counted twice, a curve φ^{18} in common possessing *four sextuple points* in S^*, S_k. Apart from the multiple points, φ^6 and φ^{18} have moreover $6 \times 18 - 4 \times 2 \times 6$ points in common; from this it ensues that *each plane is osculated by thirty curves* ϱ^5.

Mathematics. — *"Some particular bilinear congruences of twisted cubics."* By Prof. JAN DE VRIES.

(Communicated in the meeting of March 27, 1915).

The bilinear congruences of twisted cubics ϱ^3 may principally be brought to two groups.[1]) The congruences of the first group may be produced by two pencils of ruled quadrics, the bases of which have a straight line in common; the congruences of the second group consist of the base-curves of the pencils belonging to a net of cubic surfaces, which have in common a fixed point and a twisted curve of order six and genus three. REYE's congruence formed by the ϱ^3 passing through five given points F_k, belongs to both groups; it may be produced by two pencils of quadratic cones; the straight lines, connecting each of two points F_1, F_2 with each of the remaining four, are base-edges. We shall now consider some other particular cases of congruences of the first group, which may also be produced by two pencils of quadratic cones.

1. We consider the curves ϱ^3 passing through the *fundamental points* F_1, F_2, F_3, F_4 and having the lines s_1 (passing through F_1) and s_2 (passing through F_2) as chords. Each ϱ^3 is the partial intersection of a quadratic cone passing through the lines $(s_1, F_1F_2, F_1F_3, F_1F_4)$; $(s_2, F_2F_1, F_2F_3, F_2F_4)$; the congruence is consequently bilinear. Apparently s_1 and s_2 are *singular bisecants*. Any point S_1 of s_1 is singular; the ϱ^3 passing through S_1 lie on the cone of the second pencil passing through S_1. Consequently s_1, as well as s_2, is a *singular straight line of order two*.

The figures of the congruence consisting of a straight line d and a conic σ^2, may be brought to four groups.

A. The straight line $d_{12} \equiv F_1F_2$, may be combined with any σ^2 of the system of conics passing through F_3 and F_4 and resting on

1) VENERONI, *Rendiconti del Circolo matematico di Palermo*, tomo XVI, 209—229. In a short communication in vol. XXXVII, 259, of the *Rendiconti del Ist. Lombardo*, VENERONI has added to these two main types a third which by the way may be considered as a limit case of the first type. This congruence may be produced by a pencil of quadrics and a pencil of quartic surfaces, one surface of which is composed of two quadrics of the first pencil. The bases of the pencils have a straight line in common, which is nodal line for the second pencil.

the three straight lines d_{12}, s_1, s_2. These curves lie on the hyperboloid H^2, which contains the three straight lines mentioned and the points F_3, F_4 [1]).

B. The straight line $F_2F_4 \equiv d_{24}$ may be coupled to any δ^2 of the pencil in the plane (F_3s_1), which has the points F_1, F_3 and the intersections of s_2 and d_{24} as base-points. Similar systems of degenerate ϱ^3 are determined by the straight lines d_{23}, d_{13}, d_{14} with pencils lying in the planes $(F_4s_1), (F_4s_2), (F_3s_2)$.

C. The transversal g_3 of s_1 and s_2 passing through F_3 may be coupled to any δ^2 of a pencil in the plane $F_1 F_2 F_4$; the base consists of F_1, F_2, F_4 and the intersection of g_3.

Analogously with this is the system determined by the transversal g_4 of s_1, s_2 through F_4; the pencil lies in that case in the plane $F_1 F_2 F_3$.

D. In the plane $F_1 F_3 F_4$ a pencil (δ^2) is determined, the base of which consists of the intersection S_2 of s_2 and the points F_1, F_3, F_4. To each δ^2 belongs a ray d of the pencil which has F_2 as vertex and is situated in the plane $(F_2 s_1)$. In this system *both* component parts of (d, δ^2) are variable.

A system analogous with this is formed by the pencil of rays in the plane $(F_1 s_2)$, with vertex F_1, and a (δ^2) in the plane $F_2 F_3 F_4$.

Summarising we observe that the figures δ^2 form a locus of degree *ten*. In the general congruence of the first principal group the figures δ^2 form also a surface of order ten; it does, however, not consist, as in this case, of different figures.

2. We can now easily determine the order of the surface Δ formed by the ϱ^3, intersecting a given straight line l. For that purpose we observe that the intersection of Δ with the plane $F_1 F_2 F_3$ must consist of figures d and δ^2. To this belongs in the first place the δ^2 of the pencil lying in this plane, which meets l; further twice the straight line d_{12}, for l rests in its points of intersection with the hyperboloid H^2 on two δ^2; finally the two straight lines d_{13}, d_{23}, each belonging to a figure the δ^2 of which rests on l. The intersection with $F_1F_2F_3$ is therefore a figure of order six, passing four times through F_1 and F_2, thrice through F_3.

The curves ϱ^3 intersecting l consequently form a surface Δ^6, having d_{12} as *nodal line*, passing through $d_{13}, d_{23}, d_{14}, d_{24}$ and possessing

1) The conics passing through two points and resting on three arbitrary straight lines form a quartic surface. Here the planes $F_3 F_1 F_1$ and $F_3 F_4 F_2$ contain each a pencil of conics which cannot be taken into consideration so that their planes fall away.

quadruple points in F_1, F_2, *triple points* in F_3, F_4. Δ^6 further contains the straight lines g_3, g_4 and the *nodal lines* s_1, s_2; the latter arises from the observation that l intersects two curves ϱ^3 meeting s_1 or s_2 in a point S_1 or S_2 lying on them.

The intersection of the surfaces Δ belonging to two straight lines l, l' consists of: 6 curves ϱ^3, resting on l and l', the nodal lines s_1, s_2, d_{12} and the straight lines d_{13}, d_{14}, d_{23}, d_{24}, g_3, g_4.

The cubic transformation, which, in tetrahedrical coordinates is determined by

$$x_1y_1 = x_2y_2 = x_3y_3 = x_4y_4,$$

transforms this congruence into the bilinear congruence of rays, which has the images s_1^*, s_2^*, of s_1, s_2 as directrices[1]). The surface Δ passes in consequence of this into the ruled surface formed by the straight lines r, which rest on s_1^*, s_2^* and on the curve λ^3 passing through the four points F into which l is transformed. The image of Δ is apparently a ruled surface of order four with nodal lines s_1^*, s_2^*. As this, apart from the points F, has six points in common with an arbitrary curve ϱ^3 laid through those points, it is found once more that Δ must be of order six.

Now that the surface Δ is completely known, the characteristic numbers of the congruence may be found in the usual way[2]).

In an arbritary plane Φ this congruence determines a cubic involution possessing three singular points of order two (the intersections of d_{12}, s_1, s_2) and six singular points of order one (the intersections of d_{13}, d_{14}, d_{23}, d_{24}, g_3, g_4). It has been more fully described in my paper on "Cubic involutions in the plane". [3])

3. Let us now consider the congruence $[\varrho^3]$, which possesses *three fundamental points* F_1, F_2, F_3 and *four singular bisecants* s_1, s_1', s_2, s_2', of which the first two pass through F_1, the other two through F_2. Here too two pencils of quadratic cones that can produce it are easily pointed out, while the four straight lines s are again *singular straight lines of order two*.

The degenerate figures (d, δ^2) now form the following groups:

A. The conic δ^2 passes through F_3 and rests on the five straight lines $d_{12} \equiv F_1 F_2$ s_1, s_1', s_2, s_2'; the locus of δ^2 is the *ruled cubic surface* Δ^3, of which d_{12} is the nodal line, the second transversal t

[1]) This transformation may effectively be used in the investigation of REYE's congruence (see my paper in volume XI of these Proceedings, p. 84).

[2]) Cf. my paper in vol. XIV (p. 255) of these Proceedings.

[3]) These Proc. XVI, p 974 (§ 6).

of the lines s being the second directrix. It is quite determined by the lines s and the transversal out of F_3 via d_{12} and t.

B. In the plane (s_2, s_2') lies a pencil (σ^2), having F_2 and the intersections of s_1, s_1' and d_{13} as base-points. Each of these σ^2 forms with d_{13} a figure ϱ^3.

In the same way d_{23} is to be combined with a σ^2 of a pencil lying in the plane (s_1, s_1').

C. The straight line t may be coupled to any conic σ^2 passing through F_1, F_2, F_3, which meets t.

D. In the plane $(F_3 s_1)$ lies a (σ^2), having as base-points F_1, F_3 and the intersections of s_2, s_2'. The corresponding straight line d passes through F_2 and rests on s_1'. *Both* component parts of (d, σ^2) are variable.

In the same way each of the planes $(F_3 s_1')$, $(F_3 s_2)$, $(F_3 s_2')$ contains a pencil (σ^2); the corresponding pencils of rays lie in the planes $(F_1 s_1)$, $(F_1 s_2')$, $(F_1 s_2)$.

Here too the figures $d\sigma^2$ form a locus of order *ten*.

The intersection of the plane $F_1 F_2 F_3$ with the surface Δ now consists of a conic (resting on l), the straight lines $F_1 F_3$ and $F_2 F_3$ (belonging to figures $d\sigma^2$, intersecting l elsewhere) and the straight line $F_1 F_2$, which is a triple one, because Δ^3 contains three σ^2 resting on l. The straight line l consequently determines a surface Δ^7, which has d_{12} as *triple straight line*, passes through d_{13}, d_{23}, t and possesses *four nodal lines* s_1, s_1', s_2, s_2'; F_1, F_2 are *quintuple points*, F_3 is a *triple point*.

In an arbitrary plane Φ this congruence determines a cubic involution with *one* singular point of order three, *four* singular points of order two, and *three* singular points of order one. [1])

4. We shall finally consider the $[\varrho^3]$, which has F_1, F_2 as *fundamental points*, the straight lines s_1, s_1', s_1'' and s_2, s_2', s_2'' as *singular bisecants*; the first three meet in F_1, the other three in F_2.

The straight line $d_{12} \equiv F_1 F_2$ is triple directrix of a *ruled quartic surface* Δ^4, which has the six straight lines s as generators. Any plane passing through two generators intersecting on d_{12} intersects Δ^4 moreover along a conic σ^2 resting on d_{12} and on the straight lines s, consequently forms a degenerate figure ϱ^3.

In the plane $(s_1 s_1')$ lies a pencil (σ^2) having as base-points F_1 and the intersections of s_2, s_2', s_2''; each of these curves forms a figure ϱ^3 with a definite ray d of the plane pencil which has F_2 as vertex

[1]) It has been treated more fully in my paper quoted above (Proc. XVI, § 13).

and lies in the plane $(F_2 s_1'')$. *Both* component parts are variable.

There are apparently five more systems equivalent to this, each determined by a pencil (σ^2) and a pencil (d).

The locus of the conics σ^2 is therefore also here of order *ten*.

The surface A appears to be of order *eight*; it has d_{12} as *quadruple straight line*, each of the *six* straight lines s as *nodal lines*. For if the complete intersection of two surfaces A is considered, it appears that the order x is to be found from the equation $x^2 - 3x - 40 = 0$; hence $x = 8$.

In a plane Φ a cubic involution possessing one singular point of order four and six singular points of order two is determined by this $[\varrho^3]$. It has been described in § 14 of my paper quoted above.

Chemistry. — "*Equilibria in Ternary Systems*" XVIII. By Prof. SCHREINEMAKERS.

In the previous communications here and there some equilibria between solid substances and vapour have been brought in discussion already; now we will consider some of these equilibria more in detail. We may distinguish several cases according as F and G are unary, binary or ternary phases.

I. The equilibrium $F + G$; F is a ternary compound, G a ternary vapour.

The equilibrium $F + G$ is monovariant (P and T constant), this means that the vapours, which can be in equilibrium with solid F, are represented by a curve. In order to find this curve we construct a cone, which touches the vapourleaf of the ζ-surface and which has its top in the point, representing the ζ of the solid substance F. The projection of the tangent curve is the curve sought for, viz. the saturationcurve (P and T constant) of the substance F. From this deduction it is apparent also, that this curve is circumphased and that we cannot construct from F a tangent to it.

The equilibrium $F + G$ is determined by:

$$Z_1 + (\alpha - x_1)\frac{\partial Z_1}{\partial x_1} + (\beta - y_1)\frac{\partial Z_1}{\partial y_1} = \zeta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When we keep P and T constant in (1), it determines the vapour-saturationcurve (P, T) of F. When we assume that in the vapour the compound F is completely decomposed into its components and that the gas-laws are true, (1) passes into:

$$\alpha \log x_1 + \beta \log y_1 + (1-\alpha-\beta) \log (1-x_1-y_1) = C \quad . \quad . \quad (2)$$