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Mathematics. — "*Characteristic numbers for nets of algebraic curves.*" By Professor JAN DE VRIES.

(Communicated in the meeting of November 28, 1914).

1. The curves of order n, c^n, which belong to a net N, cut a straight line l in the groups of an involution of the second rank, $I_n{}^2$. The latter has $3(n-2)$ groups each with a triple element[1]; l is therefore stationary tangent for $3(n-2)$ curves of N.

Any point P is base-point of a pencil belonging to N, hence inflectional point for three curves[2]) of N.

The locus of the inflectional points of N which send their tangent i through P, is therefore a *curve* $(I)_P$ *of order* $3(n-1)$ *with triple point* P.

If the net has a base-point in B any straight line through B is stationary tangent with point of inflection in B. Consequently $(I)_P$ passes through all the base-points of the net.

We shall suppose that N has only single base-points.

On PB N determines an I^2_{n-1}; the latter has $3(n-3)$ triple elements; from which it ensues that B is an inflectional point of $(I)_P$ having PB as tangent i.

Through P pass $3(n-1)(2n-3)$ straight lines, each of which touch a singular curve in its node[3]); all these nodes D lie apparently on $(I)_P$.

2. Every c^n, which osculates l in a point I, cuts it moreover in $(n-3)$ points S. We consider the locus of the points S, which belong in this way to $(I)_P$. Since P, as base-point of a pencil, lies on $3(n-3)(n+1)$ tangents of inflexion[4]), the curve (S) has in P a $3(n-3)(n+1)$-fold point. Apart from P each ray of the pencil (P) contains $3(n-2)(n-3)$ points S; hence (S) is a curve of order $3(n-3)(2n-1)$.

Let us now consider the correspondence between the rays s and s', which connect a point M with two points S and I belonging to

[1]) If the $I_n{}^2$ is transported to a rational curve c^n and determined by the field of rays, these groups lie on the stationary tangents.

[2]) For the characteristic numbers of a pencil my paper "Faisceaux de courbes planes" may be referred to (Archives Teyler, sér. II, t. XI, 99—113). For the sake of brevity it will be quoted by T.

[3]) Cf. for instance my paper "On nets of algebraic plane curves". (Proceedings volume VII, p. 631).

[4]) T. p. 100.

the same c^n. Any ray s contains $3(n-3)(2n-1)$ points S, determines therefore as many rays s'; any ray s' contains $3(n-1)$ points I, determines therefore $3(n-1)(n-3)$ points S and consequently as many rays s. The number of rays of coincidence $s' \equiv s$ amounts therefore to $3(n-3)(2n-1) + 3(n-3)(n-1) = 3(n-3)(3n-2)$. The ray MP contains $3(n-2)$ points I, which are each associated to $(n-3)$ points S; consequently MP represents $3(n-2)(n-3)$ coincidences. The remaining $6n(n-3)$ coincidences arise from coincidences $I \equiv S$, consequently from *points of undulation* U. Through P pass consequently $6n(n-3)$ *four-point tangents* t_4; *the tangents* t_4 *envelop therefore a curve of class* $6n(n-3)$.

3. We further consider the correspondence between the rays s_1, s_2, which connect M with two points S belonging to the same point I. This symmetrical correspondence has apparently as characteristic number $3(n-3)(2n-1)(n-4)$. The ray MP contains $3(n-2)$ points of inflection, hence $3(n-2)(n-3)(n-4)$ pairs S_1, S_2; as many coincidences $s_1 \equiv s_2$ coincide with MP. The remaining coincidences pass through points of contact of tangents $t_{2,3}$ (straight lines, which touch a c^n in a point R and osculate it in a point I). *The tangents* $t_{2,3}$ *envelop therefore a curve of class* $9n(n-3)(n-4)$.

4. Let a be an arbitrary straight line; each of its points is, as base-point of a pencil, point of inflection for three c^n. The curves c^n coupled by this to a form a system $[c^n]$ with index $6(n-1)$; for the inflectional points of the curves c^n, which pass through a point P, lie on a curve of order $6(n-1)$ [1]), and the latter cuts a in $6(n-1)$ points I. The stationary tangents i, which have their point of contact I on a, form a system $[i]$ with index $3(n-1)$, for through a point P pass the straight lines i, which connect P with the intersections of a and $(I)_P$.

The systems $[c^n]$ and $[i]$ are projective; on a straight line l they determine between two series of points a correspondence which has as characteristic numbers $6(n-1)$ and $3(n-1)n$. The coincidences of this correspondence lie in the points, in which l is cut by the loci of the points I and S, which every i determines on the associated c^n. As any point of a is point of inflection for three c^n, a belongs nine times to the locus in question. Hence the points S lie on a curve $(S)_a$ of order $3(n^2+n-5)$.

For $n=3$ the number 21 is found; this is in keeping with the

[1]) T. p. 104.

well-known theorem, according to which a net of cubics contains 21 figures, composed of a conic and a straight line.

5. To the intersections of a with the curve $(S)_a$ belong the $3(n-2)$ groups of $(n-3)$ points S, arising from the curves c^n, which osculate a. In each of the remaining $3(n^2+n-5)-3(n-2)(n-3)$ intersections a point I coincides with a point S of one of the three c^n, which have I as point of inflection. The corresponding tangent i then has in common with c^n four points coinciding in I, so that I is point of undulation. *The points of undulation of the net lie therefore on a curve* (U) *of order* $3(6n-11)$.

For $n=3$ we find the 21 straight lines belonging to the degenerate cubics of the net.

As a base-point B of a net is point of inflection of ∞^1 curves c^n, there will have to be a finite number of curves, for which B is point of undulation. In order to find this number we consider the locus of the points T which any ray t passing through B has still in common with the c^n, which osculates it in B. As B is point of inflection on three c^n of the pencil which has an arbitrary point P as base-point, the curves of N falling under consideration here form a system $[c^n]$ with index three, which is projective with the pencil of rays (t).

The two systems produce a curve of order $(n+3)$, which is cut by a ray t in $(n-3)$ points T. Consequently it has in B a sextuple point, and there are six curves c^n, on which B is point of undulation.

If the net has *base-points* they are *sixfold points* on the curve (U).

For $n=3$ the curve degenerates into a sixray, which consists of parts of compound curves.

6. To each c^n, which possesses a point of undulation U we shall associate its fourpoint tangent u; the latter cuts it moreover in $(n-4)$ points V. The locus of the points forms with the curve (U) counted four times the product of the projective systems $[c^n]$ and $[u]$. In the pencil which a point P sets apart from N occur $6(n-3)(3n-2)$ curves, which possess a point U[1]); this number is therefore the index of $[c^n]$. The system $[u]$ has, as appears from § 2, the index $6n(n-3)$. In a similar way as above (§ 4) we find now for the order of (V) $6(n-3)(3n-2)+6n^2(n-3)-12(6n-11)=$ $=6(n-4)(n^2+4n-7)$.

We now associate on each straight line u the point U to each

[1]) T. p. 105.

of the $(n-4)$ points V. By this the rays of a pencil (M) are arranged into a correspondence with characteristic numbers $3(6n-11)(n-4)$ and $6(n-4)(n^2+4n-7)$. Observing that the $6n(n-3)$ fourpoint tangents, which meet in M, represent $(n-4)$ coincidences each, we find for the coincidences $U \equiv V$ the number
$(n-4)[3(6n-11)+6(n^2+4n-7)-6n(n-3)] = 15(n-4)(4n-5)$. This is therefore the *number of curves c^n with a fivepoint tangent t_5*.

Let us now consider the correspondence between two points V_1, V_2, which lie on the same tangent u. Using the correspondence arising between the rays MV_1, MV_2 we find in a similar way for the number of coincidences $V_1 \equiv V_2$ $12(n^2+4n-7)(n-4)(n-5)-6n(n-3)(n-4)(n-5) = 6(n-4)(n-5)(n^2+11n-14)$. With this the *number of curves of N has been found, which are in possession of a tangent $t_{4,2}$*, consequently of a point of undulation, the tangent of which touches the curve moreover.

7. The involution of the second rank, which N determines on a straight line l, has $2(n-2)(n-3)$ groups, each of which possesses two double elements; l is therefore bitangent for as many curves of the net. If l rotates round a point P, the points of contact R, R' will describe a curve, which passes $(n-3)(n+4)$ times through P; for P as base-point of a net lies on $(n-3)(n+4)$ curves, which are each touched in P by one of their bitangents. From this follows that the locus of the pairs R, R', which we shall indicate by $(R)_P$ is a curve of order $(n-3)(5n-4)$.

If we consider the correspondence (R, R') on the rays of the pencil (P), and, in connection with this, the correspondence between the rays MR, MR', we arrive at the number of coincidences $R \equiv R'$ and we find once more that the fourpoint tangents envelop a curve of class $6n(n-3)$.

Let us now determine the order of the locus of the groups of $(n-4)$ points S, which l has in common with the $2(n-2)(n-3)$ curves c^n, for which l is bitangent. The pencil determined by P contains $2(n-3)(n-4)(n+1)$ curves which are cut[1]) in P by one of their bitangents. This number indicates at the same time the number of branches of the curve $(S)_P$ passing through P; for its order we find therefore $2(n-3)(n-4)(n+1)+2(n-2)(n-3)(n-4)$, or $2(n-3)(n-4)(2n-1)$.

If we associate each point R to each of the points S belonging to the same c^n, a correspondence is determined in the pencil of rays

[1]) T. p. 102.

with vertex M, which correspondence has $(n-3)(5n-4)(n-4)$ and $4(n-3)(n-4)(2n-1)$ as characteristic numbers.

Since the ray MP contains $4(n-2)(n-3)$ points R, which are each associated to $(n-4)$ points S, so that MP is to be considered as $4(n-2)(n-3)(n-4)$-fold coincidence, we find for the number of coincidences $9n(n-3)(n-4)$. By this we again find the class of the curve enveloped by the tangents $t_{2,3}$ (cf. § 3).

A new result is arrived at from the correspondence between two points S_1, S_2 belonging to the same pair R, R'. The symmetrical correspondence between the rays MS_1, MS_2 has as characteristic number $2(2n-1)(n-3)(n-4)(n-5)$. Any of the groups of $(n-4)$ points S lying on MP produces $(n-4)(n-5)$ pairs S_1, S_2, so that MP represents $2(n-2)(n-3)(n-4)(n-5)$ coincidences. The remaining $[4(2n-1) - 2(n-2)](n-3)(n-4)(n-5)$ coincidences are, taken three by three, points of contact of triple tangents $t_{2,2,2}$. *Through an arbitrary point P pass consequently $2n(n-3)(n-4)(n-5)$ triple tangents.*

8. Let a again be an arbitrary straight line; each of its points is, as base-point of a pencil belonging to N, point of contact R of $(n+4)(n-3)$ bitangents d [1]). We determine the order of the locus of the second point of contact R'. The latter has in common with a the pairs of points R, R', in which a is touched by c^n, and also the points of undulation $(R' \equiv R)$, lying on a, consequently $4(n-2)(n-3) + 3(6n-11)$ or $(4n^2-2n-9)$ in all. This number is apparently the order of the curve $(R')_a$ in question.

In order to determine the locus of the points W, which each bitangent d of the system in question has moreover in common with the c^n, twice touched by it, we associate to each of those curves c^n, the bitangent d, for which the point of contact R lies on a.

To the pencil, which a point P sets apart from N, a curve of order $(n-3)(2n^2+5n-6)$ is associated, which contains the points of contact of the bitangents to the curves of that pencil [2]). By this the number of straight lines d becomes known, of which a point of contact lies on a; the system $[c^n]$ has therefore as index $(n-3)(2n^2+5n-6)$. The index of the system $[d]$ is $(n-3)(5n-4)$; for this is (§ 7) the number of intersections of a with the curve $(R)_P$. The systems $[c^n]$ and $[d]$ rendered projective, produce a locus of order $(n-3)(2n^2+5n-6) + n(n-3)(5n-4)$. To it belongs the straight

1) T. p. 102.

2) Bitangential curve; cf. T. p. 107.

line a $2(n+4)(n-3)$-times, because each of its points is point of contact of $(n+4)(n-3)$ bitangents. The curve $(R')_a$ belongs moreover twice to it. For the order of the curve $(W)_a$ we find consequently $(n-3)(7n^2+n-6)-2(n-3)(n+4)-2(4n^2-2n-9)=(n-4)(7n^2-2n-15)$.

We now consider the correspondence between the rays $r' = MR'$ and $w = MW$. A ray r' contains $(4n^2-2n-9)$ points R', consequently determines $(4n^2-2n-9)(n-4)$ rays w; to a ray w $(n-4)(7n^2-2n-15)$ rays r' are associated. Each of the $(n-3)(5n-4)$ lines d, which connect M with the intersections of a and $(R)_M$, is apparently an $(n-4)$-fold coincidence. The number of coincidences $R' \equiv W$ amounts therefore to $(n-4)[(4n^2-2n-9)+(7n^2-2n-15)-(n-3)(5n-4)]=$ $(n-4)(6n^2+15n-36)$. *This number is the order of the locus* $(R)_{2,3}$ *of the points of contact* R *of the tangents* $t_{2,3}$.

9. In order to find also the order of the locus $(I)_{2,3}$ of the *inflectional points* I of the tangents $t_{2,3}$, we return to the system $[c^n]$ considered in § 4, of which all the curves have an inflectional point I on a given line a. The points S, which the corresponding stationary tangent has moreover in common with c^n, lie on a curve $(S)_a$ of order $3(n^2+n-5)$. We consider now the correspondence between two points S_1, S_2 of the same curve. It determines in a pencil of rays (M) a symmetrical correspondence with characteristic number $3(n^2+n-5)(n-4)$. The rays connecting M with the intersections of a and $(I)_M$, are $(n-3)(n-4)$-fold coincidences; as their number amounts to $3(n-1)$ (§ 1), we find for the number of coincidences $S_1 \equiv S_2$ $[2(n^2+n-5)-(n-1)(n-3)]$ or $3(n-4)(n^2+6n-13)$. This, however, is also the number of tangents $t_{2,3}$, the point of inflection of which lies on a, consequently *the order of the locus* $I_{2,3}$ *of the points of inflection of the tangents* $t_{2,3}$.

By means of the curves $(R)_{2,3}$ and $(I)_{2,3}$, belonging to the system $[t_{2,3}]$, we can again determine the number of *fivepoint-tangents* t_5. For this purpose we associate the lines MR and MI, on account of which a correspondence with characteristic numbers $3(n-4)(2n^2+5n-12)$ and $3(n-4)(n^2+6n-13)$ arises. The $9n(n-3)(n-4)$ tangents $t_{2,3}$ converging in M are coincidences. On the remaining ones R coincides with I. So we find for the number of the t_5 $3(n-4)(3n^2+11n-25)-9n(n-3)(n-4)$ or $15(n-4)(4n-5)$.

10. We return to the system $[c^n]$ of the curves, which (§ 8) are each touched by one of their bitangents d in a point R of a straight line a.

If on a line d two of the points W coincide d becomes a triple tangent. The correspondence between two points W_1, W_2 of a same c^n determines in the pencil of rays M a symmetrical correspondence with characteristic number $(7n^2 - 2n - 15)(n - 4)(n - 5)$. As each bitangent through M having one of its points of contact on a, represents $(n-4)(n-5)$ coincidences, the number of coincidences $W_1 \equiv W_2$ amounts to $2(n-4)(n-5)(7n^2-2n-15) - (n-4)(n-5)(n-3)(5n-4) = (n-4)(n-5)(9n^2+15n-42)$. As they lie two by two on tangents $t_{2,2,2}$, *the locus of the points of contact of the triple tangents is a curve* $R_{2,2,2}$ *of order* $\frac{3}{2}(n-4)(n-5)(3n^2+5n-14)$.

We consider now the system $[c^n]$ of the curves possessing a tangent $t_{2,2,2}$, and determine the order of the locus of the points Q, which each c^n has moreover in common with its $t_{2,2,2}$. The system $[c^n]$ has as index $(n-3)(n-4)(n-5)(n^2+3n-2)$; for this is the number of c^n of the pencil determined by a point P possessing a $t_{2,2,2}$ [1]). The index of the system $[t_{2,2,2}]$ is (§ 7) $2n(n-3)(n-4)(n-5)$. To the figure produced by $[c^n]$ and $[t_{2,2,2}]$ the curve $(R)_{2,2,2}$ belongs twice. For the order of (Q) we find consequently $(n-3)(n-4)(n-5)(n^2+3n-2) + 2n^2(n-3)(n-4)(n-5) - 3(n-4)(n-5)(3n^2+5n-14)$ or $(n-4)(n-5)(n-6)(3n^2+ + 3n-8)$.

11. On each $t_{2,2,2}$ we associate each of the points of contact R to each of the intersections Q, and consider the correspondence (MR, MQ). Its characteristic numbers are $\frac{3}{2}(3n^2+5n-14)(n-4)(n-5)(n-6)$ and $3(n-4)(n-5)(n-6)(3n^2+3n-8)$. Each of the $2n(n-3)(n-4)(n-5)$ tangents $t_{2,2,2}$ converging in M, represents apparently $3(n-6)$ coincidences. Taking this into consideration we find for the number of coincidences $R \equiv Q$, consequently for the *number of tangents* $t_{2,2,3}$, $\frac{3}{2}(n-4)(n-5)(n-6)(5n^2+23n-30)$.

The correspondence between two points Q belonging to the same c^n determines in the pencil of rays (M) a symmetrical correspondence with characteristic number $(3n^2+3n-8)(n-4)(n-5)(n-6)(n-7)$. To this each of the $2n(n-3)(n-4)(n-5)$ tangents converging in M belongs $(n-6)(n-7)$-times. Paying attention to this we find for the number of coincidences $Q_1 \equiv Q_2$ $4(n-4)(n-5)(n-6)(n-7)(n-1)(n+4)$. There are consequently $(n-4)(n-5)(n-6)(n-7)(n-1)(n+4)$ *quadruple tangents*.

12. We shall now consider the system of the curves c^n possessing a tangent $t_{2,3}$, which touches it in a point R, and osculates it in a point I. In order to find the locus of the points S, which c^n has

[1]) T. p. 108.

moreover in common with $t_{2,3}$, we determine the order of the figure produced by the projective systems $[c^n]$ and $[t_{2,3}]$. The former has as index $3(n-3)(n-4)(n^2+6n-4)$ i.e. the number of c^n with a $t_{2,3}$ appearing in a pencil[1]) of N. The index of $[t_{2,3}]$ is (§ 3) $9n(n-3)(n-4)$. The figure produced contains the curve (R) twice, the curve (I) three times. For the order of S we find therefore

$$3(n-3)(n-4)(n^2+6n-4)+9n^2(n-3)(n-4)-6(n-4)(2n^2+5n-12)-$$
$$-9(n-4)(n^2+6n-13)=3(n-4)(n-5)(4n^2+7n-15).$$

By means of this result we can determine the number of twice osculating lines $t_{3,3}$. For this purpose we consider the correspondence (MR, MS). Its characteristic numbers are

$$3(2n^2+5n-12)(n-4)(n-5) \text{ and } 3(4n^2+7n-15)(n-4)(n-5).$$

Each of the $9n(n-3)(n-4)$ $t_{2,3}$ belonging to the pencil (M) is $(n-5)$-fold coincidence, hence the number of coincidences $R \equiv S$ is $(n-4)(n-5)[(6n^2+15n-36)+(12n^2+21n-45)-9n(n-3)] = (n-4)(n-5)(9n^2+63n-81)$. But then *the number of twice osculating tangents* $t_{3,3}$ amounts to $\frac{9}{2}(n-4)(n-5)(n^2+7n-9)$.

By means of the correspondence between the points I and S of the tangents $t_{2,3}$ we can find back the number of tangents $t_{2,4}$ found already in § 6. Analogously we obtain by means of the correspondence between two points S of the same $t_{2,3}$ again the number of tangents $t_{2,2,3}$ found in § 11.

13. If the net has a base-point B, the curves c^n, having an inflection in B are cut by their stationary tangents t in groups of $(n-3)$ points T, lying on a curve $(T)^{n+3}$ with sextuple point B (§ 5). This curve is of class $(n+3)(n+2)-30$; through B pass $(n^2+5n-36)$ of its tangents. In the point of contact R of such a tangent the latter is touched by a c^n, which it osculates in B; *consequently B is a $(n-4)(n+9)$-fold point on the curve* $(I)_{2,3}$.

The curves c^n, which touch in B at a ray d, form a pencil, consequently determine on d an involution of order $(n-2)$. As it possesses $2(n-3)$ coincidences there are $2(n-3)$ c^n, which have d as bitangent, of which B is one of the points of contact. The second point of contact, R, coincides with B if d becomes fourpoint tangent, consequently B point of undulation. This occurs six times; hence the locus $(R)_B$ of the points R is a curve of order $2n$, with sextuple point B.

Every straight line d cuts the c^n, which it touches in B and in R, moreover in $(n-4)$ points S. In order to determine the locus

1) T. p. 106.

$(S)_B$ of these points, we associate each ray d to the $2(n-3)$ curves c^n, to which it belongs, and consider the figure produced by the projective systems $[c^n]$ and (d) thus determined.

Through a point P passes a pencil of c^n; the base-point B is point of contact of $(n-3)(n+4)$ bitangents; this number is the index of $[c^n]$. The order of the figure produced now amounts to $(n-3)(n+4)+$ $+2n(n-3)=(n-3)(3n+4)$. To this the curve $(R)_B$ apparently belongs twice; for the order of $(S)_B$ we find therefore $(n-3)(3n+4)-4n$ or $3(n+1)(n-4)$.

As every d, outside B, contains $2(n-3)(n-4)$ points S, $(S)_B$ will have in B a multiple point of order $3(n+1)(n-4)-2(n-3)(n-4)$ or $(n+9)(n-4)$.

14. Let us now consider the correspondence (MR, MS), if R and S lie on the same ray d through B. To each ray MR belong $2n(n-4)$ rays MS, each ray MS determines $3(n+1)(n-4)$ rays MR. The ray MB contains $2(n-3)$ points R, consequently represents $2(n-3)(n-4)$ coincidences. The remaining ones, to the number of $(n-4)(2n+3n+3-2n+6)$, pass through points $R \equiv S$. So there are $3(n-4)(n+3)$ rays d, which each touch a c^n in B and osculate it in a point I; *the curve* $(R)_{2,3}$ *has consequently a* $3(n-4)(n+3)$-*fold point in* B.

Now we pay attention to the symmetrical correspondence of the rays, which connect M with two points S belonging to the same c^n. The characteristic number is here $3(n+1)(n-4)(n-5)$, while MB represents $2(n-3)(n-4)(n-5)$ coincidences. The remaining $(n-4)(n-5)$ $[6(n+1)-2(n-3)]$ lie in pairs on a *triple tangent*, which has one of its points of contact in B. From this we conclude *that the curve* $(R)_{2,2,2}$ *possesses in* B *a* $2(n+3)(n-4)(n-5)$-*fold point.*

15. Let D be node of an c^n, t one of the tangents in D, S one of the intersections of t with c^n. In order to find the locus of S, we associate to each nodal c^n its two tangents t and determine the order of the figure produced by it. The tangents t envelop the curve of ZEUTHEN; they form consequently a system with index $3(n-1)^2$; for a pencil contains $3(n-1)^2$ nodal curves. By means of the correspondence of the series of points, which the two systems determine on a line, we now find again the order of the figure produced. Considering that the locus of D belongs six times to it, we obtain as order of the curve (S) $3n(n-1)(2n-3)+6(n-1)^2-18(n-1)=$ $=3(n-1)(2n^2-n-8)$. For $n=3$ we find 42 for it; the 21 straight lines of the degenerate curves must indeed be counted twice.

We now consider the correspondence (MD, MS). Its characteristic numbers are $3(n-1).2(n-3)$ and $3(n-1)(2n^2-n-8)$, while each of the tangents t converging in M, apparently produces $(n-3)$ coincidences. The remaining ones arise from coincidences $D \equiv S$, consequently from nodal curves, for which D has an inflection on one of its branches. It now ensues from $6(n-1)(n-3)+3(n-1)(2n^2-n-8)-3(n-1)(n-3)(2n-3) = 3(n-1)(10n-23)$, that *the net contains* $3(n-1)(10n-23)$ *curves with a flecnodal point.*

ERRATUM.

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p. 870 line 15 from the bottom: Add: Supplement N°. 37 to the Communications from the Physical Laboratory at Leiden. Communicated by Prof. H. KAMERLINGH ONNES.

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