Citation:

J. de Vries, A bilinear congruence of rational twisted quintics, in:
KNAW, Proceedings, 17 III, 1914-1915, Amsterdam, 1915, pp. 1250-1256

curves A and B need at most be of degree $(r-m)$ and $(r-n)$. If this is not the case, however, the terms of the highest degree of AF_1 and BF_2 will cancel each other; as the terms of the highest degree in F_1 and F_2 have no common factor, those of AF_1 and BF_2 will be divisible by those of F_1F_2.

Let us therefore suppose

$$AF_1 \equiv A'F_1F_2 + A''F_1$$
$$BF_2 \equiv B'F_1F_2 + B''F_2,$$

in which we extend the division by F_1F_2 only so far that the terms of the highest degree in AF_1 and BF_2 have disappeared, we have

$$A' \equiv -B'.$$

So we find

$$F_3 \equiv A''F_1 + B''F_2,$$

in which A'' and B'' are of a lower degree than A and B. So we may go on till we find

$$F_3 \equiv A^{(c)}F_1 + B^{(c)}F_2$$

in which $A^{(c)}$ and $B^{(c)}$ are at most of degree $(r-m)$ and $(r-n)$.

Mathematics. — *"A bilinear congruence of rational twisted quintics".* By Professor JAN DE VRIES.

(Communicated in the meeting of March 27, 1915).

1. The base-curves of the pencils belonging to a net $[\Phi^3]$ of cubic surfaces form a *bilinear congruence*. For through an arbitrary point passes only *one* curve, and an arbitrary straight line is chord of *one* curve; for the involution I_2^3, which the net determines on that line, has *one* neutral pair of points.

We shall consider the particular net, the base of which consists of the twisted cubic σ^3, the straight line s and the points F_1, F_2, F_3. [1]) The surfaces Φ^3, which connect this basis with a point P have moreover a twisted quintic ϱ^5 in common. A *bilinear congruence* $[\varrho^5]$ is therefore determined by $[\Phi^3]$. A plane passing through s intersects two arbitrary surfaces of the net in two conics; of their intersections three lie on σ^3, the fourth belongs to ϱ^5; consequently this curve has four points in common with s, is therefore *rational*.

The straight line s is apparently a *singular quadrisecant*.

The figure consisting of s, σ^3 and ϱ^5 is, as complete intersection

[1]) Two other particular nets I have considered in two communications placed in volume XVI (p. 733 and p. 1186) of these *"Proceedings"*. They determine bilinear congruences of twisted quartics (1st and 2nd species).

of two Φ^3, of rank 36. As σ^3 is of rank four and ϱ^5 as rational curve of rank eight, while s has four points in common with ϱ^5, ϱ^5 and σ^3 will have *eight* points in common. We can therefore determine the congruence $[\varrho^5]$ as *the system of rational curves* ϱ^5 *passing through three fundamental points* F_1, F_2, F_3, *cutting the singular curve* σ^3 *eight times and having* s *as singular quadrisecant.*

It incidentally follows from this, that ϱ^5 may satisfy 20 simple conditions.

2. Let b be a bisecant of σ^3, resting on s, all Φ^3 passing through a point of b have this line in common, therefore determine a pencil the base of which consists of s, b, σ^3 and a *rational* ϱ^4, which has three points with s six points with σ^3, consequently *one* point in common with b.

There are also figures of $[\varrho^5]$ consisting of a conic ϱ^2 and a cubic ϱ^3. The plane Φ_1 passing through F_1 and s forms with the ruled surface Φ_1^2, determined by σ^3, F_2 and F_3, a Φ_1^3. Any other figure of $[\Phi^3]$ intersects Φ_1^3 along a conic ϱ_1^2 in the plane Φ_1, passing through F_1 and the intersections $S_1^{(k)}$ of σ^3, and a twisted curve ϱ_1^3 intersecting σ^3 in five points C_1', C_1', which are determined by Φ_1^2; it passes of course through F_2 and F_3.

To the curves ϱ_1^3 belong two degenerate figures each formed by the bisecant of σ^3 out of one of the points C, and the conic φ_1^2, in which Φ_1^2 is intersected by the plane that connects the points F_2 and F_3 with the other point C. Apparently φ_1^2 and the corresponding ϱ_1^2 form a degenerate curve ϱ^4.

The three degenerate conics ϱ_1^2 as well determine degenérate curves ϱ^4. For the straight line $S_1'S_1''$ is a bisecant b; hence the line F_1S_1''' forms with the corresponding ϱ_1^3 a degenerate figure ϱ^4.

3. To the net $[\Phi^3]$ belongs the surface Σ^3, which has a node in a point S of σ^3. This nodal surface determines with any other surface of the net a ϱ^5, intersecting σ^3 in S, is therefore the locus of the ϱ^5 passing through the *singular point* S.

The surfaces Σ_1^3 and Σ_2^3 have s, σ^3 and a ϱ^5 in common, consequently *one* ϱ^5 passes through two points S_1, S_2 of σ^3. The groups of eight points, which the curves of the congruence determine on σ^3 form therefore an involution of the second rank. From this ensues that σ^3 is osculated by 18 curves ϱ^5, and contains 21 pairs S_1, S_2 through which ∞^1 curves ϱ^5 pass. So there are 21 surfaces Φ^3 each possessing *two* nodes lying on σ^3.

A straight line passing through the vertex S of the *monoid* Σ^3 intersects the latter moreover in a point P and the plane φ passing through F_1, F_2, F_3 in a point P', which we shall consider as an image of P. As *one* ϱ^5 passes through any point P, the curves of the congruence lying on Σ^3 are represented by a pencil of rational curves φ^4. Every φ^4 has in common with the intersection φ^3 of Σ^3 the five points, in which the corresponding ϱ^5 intersects the plane φ; the remaining seven intersections of φ^3 with φ^4 are base-points of the pencil (φ^4). To them belong the points F_1, F_2, F_3; the remaining four are intersections of four straight lines lying on Σ^3. One of them is intersected by every ϱ^5 in S and in a point P, is therefore a *singular bisecant* p of the congruence; the involution which the ∞^1 curves ϱ^5 determine on it, is parabolic; so we might call p a *parabolic bisecant*. The remaining three straight lines d_1, d_2, d_3 passing through S are common trisecants of the curves ϱ^5; on these *singular trisecants* as well the involution of the points of support is special, for each group contains the point S.

The monoid Σ^3 contains moreover two straight lines passing through S viz. the two bisecants of σ^3 cutting s, being consequently component parts of two ϱ^5 degenerated into a straight line b and a ϱ^4.

The pencil (φ^4) has three double base-points D_1, D_2, D_3 and four single base-points E, F_1, F_2, F_3; it contains six compound figures: three figures consisting of a nodal φ^3 and a straight line and three pairs of conics.

Let us now first consider the figure formed by the straight line D_2D_3 and the φ^3, which has a nodal point in D_1 and passes through the remaining six base-points. It is the image of a figure consisting of a bisecant b and a rational curve ϱ^4; for the plane passing through d_2 and d_3 has only one straight line in common with Σ^3 so that $D_2 D_3$ cannot be the image of a conic passing through S. Consequently there lie on Σ^3 three straight lines b not passing through S, and therefore three curves ϱ^4 passing through S.

The conic passing through D_1, D_2, D_3, E, F_1 is the image of the conic ϱ^2 which the plane (F_1s) has in common with Σ^3; the conic to be associated to her passing through D_1, D_2, D_3, F_2, F_3 is the image of the ϱ^3 forming with ϱ^2 a curve of the congruence $[\varrho^5]$. There are apparently there figures (ϱ^3, ϱ^2) on Σ^3.

4. The curves ϱ^5, meeting s in a point S^* lie on the nodal surface Φ^3, which has S^* as node. The monoids Σ^{*3} belonging to two points of s, have *one* ϱ^5 in common; so the groups of four points which the ϱ^5 have in common with s form a I_2^4. There are consequently six ϱ^5 which osculate s, and three binodal surfaces Φ^3 which have their nodes on s, consequently contain ∞^1 curves ϱ^5, intersecting s in the same two points.

The ϱ^5 of the monoid Σ^{*3} are represented on the plane $\varphi \equiv F_1 F_2 F_3$ by a pencil of φ^4, which have the intersection D of s as triple point and pass through F_1, F_2, F_3. The remaining base-points E_1, E_2, E_3, E_4 of that pencil lie in the intersections of straight lines p_k of the monoid, which lines meet in S^* and apparently are parabolic singular bisecants. The sixth straight line of the monoid passing through S^* is the bisecant b of σ^3, consequently part of a degenerate ϱ^5.

The straight line DF_1 is the image of the conic $\varrho_1{}^2$, in which the monoid is moreover intersected by the plane (sF_1); the nodal φ^3 completing it into a φ^4 represents the cubic $\varrho^3{}_1$ belonging to $\varrho_1{}^2$. So three figures (ϱ^3, ϱ^2) lie on Σ^{*3}.

The straight line DE_1 forms with the nodal cubic passing through $E_2, E_3, E_4, F_1, F_2, F_3$ and twice through D, the image of a degenerate ϱ^5, consisting of the straight line b in the plane (sp_1) and a rational ϱ^4 passing through S^*. The monoid Σ^{*3} too contains therefore five figures (b, ϱ^4).

5. We can now determine the order of the locus of the rational curves ϱ^4. It has s as quadruple straight line and passes thrice through σ^3 (§ 3). Its intersection with a Σ^{*3} consists apart from these multiple lines of five curves ϱ^4, is therefore of order 33. *The rational curves ϱ^4 lie therefore on a surface of order eleven.*

The section of this surface Φ^{11} with the plane $(F_1 s)$ consists of the quadruple straight line s, and parts of degenerate figures ϱ^4. To it belong in the first place the three straight lines joining F_1 to the intersections $S_1^{(k)}$ of σ^3 (§ 2); the remaining section is formed by the two $\varrho_1{}^2$ belonging to the bisecants b out of the points C_1', C_1'' (§ 2). A straight line passing through F_1 intersects Φ^{11} four times on s and has with each of the two conics $\varrho_1{}^2$ a point of intersection not lying in F_1; so five intersections lie in F_1. *The three fundamental points F are therefore quintuple points of Φ^{11}.*

In order to determine the locus of the intersection B of a ϱ^4 with the bisecant b coupled to it, we consider on s the correspondence between its intersections with b and ϱ^4. Through any point P passes *one* b; to it are associated the three points Q, which ϱ^4 has in common with s. In each point Q, s is intersected by four curves ϱ^4; hence four points P are associated to Q. From this it appears that s contains seven points B. A plane passing through s contains three straight lines b, consequently three points B; so the

83*

points B lie on a *curve* β^{10} *with septuple secant* s. In the same way it appears that β^{10} meets σ^3 in 15 points. The surfaces Φ^{11} and $(b)^4$ have in s and σ^3 a section of order $4 + 3 \times 2 \times 3$; moreover they have β^{10} in common. The remaining section of order 12 must consist of straight lines belonging to degenerate figures ϱ^5, each composed of a ϱ^3 and *two* straight lines b intersecting it. From this it ensues that $[\varrho^5]$ *contains six figures consisting of a twisted cubic and two of its secants.*

This result may also be formulated in this way: through three points F_k pass 6 curves ϱ^3 which intersect a given σ^3 four times and a straight line s twice. Such a ϱ^3 intersects the ruled surface $(b)^4$ in two points B lying outside s and σ^3; through these points pass the two straight lines b, completing ϱ^3 into a ϱ^5.

6. Any straight line d having three points in common with a ϱ^5 is a *singular trisecant* of the congruence. For through it passes *one* Φ^3 and the remaining surfaces of the net intersect it in the triplets of an involution. From this it ensues that the trisecants of the ϱ^5 form a *congruence of order three,* as a ϱ^5 is intersected in each of its points by three trisecants. In § 3 it has been proved that any point S of σ^3 also sends out three straight lines d; on these singular trisecants, however, all the groups of the I_3 have the point S in common.

Let b be a bisecant of a ϱ^5 intersecting σ^3. Through it passes *one* Φ^3; the net therefore determines on b an involution I^2, so that b is a *singular bisecant.*

Through a point P pass *four* straight lines b. For the curve ϱ^5_P, which can be laid through P is projected out of P by a cone k^4; the latter has in common with σ^3 the eight points in which ϱ^5_P rests on σ^3. The remaining four intersections lie on edges of k^4, which have in common with ϱ^5_P two points not lying on σ^3, consequently are *singular bisecants.*

These four straight lines b lie on the surface Π, which is the locus of the pairs of points, which the curves of $[\varrho^5]$ have in common with their chords passing through the point P. Π is apparently a surface of *order six* with quadruple point P, the tangent cone of which coincides with k^4.

Π^6 contains s and σ^3, therefore has with an arbitrary ϱ^5 four points of s and eight points of σ^3 in common; of the remaining 18 points of intersection 12 lie on the 6 chords, which ϱ^5 sends through P, and 6 in the points F. Hence Π^6 has *three nodes* F_k.

With the cone $k^4 \cdot \Pi^6$ has the curve $\varrho_P{}^5$ in common; the remaining section can only consist of straight lines. To it belong the *three* parabolic bisecants PF_k and the *four* singular bisecants b. From this it ensues that the three trisecants d which $\varrho_P{}^5$ sends through P are *nodal lines* of Π^6.

For a point S of the singular curve σ^3 the surface Π^6 degenerates into the monoid Σ^3 and a cubic cone k^3, formed by singular bisecants b. The straight lines b form therefore a *congruence of order four, with singular curve* σ^3, *consequently of class nine.*

7. The surface Λ formed by the ϱ^5 intersecting a straight line l, has the ϱ^5 intersecting l twice as *nodal curve.*

As l intersects every monoid Σ^3 thrice, s and σ^3 are *triple lines* on Λ. The section of Λ with the plane $(F_1 s)$ consists of the triple straight line s and three conics $\varrho_1{}^2$; of these, one passes through the intersection of l, the other two are determined by the two curves $\varrho_1{}^3$ resting on l. So Λ is a surface of *order nine, with triple points in* F_1, F_2, F_3.

On Λ^9 lie 15 straight lines, 9 conics, 9 curves ϱ^3 and 15 rational curves ϱ^4. For l intersects 4 bisecants b, 11 curves ϱ^4; 3 conics and 6 curves ϱ^3.

A plane λ passing through l intersects Λ^9 along a curve λ^8; the latter has in common with l the points, in which l is intersected by the ϱ^5, which has l as bisecant. In each of the remaining six points λ is touched by a ϱ^5 of the congruence.

The locus of the points in which a plane φ is touched by curves ϱ^5 is therefore a curve φ^6. It is the *curve of coincidence of the quintuple involution*, which determines $[\varrho^5]$ on φ. The intersections S^*, S_1, S_2, S_3 of the singular lines s, σ^3 are apparently *nodes* of φ^6.

With the surface Λ^9 belonging to an arbitrary straight line l, φ^6 has in those intersections $4 \times 3 \times 2$ points in common; in each of the remaining intersections φ is touched by a ϱ^5 resting on l.

The curves ϱ^5 *touching* φ *form therefore a surface* Φ^{30}.

A monoid Σ^3 has in the points S^*, S_k 4×2 points in common with φ^6; on φ^6 lie therefore the points of contact of 10 curves ϱ^5 of the monoid. From this it ensues that s and σ^3 are *decuple lines* of Φ^{30}.

With the curve ψ^6, belonging to the plane ψ, Φ^{30} has, in the four nodes of ψ^6, $4 \times 2 \times 10$ points in common; in each of the remaining intersections ψ is touched by a ϱ^5, which at the same time touches the plane φ. *There are consequently* 100 *curves* ϱ^5, *touching two given planes.*

The plane φ has with Φ^{30}, besides the curve of contact φ^6 to be counted twice, a curve φ^{18} in common possessing *four sextuple points* in S^*, S_k. Apart from the multiple points, φ^6 and φ^{18} have moreover $6 \times 18 - 4 \times 2 \times 6$ points in common; from this it ensues that *each plane is osculated by thirty curves ϱ^5.*

Mathematics. — *"Some particular bilinear congruences of twisted cubics."* By Prof. JAN DE VRIES.

(Communicated in the meeting of March 27, 1915).

The bilinear congruences of twisted cubics ϱ^3 may principally be brought to two groups.[1]) The congruences of the first group may be produced by two pencils of ruled quadrics, the bases of which have a straight line in common; the congruences of the second group consist of the base-curves of the pencils belonging to a net of cubic surfaces, which have in common a fixed point and a twisted curve of order six and genus three. REYE's congruence formed by the ϱ^3 passing through five given points F_k, belongs to both groups; it may be produced by two pencils of quadratic cones; the straight lines, connecting each of two points F_1, F_2 with each of the remaining four, are base-edges. We shall now consider some other particular cases of congruences of the first group, which may also be produced by two pencils of quadratic cones.

1. We consider the curves ϱ^3 passing through the *fundamental points* F_1, F_2, F_3, F_4 and having the lines s_1 (passing through F_1) and s_2 (passing through F_2) as chords. Each ϱ^3 is the partial intersection of a quadratic cone passing through the lines $(s_1, F_1F_2, F_1F_3, F_1F_4)$; $(s_2, F_2F_1, F_2F_3, F_2F_4)$; the congruence is consequently bilinear. Apparently s_1 and s_2 are *singular bisecants*. Any point S_1 of s_1 is singular; the ϱ^3 passing through S_1 lie on the cone of the second pencil passing through S_1. Consequently s_1, as well as s_2, is a *singular straight line of order two.*

The figures of the congruence consisting of a straight line d and a conic σ^2, may be brought to four groups.

A. The straight line $d_{12} \equiv F_1F_2$, may be combined with any σ^2 of the system of conics passing through F_3 and F_4 and resting on

[1]) VENERONI, *Rendiconti del Circolo matematico di Palermo*, tomo XVI, 209—229. In a short communication in vol. XXXVII, 259, of the *Rendiconti del Ist. Lombardo*, VENERONI has added to these two main types a third which by the way may be considered as a limit case of the first type. This congruence may be produced by a pencil of quadrics and a pencil of quartic surfaces, one surface of which is composed of two quadrics of the first pencil. The bases of the pencils have a straight line in common, which is nodal line for the second pencil.