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Meteorology. — *"On the relation between the cloudiness of the sky and the duration of sunshine"*. By Dr. J. P. VAN DER STOK.

(Communicated in the meeting of September 27, 1913).

1. The subject, mentioned in the title, has but rarely and then cursorily been discussed in meteorological textbooks, periodicals and monographs, and the fact that the sum of degree of cloudiness and duration of sunshine, expressed in percentages and considered for monthly or seasonal means, differs little from 100, is then regarded as a satisfactory result concerning both methods of observing the covering of the sky.

From observations taken during many years and for three European stations in different climates we find:

	Hamburg [1])		Pola [2])		Potsdam [3])		Hamburg	Pola	Potsdam
	B	S	B	S	B	S	T	T	T
Winter	79%	14%	56%	37%	63%	31%	93%	93%	94%
Spring	63	33	49	49	63	36	96	98	99
Summer	66	31	34	69	59	45	97	103	104
Autumn	71	22	49	48	64	30	93	97	94
Year	70	25	47	51	65	33	95	98	98

B = Cloudiness, S = Duration of Sunshine, T = B + S.

In fact it does seem natural that, to a certain extent, the one factor is complimentary to the other; if the degree of cloudiness is a hundred or zero, the duration must certainly be resp. zero and a hundred, and as in temperate climates these extreme cases often occur, it is natural that the sums do not differ much from 100. From monthly means of cloudiness and duration of sunshine we find for the five Dutch stations the numbers of table I, calculated from or reduced to the period 1900—1911.

From these results it appears that, in general, the sum increases with the sun's height; in summer the sum is nearly a hundred, in winter not more than 86 and only at Helder it remains considerably

1) KÖNIG. Dauer des Sonnenscheins in Europa. Nova Acta, K. Leip. Car. D. Ak. d. Naturf. Bd. **67**, n⁰ 3, 1896.

2) FRIEDEMANN. Bewölkung und Sonnenschein des Mittelmeergebietes. Aus dem Arch. d. D. Seewarte. Bd. **36**, 1913.

3) MEISZNER. Das Wetter. 1907 en 1910.

TABLE I.

Sums of cloudiness and duration of sunshine, Monthly means.

	De Bilt	Helder	Groningen	Vlissingen	Maastricht	Mean
January . . .	90	80	90	83	86	85.8
February . . .	90	83	94	87	90	88.8
March. . .	93	80	91	89	87	88.0
April	97	84	99	93	91	92.8
May	100	88	102	94	95	95.8
June . . .	99	89	99	94	92	94.6
July	99	88	98	92	92	93.8
August . .	101	90	99	93	95	95.6
September .	95	82	94	90	88	89 8
October . . .	92	80	90	86	88	87.2
November . .	91	80	91	80	86	85.6
December . .	88	80	90	82	86	85.2
Year	97	86	97	91	92	92.6

behind the full percentage in every month. However, it is well known, (as an inspection of the hourly values given in annals also proves) that the sunshine-recorders do not register until some time after sunrise and leave off registering before sunset, so that all average values (but mostly those of the wintermonths when the total amount of sunshine is small) must be too low and that to a considerable extent.

Monthly means therefore are certainly not likely to give a proper idea of the relation between cloudiness and duration of sunshine so that in order to come to a better understanding we employ observations taken at a time when we are sure of the undisturbed influence of the sun; as the cloudiness is observed at 8 a.m., 2 p.m and 7 p.m, it is the afternoon observation which is indicated for this investigation.

As shown by tables II and III, the sum is then considerably higher than 100°/₀ for all five stations with the exception of winter for Helder and of December for all stations.

From the fact that in June and July, when the sun is highest, the sums are almost equal for all stations, we may conclude that individual conceptions in estimating the cloudiness play a subordinate

TABLE II.
Mean Cloudiness and duration of sunshine, 2 p m , 1909–1911.

	Helder		Groningen		De Bilt		Vlissingen		Maastricht	
	Clouds	Sun	Clouds	Sun	Clouds	Snn	Clouds	Sun	Clouds	Sun
January. . .	66%	25%	78%	27%	70%	35%	74%	30%	75%	28%
February . .	69	27	76	28	70	33	73	34	78	26
March . . .	66	37	77	32	70	46	70	36	75	37
April . . .	57	54	73	42	64	55	60	55	67	51
May. . . .	46	65	65	61	60	60	50	69	67	48
June. . . .	61	52	72	41	71	45	65	48	76	37
July. . . .	57	56	75	39	69	48	64	50	72	38
August. .	53	57	70	50	59	63	53	61	62	51
September. .	62	47	74	42	66	52	62	50	63	50
October. . .	69	32	74	34	66	48	67	40	71	37
November . .	72	19	76	32	72	33	71	25	78	25
December . .	78	10	83	10	79	22	79	16	79	15
Year. . . .	63	40	74	36	68	45	66	43	72	37

TABLE III.
Sum of Cloudiness and duration of sunshine, 2 pm., 1909—1911.

	Helder	Groningen	De Bilt	Vlissingen	Maastricht	Mean
January . . .	91	105	105	104	103	101.6
February . . .	96	104	103	107	104	102.8
March. . . .	103	109	116	106	112	109.2
April	111	115	119	115	**118**	115 6
May	111	**126**	**120**	**119**	115	118.2
June	113	113	116	113	113	113.6
July	113	114	117	114	110	113.6
August . . .	110	**120**	**122**	114	**113**	115.8
September . .	109	116	118	112	113	113.6
October . . .	101	108	114	107	108	107.6
November. . .	91	108	105	96	103	100.6
December. . .	88	93	101	95	94	94.2
Year	103	111	113	109	109	108.9

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part and that the differences as shown in these tables must be ascribed rather to the nature of the clouds than to this reason.

When treated in this way several peculiarities are observed which remain concealed when daily means are used e.g. the fact that near Helder the cloudiness is smaller than at de Bilt, whereas de Bilt shows 5 °/₀ more of sunshine.

Furthermore these tables show that in wintertime also, when the suns elevation is very small (in January for 52° L. N. and 2 p. m. the sun's height is only 12.25°), the sums mostly exceed 100 °/₀.

If we do not wish to ascribe the different relations at de Bilt and Helder, characterized by a difference of 10 °/₀ in the sums for the whole year, on the one hand to a difference in the estimate of cloudiness or, on the other hand, to instrumental unequalities, an explanation of these phenomena is required.

That the sum of cloudiness and duration of sunshine is more than 100 °/₀ and that this fact is not shown in monthly means because two opposite causes neutralize each other, has also been remarked by others e.g. by BESSON and COEURDEVACHE[1]).

We can now take a further step and, from the frequent occurrence of extreme values, conclude that also the numbers of tables II and III cannot be considered as a proper measure for the relation between cloudiness and duration of sunshine and we may expect that the sums must be considerably larger for average values of cloudiness.

This conclusion is fully confirmed by the data of tables IV and V, giving the results of a classification of percentages of sunshine's duration according to those of cloudiness. Table V is printed as an example of the way in which different combinations may occur.

Although the numbers of table IV do not everywhere show a regular course, they clearly demonstrate that the sums of cloudiness and duration of sunshine may amount in summer to more than 130, in winter to more than 110 °/₀.

In this communication the disagreements between different stations are not taken into consideration and the question is restricted to the problem: how to explain the absolute values, the differences between summer and winter and the position of the maxima as shown in table IV for the station de Bilt only.

2. In order to find a possible relation between the results of both methods of observing the covering of the sky, it is in the first place necessary to consider, what is understood by the term "degree

[1]) Ann. Soc. Météor. de France, 56, 1908 (73).

TABLE IV

Sums of cloudiness and duration of sunshine in percentages
De Bilt, 2 pm. 1900—1911

Cloudiness	Jan. Dec.	Febr. March	April May	June July	Aug. Sept.	Oct. Nov.
0	89	98	100	100	100	99
1	108	107	110	110	108	105
2	**111**	104	119	118	119	110
3	107	117	123	128	126	**115**
4	94	**119**	125	**131**	**130**	109
5	92	116	**125**	127	127	107
6	98	112	124	124	122	111
7	93	110	124	118	120	102
8	91	97	103	103	102	97
9	94	97	96	100	99	95
10	101	101	104	102	104	101
Mean	98	104	112	114	114	103
Number observ.	738	714	732	732	732	730

TABLE V.

Frequency of cloudiness and duration of sunshine.
De Bilt 2 pm. 1900—1911. June and July.

	Duration of sunshine.											Number	Mean duration of sunshine
Cloudiness	0	1	2	3	4	5	6	7	8	9	10		
0	—	—	—	—	—	—	—	—	—	—	15	15	100.0 %
1	—	—	—	—	—	—	—	—	—	—	32	32	100.0
2	—	—	1	—	—	—	—	—	1	—	46	48	97.9
3	—	—	—	—	—	—	1	1	—	3	58	63	98.4
4	—	—	—	1	—	1	4	—	7	5	32	50	90.6
5	1	2	2	1	3	3	3	4	7	9	23	58	77.4
6	5	4	5	5	5	4	7	8	5	8	22	78	64.0
7	13	8	9	11	4	4	16	3	15	6	7	96	47.5
8	38	11	17	11	5	6	3	4	—	5	1	101	22.7
9	58	9	5	3	4	4	2	—	1	—	—	86	9.8
10	91	9	4	—	1	—	—	—	—	—	—	105	2.0
Sum	206	43	43	32	22	22	36	20	36	36	236	732	51.2

Mean cloudiness 62.6 %

of cloudiness", and, if possible, to substitute a simpler definition, suitable for mathematical treatment, for this evidently complicated conception.

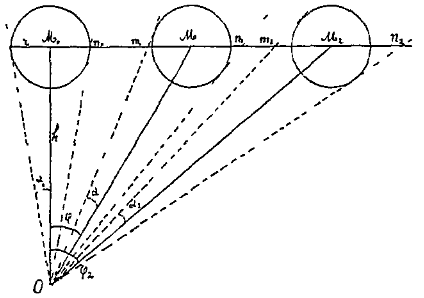

Fig. 1.

In fig.(1) the plane of the drawing represents a plane going through the observer in O and the zenith in M_0; the circles are projections of spherical clouds upon this plane, the diameter of the clouds is denoted by d, their mutual distance by Δ.

The apparent proportion of white to white and blue in the sky will be different according to the part of the arc $\varphi_1, \varphi_2-\varphi_1\; \varphi_{n+1}-\varphi_n$ considered; obviously the apparent cloudiness of the n^{th} order is

$$\frac{\alpha_{n+1}+\alpha_n}{\varphi_{n+1}-\varphi_n}$$

As we have here to do with proportions, and angular values are unsuitable for mathematical treatment, we substitute for the projection upon the sky-arc the projection upon the straight line $M_0\, M_1 \ldots M_n$; then the apparent cloudiness is defined as:

$$S_n = \frac{M_{n+1}\, m_{n+1} + M_n\, n_n}{M_{n+1}\, M_n} = \frac{d}{\Delta}\, \frac{\cos(\varphi_{n+1}-\alpha_{n+1}) + \cos(\varphi_n + \alpha_n)}{2\cos(\varphi_{n+1}-\alpha_{n+1})\cos(\varphi_n+\alpha_n)}.$$

Here $\frac{d}{\Delta}$ denotes the true cloudiness W of the sky, the cloudiness being defined as a proportion between linear quantities. When the spheres are not very large we can assume an average value for n and $n+1$, and the apparent cloudiness of the n^{th} order becomes:

$$S_n = W \frac{\cos\varphi_n \cos\alpha_n}{\cos^2\varphi_n - \sin^2\alpha_n}$$

or, as $\sin\alpha_n = \sin\alpha_0 \cos\varphi_n$

$$\cos^2\varphi_n = \frac{1}{1+n^2 q^2} \qquad q = \frac{\Delta}{h}$$

$$S_n = W \frac{\sqrt{n^2 q^2 + \cos^2\alpha_0}}{\cos^2\alpha_0}$$

If the clouds are not large so that $\cos^2\alpha = 1$ may be assumed as equal to unity, this expression becomes, owing to the relation $nq = \tan\varphi_n$,

$$S_n = \frac{W}{\cos \varphi_n}$$

As $\cos^2 6 = 0.99$, this will hold good for pretty large clouds which, when seen in the zenith, are measured by an angular value of 12°.

At a zenith's distance

$$\varphi = arc \cos (W)$$

the apparent cloudiness becomes equal to unity, i.e. the cloudbank begins there, which is always more or less manifest when the clouds are regularly spread over the sky.

In the foregoing considerations the true cloudiness (W) is regarded as a proportion of linear quantities, whereas in reality the proportion between white and blue in the plane going through $M_0\,M_1\,M_2$ and at right angle to the plane of the drawing of fig. (1) ought to be considered, and the question arises in how far it is permissible to substitute a fictitious linear repartition along a line for a real cloudiness as measured in a plane.

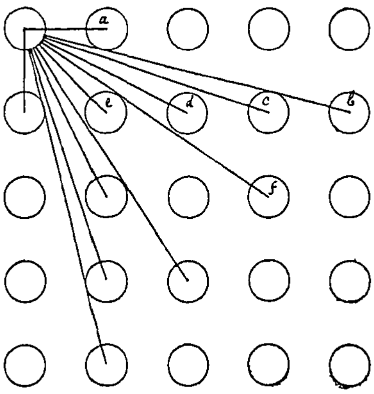

Fig. 2.

In fig. (2) the clouds, assumed to be spherical, are projected upon the plane going through their centra and an observer, looking at this plane from a point 0 below the origin of coordinates, will notice totally different proportions between white and blue in different directions.

If, as in fig. (2), the mutual distance Δ is equal to twice the diameter d, he will notice six different degrees of cloudiness within an angular distance of 45°, as denoted by full lines, and the alignments a and e will be repeated 4 times, the other 6 times over the whole sky.

The average cloudiness derived from this regular arrangement is then

$$\frac{1}{20}\left[1 + \frac{\sqrt{2}}{2} + \frac{2\sqrt{5}}{5} + \frac{2\sqrt{10}}{10} + \frac{2\sqrt{13}}{13} + \frac{2\sqrt{17}}{17}\right] = 0.214.$$

From the proportion between the surface of the circles and the total surface of the corresponding plane we find for the true cloudiness, according to the formula:

$$Cl. = \frac{\pi}{4}\frac{d^2}{\Delta^2} = \frac{\pi}{16} = 0.196$$

showing a difference of 1.8 %.

Although this calculation (given as an example, not as a proof) is applicable only to moderate and small degrees of cloudiness, because a cloudiness 10 cannot be represented by circles, we may conclude from this example that it is permissible to substitute a linear for a surface-cloudiness of which it is impossible to give a final definition because a given proportion of white and blue in a plane can be represented in very different ways.

3. In the following calculations we assume the clouds to have the shape of an ellipse rotating about a vertical axis and that these clouds, if average values derived from many observations are used, may be considered as small so that the tangents, drawn to the ellipse in a vertical plane, may be regarded as parallel. The well known fact that the dimensions of all objects (mountains, constellations, sun, moon) when seen near the horizon appear strongly exaggerated, is taken into account by introducing a physiological factor $f(\varphi)$.

The relation between true and apparent cloudiness in a point at a zenith's distance φ then becomes:

$$W_s = W f(\varphi) \sqrt{1 + k^2 \tan^2 \varphi}, \quad \ldots \ldots \quad (1)$$

where k denotes the proportion between the vertical and horizontal axes of the ellipsoid i. e. greater than unity when pointed upwards, smaller than unity when oblate.

We further assume that an object at the horizon is seen twice as large as in the zenith, and as $f(\varphi)$ must satisfy the condition that $f = 1$ for $\varphi = 0$, the simplest forms which this function can be given are:

$$f_1 = 1 + \sin \varphi \,;\; f_2 = 2 - \cos^2 \varphi \,,\; f_3 = 2 - \cos \varphi.$$

Of these functions the first corresponds with the greatest, the last with the smallest augmentation for values of φ near 45°.

The value of the angle β, the zenith's distance where, apparently, the blue disappears out of the sky and the perspective cloudbank begins, is then determined by the formula:

$$W . f(\beta) . \sqrt{1 + k^2 \tan^2 \beta} = 1 \quad \ldots \ldots \quad (2)$$

and the sum of the cloudiness in the arc $\varphi = 0$ to $\varphi = \beta$ becomes:

$$I W = W \int_0^\beta f(\varphi) . \sqrt{1 + k^2 \tan^2 \varphi}\, d\varphi \quad \ldots \ldots \quad (3)$$

The apparent cloudiness corresponding to the true cloudiness W is then:

$$W_s = 1 - 2/\pi\, (\beta - I W) \quad \ldots \ldots \quad (4)$$

If Z denotes the duration of sunshine as registered at the same time, then the expression holds good :

$$1 - Z = R' = W \sqrt{1 + k^2 \, tang^2 \, \varrho} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where ϱ is the sun's zenith's distance.

W_s and R being known by observation, it is, theoretically, possible to calculate the two unknown quantities W and k from (4) and (5) with the help of (2) and (3).

Practically the only possibility to come to a result is to give k different values and to calculate W and R for a series of values of the auxiliary quantity β.

In the following calculations the zenith's distance of the sun at 2 p.m. has been assumed to be in summer (June—July) 37°42'5, in winter (December—January) 78°37'.

Table VI shows the results of the calculation for the summer-months and spherical clouds ($k = 1$); it appears then that, starting from the assumed supposition an explanation can be given of the results of observation concerning the magnitude of the sum or cloudiness and duration of sunshine. The first augmenting factor, $1 + \sin \varphi$, leads to values which are somewhat too large; the second, $2 - \cos^2 \varphi$, to an exact maximum; the last, $2 - \cos \varphi$, to a value somewhat smaller than 130.

The position of the maximum is less satisfactory; it corresponds in table IV to an apparent cloudiness of 4, whereas in Table VI the greatest values correspond to an apparent cloudiness of 6.5 and 5.5. The augmenting factor $2 - \cos \varphi$ gives the best results because then the calculated duration of sunshine for the heavy cloudiness 7—8 is smaller than in the foregoing tables and shows a better agreement with the numbers of table IV.

For this reason the latter augmenting factor is chosen in calculating the cloudiness and duration of sunshine for three values of k viz. 1.305, 1.5 and 2, all greater than unity and therefore corresponding with clouds pointed upwards. For values smaller than unity the shifting of the maximum would certainly occur in the wrong direction i. e. to the side of the heavy cloudiness.

In fact these increased values of k lead to an improvement of the results, the maximum now corresponding with an apparent cloudiness of 5.

As might have been expected, the influence of a variation of k upon the position of the maximum is not great when the value of ϱ is small, because in that case R varies little with m.

A more complete agreement between calculation and observation might be obtained, but only by using a more complicated augmenting factor.

When the sun's height decreases or φ increases, the influence of k is much stronger; values as $k = 0.5$ or $k = 0.12$ appeared to be of no use in calculating the duration of sunshine in winter and, after some trials, only the value $k = 0.25$ proved to give satisfactory results.

As for the summermonths the factor $2 - \cos\varphi$ appeared to be the best, although the maximum value happens to be somewhat too small; on the other hand the position of the maximum is almost accurate, corresponding with an apparent cloudiness of 2.

For the wintermonths, therefore, an almost complete agreement between calculation and observation might be obtained by increasing the augmenting factor and assuming e.g. $2.5 - 1.5 \cos\varphi$, which would agree with the experiment, the augmentation near the horizon being certainly more than twice.

The integration-constants $1 + k$ and $+1$ in the formula for I_1 (table VIII) had to be added in order to ensure the condition:

$$I_1 = o \text{ for } x = 0$$

In calculating the elliptical integrals in the expressions for I_3, the "Funktionentafeln mit Formeln und Kurven von JAHNKE und EMDE". Teubner 1909, have been made use of.

4. The results of this inquiry can be summed up as follows; it is possible to explain the relation between cloudiness and duration of sunshine as found by experiment by means of theoretical reasoning and simple assumptions. In this way a numerical measure of the specific influence of the cloudiness upon radiation, received and emitted, can be obtained according to the nature of the clouds in different seasons.

In the considerations made use of in treating this problem, it is assumed that in calculating the estimated cloudiness W_s and also the apparent cloudiness R, as derived from the duration of sunshine, the same value W of the true cloudiness obtains.

This supposition is certainly not quite justified; the former observation is made with incident rays, or more accurately diffuse reflection, the latter with transmitted light and probably (at least for Cirri, Pseudo-cirri and Fracto-Cumuli) the loose, flocky clouds and cloud-borders transmit some light, whereas, with reflected light, they make the impression of an entire covering of that cloudy part of the sky; on the other hand the sunshine-recorders readily discontinue registering when the sun's rays are absorbed. The fact also that the sun's image as formed by the glass sphere or the slit of the sunshine-recorder not being small with respect to the time-scale, may give rise to a smaller value of W than is derived from the estimated cloudiness, because a more or less continuous discoloration can be observed when light and shadow are in reality varying in a discontinuous way. It will be difficult to take into account either factor as they are dependent on the velocity of the cloudmotion by which the possibility of a registration of relatively feeble rays of light is determined. The influence of these factors can be determined only by experiment, which however would be difficult to carry out. It would be necessary to extend it to a great number of different cases and, in order to inquire in how far compensation happens to occur between too large a percentage of sunshine in the case of slowly moving clouds and too small a percentage when the clouddrift is more rapid, the experiment would have to be extended to a great number of different cases.

Finally it may be noticed that the fact that clouds generally show a flat base and a pointed apex, might be taken into account by considering the upper half of ellipsoids, the equatorial plane being situated in the plane $M_s\, M_1 \ldots M_n$. Then the duration of sunshine corresponding to a given value of W increases, but as, on the other hand, the apparent cloudiness decreases, the sums remain nearly the same.

TABLE VI.

Calculation for spherical clouds.

De Bilt, June + July, $\alpha = 37°42'.5$

β	W	W_s	Z	$W_s + Z$	
30°	0.577	0.922	0.270	119.2	$k = 1$
35	0.521	0.894	0.342	123.6	
40	0.466	0.862	0.411	127.3	$1 = \frac{1+\sin\beta}{\cos\beta}\, W$
45	0.414	0.824	0.476	130.0	
50	0.364	0.781	0.540	132.1	$R = \frac{W}{\cos\beta}$
55	0.315	0.732	0.601	133.3	
60	0.268	**0.677**	0.661	**133.8**	$l_1 = \frac{1}{2} \ln \frac{1+\sin\beta}{1-\sin\beta} - \ln \cos\beta$
65	0.222	0.611	0.720	133.1	
70	0.176	0.538	0.777	131.5	
75	0.132	0.450	0.833	128.3	
80	0.088	0.345	0.889	123.4	
85	0.044	0.211	0.945	115.6	

β	W	W_s	Z	$W_s + Z$
30°	0.693	0.931	0.124	105.5
35	0.616	0.899	0.221	112.0
40	0.542	0.861	0.315	117.6
45	0.471	0.817	0 405	122.2
50	0.405	0.769	0.488	125.7
55	0.343	0.714	0.566	128.0
60	0.286	0.655	0.638	129.3
65	0.232	**0.589**	0.707	**129.6**
70	0.182	0.515	0.770	128.5
75	0.134	0.430	0.831	126.1
80	0.088	0.329	0.889	121.8
85	0.044	0.202	0.944	114.6

$$1 = \frac{2 - \cos^2\beta}{\cos\beta} W$$

$$R = \frac{W}{\cos\beta}$$

$$I_2 = \ln\frac{1+\sin\beta}{1-\sin\beta} - \sin\beta$$

30	0.764	0.947	0.035	98.2
35	0.694	0.919	0.123	104.2
40	0.621	0.883	0.215	109.8
45	0.547	0.841	0.309	115.0
50	0.474	0.791	0.401	119.2
55	0.402	0.734	0.492	122.6
60	0.333	0.670	0.579	124.9
65	0.268	**0.598**	0.661	**125.9**
70	0.206	0.518	0.739	125.7
75	0.149	0.427	0.812	123.9
80	0.095	0.322	0.880	120.2
85	0.046	0.194	0.942	113.6

$$1 = \frac{2 - \cos\beta}{\cos\beta} W$$

$$R = \frac{W}{\cos\beta}$$

$$I_3 = \ln\frac{1+\sin\beta}{1-\sin\beta} - \beta$$

TABLE VII

Calculation for clouds pointed upwards: $k > 1$. De Bilt, June—July, $\nu = 37^\circ 42'.5$.

β	W	W_s	Z	Sum
30°	0.704	0.933	0.002	93.5%
35	0.625	0.900	0.114	101.4
40	0.547	0.860	0.225	108.5
45	0.470	0.814	0.333	114.7
50	0.399	0.761	0.435	119.6
55	0.332	0.701	0.530	123.1
60	0.270	0.639	0.618	125.7
65	0.213	**0.567**	0.698	**126.5**
70	0.162	0.492	0.770	126.2
75	0.116	0.406	0.836	124.2
80	0.073	0.307	0.896	120.3
85	0.035	0.189	0.967	115.6

$k = 1.305$

$$1 = \sqrt{1+m^2 \sin^2 \beta}\; \frac{2-\cos\beta}{\cos\beta}\, W$$

$$R = \frac{\sqrt{1+m^2 \sin^2 \rho}}{\cos\rho}\, W$$

$$l_3 = k \ln \frac{\sqrt{1+m^2x^2}+kx}{\sqrt{1+m^2x^2}-kx} - m \ln \frac{\sqrt{1+m^2x^2}+mx}{\sqrt{1+m^2x^2}-mx} - \int_{\pi/2-\beta}^{\pi/2} d\varphi \sqrt{1-n^2 \sin^2 \varphi}$$

$$m^2 = k^2 - 1 \quad n^2 = \frac{k^2-1}{k^2} \quad x = \sin\beta$$

β	W	W_s	Z	Sum
30°	0.667	0.925	0.000	92.5%
35	0.584	0.889	0.106	99.5
40	0.504	0.847	0.228	107.5
45	0.429	0.800	0.343	114.3
50	0.360	0.749	0.449	119.8
55	0.297	0.688	0.546	123.4
60	0.245	0.625	0.633	125.8
65	0.188	**0.556**	0.711	**126.7**
70	0.142	0.481	0.782	126.3
75	0.101	0.397	0.845	124.2
80	0.064	0.301	0.902	120.3
85	0.030	0.183	0.953	113.6

$f_3 = 2 - \cos\varphi$

$k = 1.5$

β	W	W_s	Z	Sum	
30°	0.577	0 905	0.000	90 5%	
35	0.492	0.866	0.094	96.0	$f_3 = 2 - \cos\rho$
40	0.415	0.821	0.236	105.7	
45	0.346	0.768	0.363	113.1	
50	0.285	0.719	0.475	119.4	$k = 2$
55	0 232	0.662	0.573	123.5	
60	0 185	0.601	0.659	126 0	
65	0 144	**0.536**	0.735	**127.1**	
70	0.108	0.464	0.801	126.5	
75	0 076	0.384	0 860	124.4	
80	0.048	0.292	0.911	120.3	
85	0.023	0.179	0.958	113.7	

TABLE VIII.

Calculation for flat clouds: $k < 1 = 0.25$.

De Bilt. December + January, $\alpha = 78° 37'$.

β	W	W_s	Z	*Sum*	
40	0.596	0.912	0.052	96.4	
45	0.568	0.888	0.094	98.2	$1 = \sqrt{1 - m^2 \sin^2\beta} \quad \frac{1 + \sin\beta}{\cos\beta} W$
50	0.543	0.875	0.135	101.0	
55	0.518	0.854	0.175	102.9	$R = \frac{\sqrt{1 - m^2 \sin^2\rho}}{\cos} W$
60	0.492	0.829	0.216	104.5	
65	0.462	0.798	0.263	106.1	$I_1 = \frac{k}{2} \ln \frac{\sqrt{1 - m^2 x^2} + kx}{\sqrt{1 - m^2 x^2} - kx}$
70	0.425	0.753	0.323	107.6	
75	0.372	0.683	0.407	109.0	$+ m \, bg \sin(mx)$
80	0.290	**0.563**	0.537	**110.0**	$+ k \ln \frac{\sqrt{1 - m^2 x^2} + k}{(1 + k)\sqrt{1 - x^2}}$
82⁵	0.230	0 464	0.634	109.8	
85	0.166	0.353	0 736	108.9	$+ 1 - \sqrt{1 - m^2 x^2}$
87⁵	0.086	0.202	0.863	106.5	$m^2 = k^2 - 1 \qquad x = \sin\beta$

β	W	W_s	Z	Sum
40	0.693	0.912	0.000	91.2
45	0.647	0.886	0.000	88.6
50	0.604	0.859	0.037	89.6
55	0.564	0.829	0.102	93 0
60	0.524	0.797	0.164	96.1
65	0.495	0.759	0.229	98 8
70	0.438	0.710	0.302	101.2
75	0 378	0.639	0.397	103.6
80	0.293	0.525	0.534	105.9
82^5	0.235	0.441	0.625	106.6
85	0.166	**0.331**	0.736	**106.7**
87^5	0.086	0.202	0.863	106.5

$$1 = \sqrt{1-m^2\sin^2\beta}\quad\frac{2-\cos^2\beta}{\cos\beta}W$$

$$R = \frac{\sqrt{1-m^2\sin^2\rho}}{\cos\rho}W$$

$$I_2 = k\ln\frac{\sqrt{1-m^2x^2}+kx}{\sqrt{1-m^2x^2}-kx}$$

$$-\tfrac{1}{2}x\sqrt{1-m^2x^2}$$

$$+\frac{4m^2-1}{2m}\,bg\sin(mx)$$

$$m^2 = k^2-1 \qquad x = \sin\beta$$

40	0.793	0.949	0.000	94.9
45	0.750	0.901	0.000	90.1
50	0.706	0.890	0.000	89.0
55	0.660	0.860	0.000	86 0
60	0.612	0.823	0.026	84.9
65	0.559	0.778	0.109	88.7
70	0.497	0.719	0.207	92.6
75	0.420	0.637	0.341	97.8
80	0.316	0.513	0.497	101.0
82^5	0.249	0.421	0.603	102.4
85	0.173	**0.315**	0.724	**103 9**
87^5	0.088	0.178	0.860	103.8

$$1 = \sqrt{1-m^2\sin^2\beta}\quad\frac{2-\cos\beta}{\cos\beta}W$$

$$R = \frac{\sqrt{1-m^2\sin^2\rho}}{\cos\rho}W$$

$$I_3 = k\ln\frac{\sqrt{1-m^2x^2}+kx}{\sqrt{1-m^2x^2}-kx}$$

$$+2m\,bg\sin(mx)$$

$$-\int_0^\beta d\varphi\sqrt{1-m^2\sin^2\varphi}$$

$$m^2 = k^2-1 \qquad x = \sin\beta$$