the indications given by LANGEVIN. [1]) For the experimental investigation we may refer to FINDLAY "Der osmotische Druck" (Dresden 1914).

Remarks.

For the pressure on a semi-permeable membrane in the case of very dilute solutions it is, as we see immaterial whether or no there is interaction between the molecules S and the molecules W. Certain other effects of the osmotic pressure, can, however, only be brought about in consequence of such interaction: e.g. the difference of level that comes about between the solution and the pure water, when they are in tubes open at the top, and are in communication through a semi-permeable membrane. Let us consider the following imaginary case: The "sugar" molecules have no interaction at all with the water molecules. It is clear that there cannot ensue a difference of level — the sugar simply evaporates from the solution. When a glass bell-jar is put over the two communicating tubes, the following state of equilibrium is obtained: two solutions of the same concentration on either side with an equally high level. If the two tubes are placed each under a bell-jar of its own, sugar-vapour is formed over the solution with a pressure of the same value as the osmotic pressure of the solution and no difference of level appears then either.

If the difference in level in question is to make its appearance, none of the three following factors can, indeed, be omitted: first the tendency of the sugar to spread (its kinetic pressure), secondly the cohesion of the water, thirdly the interaction of the molecules S and W, without which it would not be possible for the sugar to lift up the water.

Mathematics. — *"On* NÖTHER's *theorem".* By Dr. W. VAN DER WOUDE. (Communicated by Prof. JAN DE VRIES).

(Communicated in the meeting of March 27, 1915).

§ 1. BRILL and NÖTHER's well-known paper on algebraic functions [2]) has as starting-point a theorem [3]) shortly before pronounced by NÖTHER. Its meaning may principally be indicated as follows:

"A curve F_3 may be represented by the form

$$F_3 \equiv AF_1 + BF_2,$$

[1]) loc. cit.

[2]) Math. Annalen, 7 (p. 271.)

[3]) Math. Annalen, 6 (p. 351): "Ueber einen Satz aus der Theorie der algebraischen Funktionen."

if it has a $(p+q-1)$-fold point in each point of intersection in which F_1 possesses a p-fold and F_2 a q-fold-point, and if no tangents at F_1 and F_2 coincide in the points of intersection".

After the simplest case[1]), in which F_1 and F_2 have only single points of intersection, had been treated by NÖTHER before, he gives in the above mentioned article a proof of the general case. Further proofs have appeared from the hands of HALPHEN[2]) and VOSS[3]). Yet the importance of the theorem may justify the attempt made here to deduce it once more in a most simple way.

§ 2. We understand by F_1 and F_2, curves respectively of order m and n, which may also be degenerate, however, not in such a way that F_1 and F_2 possess a common divisor. We suppose that x^m occurs in F_1, x^n in F_2; further that no intersections of the two curves are at infinity and no points of intersection are connected by a line parallel to one of the axes of coordinates. By these suppositions which are always to be arrived at by a linear transformation and a fit supposition of the axes, nothing is done to diminish the universality.

The curves are represented by

$$F_1(x, y) \equiv a_0 x^m + a_1 x^{m-1} + \ldots + a_m = 0 \quad . \quad . \quad . \quad (1)$$

$$F_2(x, y) \equiv b_0 x^n + b_1 x^{n-1} + \ldots + b_n = 0 \quad . \quad . \quad . \quad (2)$$

From these two equations we find by elimination of x the resultant

$$\varrho(y) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which

$$\varrho(y) \equiv \begin{vmatrix} a_0 & a_1 & a_2 & \ldots & 0 \\ 0 & a_0 & a_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_m \\ b_0 & b_1 & b_2 & \ldots & 0 \\ 0 & b_0 & b_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & b_n \end{vmatrix} \equiv \begin{vmatrix} a_0 & a_1 & a_2 & \ldots & x^{n-1} F_1 \\ 0 & a_0 & a_1 & \ldots & x^{n-2} F_1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & F_1 \\ b_0 & b_1 & b_2 & \ldots & x^{m-1} F_2 \\ 0 & b_0 & b_1 & \ldots & x^{m-2} F_2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & F_2 \end{vmatrix}$$

1) The proof for this case occurs in J. BACHARACH's dissertation: "Ueber Schnittpunktsysteme algebraischer Curven" (Erlangen 1881) and in a paper of the same author in the Math. Annalen, 26 (p. 275).

2) Bulletin de la Société Math. de France, V (p. 160): "Sur un théorème d'Algèbre."

3) Math. Annalen, 27 (p. 527): "Ueber einen Fundamentalsatz aus der Theorie der algebraischen Funktionen."

The second determinant is deduced from the first by multiplying the terms of the first column by x^{m+n-1}, those of the second by $x^{m+n-2}, \dots,$ and by adding them afterwards to those of the last column. At the same time appears from this the well-known identity:

$$\varrho \equiv PF_1 + QF_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which P and Q are of no higher order than $(n-1)$ and $(m-1)$ in x.

Let now F_3 be an arbitrary integral function of x and y of degree r in x $(r \geq m,\ r \geq n)$; we arrange F_1, F_2 and F_3 according to the descending powers of x and divide F_3 by F_1F_2, let us call the quotient q and the rest F_3', then

$$F_3 \equiv q\,F_1\,F_2 + F_3' . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The function F'_3 is in x of no higher order than $(m+n-1)$. From (4) follows

$$\varrho F'_3 \equiv PF'_3F_1 + QF'_3F_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The terms in the second member whose degrees in x are higher than $(m+n-1)$ must disappear.

If we therefore divide PF'_3 and QF'_3 by F_1F_2, after arrangement according to the descending powers of x, the quotients will be each other's opposites.

Hence:

$$PF'_3F_1 \equiv q_1F_1F_2 + RF_1$$
$$QF'_3F_2 \equiv -qF_1F_2 + SF_2$$

By this (6) is reduced to

$$\varrho F'_3 \equiv RF_1 + SF_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which R and S are of no higher degree than $(n-1)$ and $(m-1)$ in x.

From this identity NÖTHER's theorem is simply and generally to be deduced.

§ 3. We suppose now *provisionally* that all the points of intersection of F_1 and F_2 are simple ones. Is $F_3 = 0$ the equation of a curve passing through all the points of intersection of F_1 and F_2, the same is true according to (5) for the curve represented by $F_3' = 0$. We will prove that now in the identity

$$\varrho F_3' \equiv RF_1 + SF_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the functions R and S are divisible by ϱ.

We take for convenience sake one of the points of intersection 0 as origin of our system of coordinates, y is then a factor of ϱ. As F_3' also passes through 0, $\varrho F_3'$ has a node in 0, while F_1 and F_2 possess only simple points there with different tangents. This is only possible if R and S also pass through 0.

Further has F_1 apart from 0 moreover $(m-1)$ points of intersection with the X-axis, which intersections do not lie on F_2, so they do on S; in the same way the points of intersection of F_2 with the X-axis lie to the number of $(n-1)$ on R.

Now the X-axis has already n points of intersection with R, and m points of intersection with S all situated in the finite while R and S are respectively of degree $(n-1)$ and $(m-1)$ in x. So R and S are both divisible by y. We may, however, prove in the same way that R and S are divisible by all the other factors of y, so that we find:

$$F_3' \equiv R'F_1 + S'F_2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (5) it further ensues

$$F_3 \equiv AF_1 + BF_2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

§ 4. The preceding proof undergoes only a slight change if F_1 and F_2 show contact in one or more of their points of intersection, or possess multiple points there. We suppose in the first place that F_1 and F_2 touch each other in a point 0, which we again take as origin of the system of coordinates; moreover that F_3 too has in that point the same tangent l as F_1 and F_2.

Let us now again consider the identity

$$\varrho F_3' \equiv RF_1 + SF_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we suppose that the curves R and S do not both pass through 0, we might determine by RF_1 and SF_2, which have in 0 a tangent l in common, a pencil of which one of the curves K has a node in 0; K would however not be touched by l in that case.

For in that case K would have one point of intersection more there with RF_1 or SF_2 than these two possess there between them. Now $\varrho F_3'$ is a curve, however, from the pencil determined by RF_1 and SF_2 and one of its tangents in 0 coincides with the common tangent of F_1 and F_2. Consequently R and S must pass through 0.

As in § 3 it appears further that R and S are divisible by y and by all the other factors of ϱ. Consequently the identities (8) and (9) remain in force.

In the same way it appears that (9) remains in force if F_1 and F_2 have in any point contact of higher order, provided that they show there contact of the same order with F_3' as well.

Let us finally suppose that in a point 0, which we again take as origin of the system of coordinates, the curve F_1 possesses a p-fold, F_2 a q-fold point; we provisionally suppose that F_1 and F_2 have no common tangent in 0. F_3 is supposed to pass through all the points of intersection of F_1 and F_2 and to possess in 0 a $(p+q-1)$-fold point; it is to be seen at once that the curve F_3' determined by (5) satisfies the same requirements.

Let us again consider the identity

$$\varrho F_3' \equiv RF_1 + SF_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The resultant ϱ contains the factor y^{pq}; F_3' has no terms of a lower degree than $(p+q-1)$.

Let us write the equations of F_1 and F_2 thus:

$$F_1 \equiv (y - \alpha_1 x)(y - \alpha_2 x) \ldots (y - \alpha_p x) + u_{p+1} + u_{p+2} + \ldots + u_m = 0$$
$$F_2 \equiv (y - \beta_1 x)(y - \beta_2 x) \ldots (y - \beta_q x) + v_{q+1} + v_{q+2} + \ldots + v_n = 0,$$
$$\textit{in which } \alpha_i \neq \beta_k,$$

then it appears from (7) that the terms of the lowest degree in R, must at least be of degree q, those in S at least of degree p. For if R or S were of a lower degree the terms of the lowest degree in RF_1 and SF_2 could not neutralize each other and in RF_1+SF_2 terms of a lower degree than $(p+q)$ must consequently occur. So R has a q-fold, S a p-fold point in 0. R passes moreover through all the points which F_2 — to the number of $(n-q)$ — has in common with the X-axis apart from 0; the function R contains therefore the factor y. At the same time y is a factor of S and so we may divide both members of (7) by y. After that we may however, follow the same reasoning once more and going on in that way, prove that both members of (7) are divisible by y^{pq}.

Consequently all the factors of ϱ are again divisible on R and S and in this case too we find again

$$F_3 \equiv AF_1 + BF_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

To wind up with we may suppose that F_1 possesses in 0 a p-fold point, F_2 a q-fold, that they have moreover in 0 at one of the branches contact of an arbitrary order. Reasoning in the same way as above, we find, that even now the identity (9) remains in force, if only F_3 has a $(p+q-1)$-fold point in 0 and moreover in 0 contact with F_1 and F_2 of the same order as they have between them.

Observation. We have supposed that in the points of intersection of F_1 and F_2 neither curve has a multiple point with coinciding tangents. NOTHER has already shown how that case may be reduced to one of those treated here.

§ 5. If F_3 is a curve of degree r, we can observe that the curves A and B need at most be of degree $(r-m)$ and $(r-n)$. If this is not the case, however, the terms of the highest degree of AF_1 and BF_2 will cancel each other; as the terms of the highest degree in F_1 and F_2 have no common factor, those of AF_1 and BF_2 will be divisible by those of F_1F_2.

Let us therefore suppose

$$AF_1 \equiv A'F_1F_2 + A''F_1$$
$$BF_2 \equiv B'F_1F_2 + B''F_2,$$

in which we extend the division by F_1F_2 only so far that the terms of the highest degree in AF_1 and BF_2 have disappeared, we have

$$A' \equiv -B'.$$

So we find

$$F_3 \equiv A''F_1 + B''F_2,$$

in which A'' and B'' are of a lower degree than A and B. So we may go on till we find

$$F_3 \equiv A^{(c)}F_1 + B^{(c)}F_2$$

in which $A^{(c)}$ and $B^{(c)}$ are at most of degree $(r-m)$ and $(r-n)$.

Mathematics. — *"A bilinear congruence of rational twisted quintics".* By Professor JAN DE VRIES.

(Communicated in the meeting of March 27, 1915).

1. The base-curves of the pencils belonging to a net $[\Phi^3]$ of cubic surfaces form a *bilinear congruence.* For through an arbitrary point passes only *one* curve, and an arbitrary straight line is chord of *one* curve; for the involution I_2^3, which the net determines on that line, has *one* neutral pair of points.

We shall consider the particular net, the base of which consists of the twisted cubic σ^3, the straight line s and the points F_1, F_2, F_3. [1]) The surfaces Φ^3, which connect this basis with a point P have moreover a twisted quintic ϱ^5 in common. A *bilinear congruence* $[\varrho^5]$ is therefore determined by $[\Phi^3]$. A plane passing through s intersects two arbitrary surfaces of the net in two conics; of their intersections three lie on σ^3, the fourth belongs to ϱ^5; consequently this curve has four points in common with s, is therefore *rational.*

The straight line s is apparently a *singular quadrisecant.*

The figure consisting of s, σ^3 and ϱ^5 is, as complete intersection

[1]) Two other particular nets I have considered in two communications placed in volume XVI (p. 733 and p. 1186) of these "*Proceedings*". They determine bilinear congruences of twisted quartics (1st and 2nd species).