)_a$ with a belong the $4(n-3)$ groups of

[1]) N bl. 937.

$(n-4)$ points S arising from the c^n having a as tangent t_4. In each of the remaining intersections a point R_4 has coincided with a point S into a point R_5. *The points, where a c^n possesses a five-point tangent, lie therefore on a curve (R_5) of order* $(40n-105)$.

For $n=4$, the number 55 is duly found.

3. To each c^n possessing a tangent t_5, we associate that tangent; the latter cuts it moreover in $(n-5)$ points V. The locus of the points V, together with the curve (R_5) to be counted five times, is produced by the projective systems $[c^n]$ and $[t_5]$. The system $[t_5]$ has (§ 1) as index $10n(n-4)$. The curves c^n passing through a point P form a net; in this net occur $15(n-4)(4n-5)$ curves with a t_5 [1]; this number is the index of $[c^n]$. For the order of the curve (V) is found $15(n-4)(4n-5)+10n^2(n-4)-5(40n-105)=5(n-5)(2n^2+14n-33)$.

In the pencil of rays (M) the pairs of points R_5, V determine a correspondence with characteristic numbers $(n-5)(40n-105)$ and $(n-5)(10n^2+70n-165)$. The $10n(n-4)$ tangents t_5 passing through M produce each $(n-5)$ coincidences. As the remaining ones arise from the coinciding of R_5 with V, it appears that Γ *contains* $30(n-5)(5n-9)$ *curves with a six-point tangent* t_6.

The symmetrical correspondence (MV, MV') has as characteristic number $(10n^2+70n-165)(n-5)(n-6)$, while the $10n(n-4)$ tangents t_5 passing through M represent each $(n-5)(n-6)$ coincidences. From this ensues that Γ possesses $10(n-5)(n-6)(n^2+18n-33)$ *curves with a tangent* $t_{5,2}$.

4. The I_n^3, which Γ determines on l, possesses $6(n-3)(n-4)$ groups in which a triple element occurs beside a twofold one; consequently is l for $6(n-3)(n-4)$ curves a tangent $t_{2,3}$. If l rotates round P, the points of contact R_2 and R_3 describe two curves $(R_2)_{2,3}$ and $(R_3)_{2,3}$. P is as base-point of a net, point of contact R_2 for $3(n-4)(n+3)$ [2], point of contact R_3 for $(n-4)(n+9)$ [3] curves. So $(R_2)_{2,3}$ is of order $(n-4)(3n+9)+6(n-3)(n-4)$ or $(n-4)(9n-9)$ and $(R_3)_{2,3}$ of order $(n-4)(n+9)+6(n-3)(n-4)$ or $(n-4)(7n-9)$.

From the correspondence (MR_2, MR_3) may be deduced again that t_5 envelops a curve of class $10n(n-4)$. (See § 1).

Each tangent $t_{2,3}$ passing through P cuts the corresponding c^n in $(n-5)$ points W; on a ray passing through P lie therefore $6(n-3)$

1) N p. 938.
2) N p. 943.
3) N p. 942.

$(n-4)(n-5)$ points of the curve $(W)_P$. The c^n passing through P form a net, of which the tangents $t_{2,3}$ envelop a curve of class $9n(n-3)(n-4)$. [1])

As P is point of contact R_2 for $(n-4)(3n+9)$ and point of contact R_3 for $(n-4)(n+9)$ curves of the net, P lies, as point W, on $9n(n-3)(n-4)-2(n-4)(3n+9)-3(n-4)(n+9) = 9(n-4)(n-5)(n+1)$ tangents $t_{2,3}$. The order of $(W)_P$ amounts therefore to $6(n-3)(n-4)(n-5)+9(n-4)(n-5)(n+1)$ or $3(n-4)(n-5)(5n-3)$.

Starting from the correspondence (R_3, W) we arrive again at the class of the curve enveloped by $t_{4,2}$ (§ 1).

A new result is produced by the correspondence of the rays MR_2, MW. Its characteristic numbers are $(9n-9)(n-4)(n-5)$ and $(15n-9)(n-4)(n-5)$. The ray MP represents $6(n-3)(n-4)(n-5)$ coincidences. The remaining $18n(n-4)(n-5)$ arise from coincidences $R_2 \equiv W$, consequently from tangents $t_{3,3}$. As each $t_{3,3}$ determines two coincidences, *the twice osculating tangents* $t_{3,3}$ *envelop a curve of class* $9n(n-4)(n-5)$.

The symmetrical correspondence between the rays connecting M with the pairs of points W, W', belonging to the same c^n, has as characteristic number $(n-4)(n-5)(n-6)(15n-9)$. As MP represents $6(n-3)(n-4)(n-5)(n-6)$ coincidences, and the remaining ones arise in pairs from tangents $t_{2,2,3}$, *the tangents* $t_{2,2,3}$ *envelop a curve of class* $12n(n-4)(n-5)(n-6)$.

5. The I_n^3, which Γ determines on l, contains $\frac{4}{3}(n-3)(n-4)(n-5)$ groups with three double elements; as many curves c^n have l as triple tangent $t_{2,2,2}$. In the net determined by P occur $2(n+3)(n-4)(n-5)\,c^n$, on which P is point of contact of a triple tangent.[2]) If l rotates round P. the points of contact describe therefore a curve of order $4(n-3)(n-4)(n-5)+2(n+3)(n-4)(n-5)$ or $6(n-4)(n-5)(n-1)$.

We further determine the order of the locus of the groups of $(n-6)$ points Q, which l has moreover in common with the c^n, which it touches three times. The $t_{2,2,2}$ belonging to the net with base-point P envelop a curve of class $2n(n-3)(n-4)(n-5)$. [3]) As P is point of contact for $2(n+3)(n-4)(n-5)\,c^n$, the number of c^n intersecting their $t_{2,2,2}$ in P amounts to $2n(n-3)(n-4)(n-5) - 4(n+3)(n-4)(n-5)$ or $2(n+1)(n-4)(n-5)(n-6)$. The order

1) N. p. 936.
2) N. p. 943.
3) N. p. 939.

of (Q) is therefore equal to $2(n+1)(n-4)(n-5)(n-6)+\frac{4}{3}(n-3)(n-4)(n-5)(n-6)$ or $\frac{2}{3}(5n-3)(n-4)(n-5)(n-6)$.

The correspondence (MR, MQ) produces again the class of the envelop of $t_{2,2,3}$ (§ 4).

From the symmetrical correspondence (MQ, MQ'), which has as characteristic number $\frac{2}{3}(5n-3)(n-4)(n-5)(n-6)(n-7)$ and has in MP $\frac{1}{3}(n-3)(n-4)(n-5)(n-6)(n-7)$ coincidences, we find that the *quadruple tangents* $t_{2,2,2,2}$ *envelop a curve of class* $\frac{4}{3}n(n-4)(n-5)(n-6)(n-7)$.

6. Any point of the arbitrary straight line a, is, as base-point of a net, point of contact R_3 of $(n-4)(n+9)$ tangents $t_{2,3}$. [1]) The locus $(R_2)_a$ of the corresponding points of contact R_2 has two groups of points in common with a, the first group contains the $(40n-105)$ intersections with the curve (R_5), the second contains the $6(n-3)(n-4)$ points R_2, where a is touched by the curves c^n, osculating it in a point R_3. From this ensues that $(R_2)_a$ is of order $(6n^2-2n-33)$.

In order to find the order of the locus of the points W, which each $t_{2,3}$ has in common with its c^n, we consider the figure produced by projective association of the corresponding systems $[c^n]$ and $[t_{2,3}]$. The curves c^n, of the net determined by P, which possess a $t_{2,3}$, have their points of contact R_3 on a curve of order $3(n-4)(n^2+6n-13)$[2]); the latter intersects a in the points R_3 of the curves of $[c^n]$ passing through P. The index of $[t_{2,3}]$ is, see § 4, $(n-4)(7n-9)$. Considering that the figure produced is composed of $3(n-4)(n+9)$ times the straight line a, twice the curve $(R_2)_a$ and the locus $(W)_a$, we find for the order of the last-mentioned curve

$$(n-4)(3n^2+18n-39)+n(n-4)(7n-9)-3(n-4)(n+9)-$$
$$-2(6n^2-2n-33)=(n-5)(10n^2+4n-66).$$

The curve $(W)_a$ cuts a in $6(n-3)(n-4)$ groups of $(n-5)$ points W; in each of the remaining intersections a c^n has a four-point contact with a line $t_{4,2}$. *Consequently the points of contact* R_4 *of the tangents* $t_{4,2}$ *lie on a curve of order* $(n-5)(4n^2+46n-138)$.

The pairs of points R_2, W determine in a pencil of rays (M) a correspondence with characteristic numbers $(n-5)(10n^2+4n-66)$ and $(n-5)(6n^2-2n-33)$. The $(n-4)(7n-9)$ rays $t_{2,3}$ passing through M, which have their point of contact R_3 on a, represent each $(n-5)$ coincidences. From this ensues that *the points of contact* (inflectional points) *of the twice osculating lines are situated on a curve of order* $(n-5)(9n^2+39n-135)$.

1) N. p. 942.
2) N. p. 940.

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The symmetrical correspondence (MW, MW') has as characteristic number $(n-5)(n-6)(10n^2+4n-66)$ and possesses $(n-5)(n-6)$ coincidences in each of the $(n-4)(7n-9)$ rays MR_3. The remaining ones arise in pairs from tangents $t_{2,2,3}$. So we find that *the inflectional points R_3 of the tangents $t_{2,2,3}$ lie on a curve of order* $\frac{1}{2}(n-5)(n-6)(13n^2+45n-168)$.

7. Let us now consider the system $[c^n]$ of the curves which have the point of contact R_2 of their tangent $t_{2,3}$ on the straight line a. The curve $(R_3)_a$ cuts a in $(40n-105)$ points R_5 and in $6(n-3)(n-4)$ points R_3, where a osculates a c^n, for which it is tangent $t_{2,3}$. Consequently $(R_3)_a$ is of order $(6n^2-2n-33)$.

The system $[c^n]$ has as index $(n-4)(6n^2+15n-36)$ [1]; for $[t_{2,3}]$ the index is, see § 4, $(n-4)(9n-9)$. The figure produced by these projective systems consists of $2(n-4)(3n+9)$ times the straight line a, three times the curve $(R_3)_a$ and the locus of the points W^*, which each $t_{2,3}$ has moreover in common with its c^n. For the order of $(W^*)_a$ we find $(n-4)(6n^2+15n-36)+n(n-4)(9n-9)-2(n-4)(3n+9)-3(6n^2-2n-33)$ or $(n-5)(15n^2-3n-63)$.

The number of the intersections of $(W^*)_a$ with a again produces the order of the curve $(R_3)_{3,3}$.

The correspondence (MR_3, MW^*) has as characteristic numbers $(n-5)(15n^2-3n-63)$ and $(n-5)(6n^2-2n-33)$, while each of the rays $t_{2,3}$ passing through M represents $(n-5)$ coincidences. From $(n-5)[(15n^2-3n-63)+(6n^2-2n-33)-(n-4)(9n-9)]$ we find now that *the points of contact R_2 of the tangents $t_{2,4}$ are situated on a curve of order* $(n-5)(12n^2+40n-132)$.

The symmetrical correspondence (MW_1^*, MW_2^*) furnishes in the same way from $(n-5)(n-6)[(30n^2-6n-126)-(n-4)(9n-9)]$ the result, that *the points of contact R_2 of the tangents $t_{2,2,3}$ lie on a curve of order* $(n-5)(n-6)(21n^2+39n-162)$.

8. Let us now consider the system $[c^n]$ of the curves with triple tangent of which one of the points of contact, R_2, lies on the straight line a. The other two points of contact T_2, lie on a curve $(T_2)_a$, which has two groups of points in common with a. The former contains the $(n-5)(4n^2+46n-138)$ points R_4 of tangents $t_{1,2}$ (§ 6), the latter the groups of three points of contact T_2 lying on the curves c^n, for which a is triple tangent; these points are apparently to be counted twice. Consequently $(T_2)_a$ is of order $(n-5)(4n^2+46n-138)+8(n-5)(n-4)(n-3)$ or $(n-5(12n^2-10n-42)$.

[1]) N. p. 940.

We now consider again the figure produced by the projective systems $[c^n]$ and $[t_{2,2,2}]$. The former has as index $\frac{3}{2}(n-4)(n-5)(3n^2+$ $+5n-14)$ [1]), the latter, see § 5, $6(n-4)(n-5)(n-1)$. As the figure produced consists of $4(n+3)(n-4)(n-5)$ *times the line* a [2]), twice the curve $(T_2)_a$ and the locus of the points Q, which each c^n has moreover in common with its $t_{2,2,2}$, we find for the order of $(Q)_a$ $\frac{3}{2}(n-4)(n-5)(3n^2+5n-14)+6(n-4)(n-5)(n-1)n-4(n-4)$ $(n-5)(n+3)-2(n-5)(12n^2-10n-42)$ or $\frac{1}{2}(n-5)(n-6)(21n^2-$ $11n-72)$.

The curve $(Q)_a$ is cut by a in $\frac{1}{3}(n-3)(n-4)(n-5)$ groups of $(n-6)$ points Q, which are each to be counted thrice, and in a number of points (T'_3), where a c^n is osculated by a tangent $t_{3,2,2}$. From $\frac{1}{2}(n-5)(n-6)(21n^2-11n-72)-4(n-3)(n-4)(n-5)(n-6)$ ensues again (§ 6), that *the points of contact* T'_3 *of the tangents* $t_{3,2,2}$ *are situated on a curve of order* $\frac{1}{2}(n-5)(n-6)(13n^2+45n-168)$.

The correspondence between the points T_2, outside a, and the corresponding points Q, produces again the order of the curve (R_2) belonging to the tangents $t_{2,2,3}$ (§ 7).

The symmetrical correspondence (MQ, MQ') has as characteristic number $\frac{1}{2}(n-5)(n-6)(n-7)(21n^2-11n-72)$ and in each $t_{2,2,2}$ passing through M $(n-6)(n-7)$ coincidences. From $(n-5)(n-6)(n-7)[(21n^2-$ $11n-72)-6(n-1)(n-4)]$ ensues that the *locus of the points of contact of the quadruple tangents is a curve of order* $\frac{1}{2}(n-5)(n-6)$ $(n-7)(15n^2+19n-96)$.

9. Let us now consider the figure determined by the projectivity between the curves c^n, which possess a $t_{2,2,2,2}$ and those quadruple tangents. The system $[c^n]$ has as index $(n-1)(n+4)(n-4)(n-5)$ $(n-6)(n-7)$ [3]), the tangents $t_{2,2,2,2}$ form (§ 5) a system with index $\frac{1}{3}n(n-4)(n-5)(n-6)(n-7)$. The figure produced consists of twice the locus of the points of contact (§ 8) and the curve (S) of the intersections of the c^n with its quadruple tangents. For the order of (S) we now find $\frac{1}{3}(n-4)(n-5)(n-6)(n-7)(7n^2+9n-12)-\frac{1}{3}(n-5)$ $(n-6)(n-7)(30n^2+38n-192)$ or $\frac{1}{3}(n-5)(n-6)(n-7)(n-8)(7n^2+$ $7n-30)$.

The correspondence (T_2, S) determines in the pencil of rays (M) a correspondence with characteristic numbers $\frac{1}{3}(n-5)(n-6)(n-7)$ $(n-8)(15n^2+19n-96)$ and $\frac{4}{3}(n-5)(n-6)(n-7)(n-8)(7n^2+7n-30)$. As the tangents $t_{2,2,2,2}$ passing through M each represent $4(n-8)$

[1]) N p. 941.
[2]) N p. 943.
[3]) N p. 941.

coincidences, we find that *the complex contains* $(n-5)(n-6)(n-7)(n-8)(9n^2+37n-72)$ *curves with a tangent* $t_{2,2,2,3}$.

The correspondence (MS, MS') has as characteristic number $\frac{1}{3}(n-5)(n-6)(n-7)(n-8)(n-9)(7n^2+7n-30)$; each tangent $t_{2,2,2,2}$ passing through M represents $(n-8)(n-9)$ coincidences. From $\frac{2}{3}(n-5)(n-6)(n-7)(n-8(n-9)[(7n^2+7n-30)-2n(n-4)]$ ensues that $\frac{10}{3}(n-5)(n-6)(n-7)(n-8(n-9)(n^2+3n-6)$ *curves of* Γ *possess a quintuple tangent* $t_{2,2,2,2,2}$.

10. The curves c^n with a twice osculating tangent $t_{3,3}$ form a system with index $\frac{3}{2}(n-4)(n-5)(n^2+7n-9)$ [1]), their tangents $t_{3,3}$ (§ 4) a system with index $9n(n-4)(n-5)$. The figure produced by these projective systems consists of three times the curve $(R_3)_{3,3}$, containing the points of contact (§ 6) and the locus of the points O, which each c^n has moreover in common with its $t_{3,3}$. For the order of (O) we find $\frac{3}{2}(n-4)(n-5)(n^2+7n-9)+9n^2(n-4)(n-5)-3(n-5)(9n^2+39n+135) = \frac{9}{2}(n-5)(n-6)(3n^2+7n-21)$.

The correspondence (MR_3, MO) has as characteristic numbers $(n-5)(n-6)(9n^2+39n-135)$ and $9(n-5)(n-6)(3n^2+7n-21)$; each $t_{3,3}$ passing through M represents $2(n-6)$ coincidences. From this we find, that *the complex contains* $6(n-5)(n-6)(3n^2+29n-54)$ *curves with a tangent* $t_{3,4}$.

The correspondence (MO, MO') has as characteristic number $\frac{9}{2}(n-5)(n-6)(n-7)(3n^2+7n-21)$ and possesses in each $t_{3,3}$ passing through M an $(n-6)(n-7)$-fold coincidence. From this ensues that Γ possesses $9(n-5)(n-6)(n-7)(2n^2+11n-21)$ *curves with a tangent* $t_{3,3,2}$.

11. The curves c^n with a tangent $t_{4,2}$ form a system with index $6(n-4)(n-5)(n^2+11n-14)$ [2]), their tangents $t_{4,2}$ (§ 1) a system with index $16n(n-4)(n-5)$. These projective systems produce a figure, composed of four times the curve $(R_4)_{4,2}$, see § 6, twice the curve $(R_2)_{4,2}$, see § 7, and the locus of the points S, which each c^n has moreover in common with its $t_{4,2}$.

For the order of (S) we find $6(n-4)(n-5)(n^2+11n-14) + 16n^2(n-4)(n-5) - 4(n-5)(4n^2+46n-138) - 2(n-5)(12n^2+40n-132) = (n-5)(n-6)(22n^2+70n-192)$.

From (MR_4, MS) we find again the number of the $t_{5,2}$ (§ 3), from (MR_2, MS) the number of the $t_{3,4}$ (§ 10).

The symmetrical correspondence (MS, MS') produces *one* new characteristic number. Its characteristic number is apparently $(n-5)(n-6)(n-7)(22n^2+70n-192)$, while the $16n(n-4)(n-5)$ lines $t_{4,2}$

[1]) N p. 942.
[2]) N p. 938.

passing through M represent each $(n-6)(n-7)$ coincidences. From the remaining ones we find, that Γ possesses $(n-5)(n-6)(n-7)(14n^2+102n-192)$ *curves with a tangent* $t_{4,2,2}$.

12. Any point is, in general, node of *one* c^n belonging to Γ. We consider the system of the c^n having their node D on a straight line a. The straight line connecting D with the arbitrary point P, intersects c^n moreover in $(n-2)$ points E. The nodal curves of which a point E lies in P belong to the net with base-point P. Now the locus J of the nodes of the net (JACOBI's curve) is a curve of order $3(n-1)$, with node in P. The locus (E) has therefore a $3(n-1)$-fold point in P; so it is of order $(4n-5)$. In each intersection of (E) with a, a c^n has a node D, of which one of the tangents d passes through P. Consequently the locus $(D)_P$ of the nodes of which one of the tangents passes through P is a curve of order $(4n-5)$ having a node in P. Hence a straight line passing through P contains moreover $(4n-7)$ points D; *any straight line is therefore tangent in the node for* $(4n-7)$ *nodal curves.*

On a straight line l the tangents d of the nodal curves of which the node lies on a, determine a symmetrical correspondence (L, L'); its characteristic number is apparently $(4n-5)$. The intersection of a and l represents two coincidences, for the c^n, which has a node there, determines two points L each coinciding with the corresponding point L'. The remaining coincidences are produced by coinciding tangents d, d'. *So the locus* (C) *of the cusps* (cusp-locus) *of* Γ *is a curve of order* $4(2n-3)$.

13. The curves $(D)_P$ and $(D)_Q$ see § 12, have the $(4n-7)$ points D in common, for which PQ is one of the tangents. The remaining $(4n-5)^2-(4n-7)=16n^2-44n+32$ intersections are nodes of curves c^n, of which the lines d and d' pass through P and Q.

We now consider the system of the nodal c^n, of which a tangent d passes through P. The pairs of tangents d, d' determine on a straight line l a correspondence (L, L'). Any ray d is tangent for $(4n-7)$ curves; to its intersection L correspond therefore $(4n-7)$ points L'. Through L' pass $(16n^2-44n+32)$ tangents d'; as many points L have been associated to L'. The coincidences of (L, L') form two groups; the first contains the $(4n-5)$ points D situated on l, for which d passes through P. The remaining ones arise in consequence of d' coinciding with d; *the tangents in the cusps of the complex envelop therefore a curve of class* $(16n^2-44n+30)$.

14. To each nodal c^n, of which the node D lies on a we associate its tangents d,d', and consider the figure produced by those projective systems. As the c^n passing through a point P form a net, $3(n-1)$ nodal curves of the system in question pass through P. The index of the system $[d,d']$ is, as appeared above, $(4n-5)$. To the produced figure the straight line a belongs six times. So the order of the locus of the points E, which c^n has moreover in common with d,d', is a curve of order $n(4n-5)+6(n-1)-6=$ $=4n^2+n-12$.

For $n=3$ we find 27; in this case (E) consists apparently of 27 straight lines. If Γ has six base-points, this result is confirmed as follows. Each c^n passing through 5 base-points intersects a in two points D; the lines connecting these points with the 6th base-point form each with c^2 a c^3 of Γ, and belong to (E); in this way 12 straight lines are found. The connecting line b of two base-points cuts a in a point D, which determines with the remaining four base-points a c^2; the 15 lines b belong apparently also to (E).

The curve (E) cuts a in $(4n-7)$ groups of $(n-3)$ points E arising from the nodal curves which have a for tangent in their nodes. In each of the remaining intersections a nodal c^n has a three-point contact with one of its tangents d. From this ensues that *the locus* (F) *of the flecnodes is a curve of order* $(20n-33)$.

In the above case $n=3$ this figure consists of six conics and fifteen straight lines.

15. The tangents d,d' in the nodes of the nodal curves of a net envelop a curve of class $3(n-1)(2n-3)$ [1]. If the net has a base-point B there is a c^n having a node in B. Through B pass then $3(n-1)(2n-3)-6$ tangents d of nodal curves of which the node does not lie in B. In order to understand this we consider a net of cubics with seven base-points. Through the base-point B pass no tangents of proper nodal curves. But the straight line connecting B with another base-point B', forms with the c^2 passing through the remaining base-points a binodal c^3; the straight line BB' represents therefore two tangents d. For $n=3$ we have $3(n-1)(2n-3)=18$; as the 6 straight lines BB' represent 12 tangents d, the tangents d of the nodal c^3 having its node in B are each to be counted thrice.

We now consider the system of the nodal curves c^n, which send one of their tangents d through P. Any ray passing through P is tangent d for $(4n-7)$ curves (§ 12) and is moreover cut by those

[1]) Cf. for instance my paper "*On nets of algebraic plane curves.*" (These Proceedings VII, 631—633).

curves in $(4n-7)(n-3)$ points G. As base-point of a net belonging to Γ P lies on $(6n^2-15n+3)$ tangents d of nodal curves passing through P; so the locus (G) has in P a $(6n^2-15n+3)$-fold point and is therefore of order $(6n^2-15n+3)+(4n-7)(n-3)=10n^2-34n+24$.

The correspondence (MD, MG) has as characteristic numbers $(4n-5)(n-3)$ and $(10n^2-34n+24)$; the ray MP represents $(4n-7)(n-3)$ coincidences. As the remaining ones arise from coincidences $D\equiv G$, it ensues that *the inflectional tangents of the flecnodes envelop a curve of class* $(10n^2-32n+18)$.

16. Let the complex be given by the equation

$$\alpha A+\beta B+\gamma C+\delta D=0.$$

If the derivative of A with regard to x_k is indicated by A_k it ensues from the equations

$$\alpha A_k+\beta B_k+\gamma C_k+\delta D_k=0 \quad (k=1,2,3)$$

that an arbitrary point is node of *one* c^n, unless

$$\left\|\begin{array}{cccc} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{array}\right\|=0.$$

be satisfied.

The exceptional points in question K (*critical points*) are consequently common points of the four curves of Jacobi belonging to the nets $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$.

To the intersections of $|A_kB_kC_k|=0$ with $|B_kC_kD_k|=0$ belong the points, for which we have

$$\left\|\begin{array}{ccc} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{array}\right\|=0$$

and they are not situated on the two other curves J. The last mentioned relation is apparently satisfied by $2^2(n-1)^2-(n-1)^2=3(n-1)^2$ points; consequently the number of *critical points* amounts to $3^2(n-1)^2-3(n-1)^2$ or $6(n-1)^2$.

17. If Γ has a base-point B this is as base-point of any net of Γ, node of the curves J, consequently represents *four* points K. *The number of critical points of a complex with* b *base-points amounts therefore to* $6(n-1)^2-4b$.

Any point K is node of ∞^1 curves forming a pencil, hence cusp of *two* curves; the cuspidal tangents are the double rays of the involution formed by the pairs d, d'. So K *is node of the locus* (C) *of the cusps.*

All c^n passing through an arbitrary point P form a net, N. The curve J of N has a node in B and passes through all the points K; for through B passes *one* c^n of the pencil of nodal curves determined by K. The curves (C) and J have two points in common in each point K; they further intersect in the $12\,(n-1)\,(n-2)$ cusps of N; the remaining intersections are found in B. From $4\,(2n-3)\,(3n-3)-2\,[6(n-1)^2-4]-12\,(n-1)\,(n-2)=8$ it appears that the *curve* (C) *has a quadruple point in* B.

B is node for all c^n of a pencil, consequently cusp of two c^n; from this follows that each of the two cusptangents is touched by two branches of (C).

Any point K is flecnode for *five* c^n. In order to understand this we consider the curve which arises if to each nodal c^n of the pencil (K) its tangents d, d' are associated. The c^{n+2} thus produced has with a line d only $(n-3)$ points outside K in common.

The locus (F) *of the flecnodes passes therefore five times through each of the critical points.*

The locus J of the nodes in the net N, which is set apart from Γ by an arbitrary point P, has with (F) five intersections in each point K. They further have the $3\,(n-1)(10n-23)$ flecnodes [1]) of N in common, the remaining intersections lie in the b base-points. From $3\,(n-1)\,(20n-33)-5[6(n-1)^2-4b]-3\,(n-1)\,(10n-23)=20b$ it appears that *the curve* F *passes ten times through each of the base-points.*

Each of the inflectional tangents f of the five c^n, having a flecnode in B, touches two of the branches.

18. The curves (C) and (F) have in the critical points K and the base-points B of Γ $10\,[6\,(n-1)^2-4b]+40b$, or $60\,(n-1)^2$ points in common. Each of the remaining $(20n-33)\,(8n-12)-60\,(n-1)^2$ intersections is a cusp with a four-point tangent and at the same time to be counted twice as flecnode. *In* Γ *occur therefore* $(50n^2-192n+168)$ *cusps with four-point tangent.*

If we have $n=3$, $b=6$, these particular curves are easy to determine. Any line BB' is tangent of two conics passing through the remaining four base-points; through each point B pass two tangents to the conic of the remaining five base-points. All in all

[1]) N p. 944.

we find therefore $15 \times 2 + 6 \times 2 = 42$ figures (c^2, c^1), satisfying the condition.

The tangents in a flecnode we shall indicate by f and d; f indicates the inflectional tangent. We shall determine the index of the system $[d]$.

The curve $(D)_P$, containing the nodes which send one of their tangents through P (§ 12), passes through the points K and twice through the points B. With (F) it has, apart from those points, $(4n-5)(20n-33)-30(n-1)^2$ or $50n^2-172n+135$ intersections. As many tangents f and d pass through P. The number of lines f amounts according to § 15 to $(10n^2-32n+18)$, hence $[d]$ has as index $(40n^2-140n+117)$.

In order to find the locus of the points G, which any flecnodal c^n has in common with its tangent d we consider the product of the projective systems $[c^n]$ and $[d]$. Their indices are $3(n-1)(10n-23)$, i.e. the number of flecnodal c^n in a net, and $(40n^2 - 140n + 117)$. Since the curve (F) belongs three times to the figure produced, we find for the order of (G) $3(n-1)(10n-23) + n(40n^2-140n+117)-$ $-3(20n-33)$ or $(40n^3-110n^2-42n+168)$.

Let us now consider the correspondence (MF, MG). The straight lines d passing through M produce each $(n-3)$ coincidences; the number of the remaining ones amounts to $(20n-33)(n-3)+(40n^3-$ $110n^2-42n+168)-(40n^2-140n+117)(n-3)=170n^2-672n+618$.

To the coincidences $F \equiv G$ determined by this belong in the first place the cusps with tangent t_4; the remaining ones arise in pairs from nodal curves with two inflectional tangents f. Their number amounts therefore to $\frac{1}{2}\left[(170n^2-672n+618)-(50n^2-192n+168)\right]$; *the complex contains* $(60n^2 - 240n + 225)$ *curves with a fleflecnode.*

In the case $n=3$, $b=6$ we find 45 for it. Each of the trilaterals belonging to Γ is apparently to be considered as a figure with three fleflecnodes.

19. In a similar way as in N § 5, 13, 14 it may be determined how many times a base-point B of the complex is point of contact of a particular tangent. We find then in the first place that B is point of contact R_5 of *ten* tangents t_5. It is further consecutively found that B is point of contact R_4 of $(n-5)(n+16)$ tangents $t_{1,2}$, point of contact R_3 of $3(n-5)(n+6)$ tangents $t_{3,3}$ and of $2(n-5)$ $(n-6)(n+6)$ tangents $t_{3,2,2}$, point of contact R_2 of $2(n-5)(3n+8)$ tangents $t_{2,4}$, of $3(n-5)(n-6)(3n+8)$ tangents $t_{2,3,2}$, and $\frac{2}{3}(n-5)$ $(n-6)(n-7)(3n+8)$ tangents $t_{2,2,2,2}$.