Mathematics. — "*On the rank of the section of two algebraic surfaces.*" By Dr. W. A. VERSLUYS. (Communicated by Prof. P. H. SCHOUTE.

1. In this paper I intend to prove the relation new to me

$$r = m_1 n_2 + m_2 n_1 - 2\sigma - 3\chi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

where r is the rank of the curve of intersection s of two algebraic surfaces S_1 and S_2, respectively of the degree n_1 and n_2 and of the class m_1 and m_2 and possessing in σ points an ordinary contact and in χ points a stationary contact. Some applications of this formula are given too.

Formerly I proved [1]) the following extension of a well-known formula [2])

$$r = n_1 n_2 (n_1 + n_2 - 2) - 2 (n_1 \xi_2 + n_2 \xi_1 + \sigma) - 3 (n_1 \nu_2 + n_2 \nu_1 + \chi), \quad . \quad (B)$$

where $\xi_1, \xi_2, \nu_1, \nu_2$ represent the degrees of the nodal and cuspidal curves of the two surfaces S_1 and S_2. Formula (A) shall first be proved for the case that S_1 and S_2 are developables. If we wish to apply formula (B) to developables the numbers of double generating lines ω_1 and ω_2 must be added to the orders ξ_1 and ξ_2 of the nodal curves and the numbers of stationary generating lines v_1 and v_2 to the orders ν_1 and ν_2 of the cuspidal curves.

Formula (B) becomes

$$r = n_1 n_2 (n_1 + n_2 - 2) - 2 \{n_1 (\xi_2 + \omega_2) + n_2 (\xi_1 + \omega_1) + \sigma\} -$$
$$- 3 \{n_1 (\nu_2 + v_2) + n_2 (\nu_1 + v_1) + \chi\} \quad . \quad . \quad . \quad . \quad (C)$$

2. Let $\Delta^2 S$ be the second polar surface of the degenerated surface $S_1 + S_2$ with respect to the arbitrary point P. This surface $\Delta^2 S$ is of the degree $(n_1 + n_2 - 2)$ and meets the curve of intersection s of S_1 and S_2, this curve being of the degree $n_1 n_2$, in $n_1 n_2 (n_1 + n_2 - 2)$ points.

These points of intersection are 1st the triple points of $S_1 + S_2$ through which the curve s passes and 2nd the points of s for which the tangent plane to one of the two surfaces passes through P. The triple points of $S_1 + S_2$ through which the curve s passes are the points in which a double line of one of the two surfaces meets the other surface. So these triple points are:

1st. The $(n_1 \nu_2 + n_2 \nu_1)$ points in which a cuspidal curve of one of the surfaces meets the other one. These points are cusps on the curve of intersection s, they are indicated by CREMONA as points λ

1) VERSLUYS, Mémoires de Liége, 3me serie, t. VI. Sur les nombres Plückériens etc.

2) E. PASCAL, Rep. di Mat. Sup. II, p- 325.

and must count according to him for three points of intersection of the nodal curve, thus here of the curve s with $\Delta^2 S$ [1]).

2nd. The $(n_1 v_2 + n_2 v_1)$ points in which a stationary generating line of one of the surfaces meets the other one. These points, also cusps on the curve s, are indicated by CREMONA as points r which must count according to him for three points of intersection of the nodal curve s with $\Delta^2 S$ [2]).

3rd. The $(n_1 \xi_2 + n_2 \xi_1)$ intersections of S_1 or S_2 with the nodal curve of the other surface. According to CREMONA each of the branches of the nodal curve meets $\Delta^2 S$ [3]) one time in a triple point τ. Through each of these points τ pass two branches of s, which is a nodal curve on $S_1 + S_2$; so each of these triple points counts for two points of intersection of s with $\Delta^2 S$.

4th. The $(n_1 \omega_2 + n_2 \omega_1)$ nodes of s in which a double generator of one of the surfaces S_1 or S_2 meets the other one. According to CREMONA the nodal curve s meets $\Delta^2 S$ [4]) two times in such a triple point r.

The surface $S_1 + S_2$ possesses still more triple points, among others the cusps β of the cuspidal curves; these points do lie on $\Delta^2 S$, but on the curve of intersection s they do not, so they do not belong to the points of intersection of s with $\Delta^2 S$.

3. Through P pass m_1 tangent planes of the surface S_1. A generator of S_1, along which one of the m_1 tangent planes through P touches S_1, meets n_2 times the surface S_2. Each of these points of intersection is a point on s also situated on $\Delta^2 S$. Such a point of s and of $\Delta^2 S$ counts for *one* point of intersection, according to CREMONA [5]). So $\Delta^2 S$ is met by the curve s in $(m_1 n_2 + m_2 n_1)$ points for which one of the tangent planes passes through P.

This gives the relation:

$$n_1 n_2 (n_1 + n_2 - 2) = m_1 n_2 + m_2 n_1 + 2\{n_1(\xi_2 + \omega_2) + n_2(\xi_1 + \omega_1)\} + \\ + 3\{n_1(\nu_2 + v_2) + n_2(\nu_1 + v_1)\} \quad \ldots \quad (D)$$

Comparing the equations (C) and (D) we get immediately

$$r = m_1 n_2 + m_2 n_1 - 2\delta - 3\chi. \quad \ldots \quad (A)$$

The degree of a developable being the rank of its cuspidal curve, we can write for this formula:

$$r = m_1 r_2 + m_2 r_1 - 2\delta - 3\chi.$$

1) CREMONA—CURTZE, Oberflächen § 108.
2) CREMONA—CURTZE, loc. cit. § 100.
3) CREMONA—CURTZE, loc. cit. § 109.
4) CREMONA—CURTZE, loc. cit. § 101.
5) CREMONA—CURTZE, loc. cit. § 99.

4. The formula (D) and hence also the formula (A), which is now proved for the case that the two surfaces are developables holds still good when S_1 and S_2 are arbitrary algebraic surfaces. Let ξ_1 and ν_1 represent the degree of the total nodal curve and total cuspidal curve of S_1, likewise ξ_2 and ν_2 for S_2. One of the formulae of PLÜCKER applied to an arbitrary plane section of S_1 gives

$$m_1 = n_1^2 - n_1 - 2\,\xi_1 - 3\,\nu_1\,,$$

or

$$0 = n_1^2 - n_1 - m_1 - 2\,\xi_1 - 3\,\nu_1.$$

In like manner an arbitrary plane section of S_2 gives

$$0 = n_2^2 - n_2 - m_2 - 2\,\xi_2 - 3\,\nu_2$$

hence

$$0 = n_2\,(n_1^2 - n_1 - m_1 - 2\,\xi_1 - 3\,\nu_1) + n_1\,(n_2^2 - n_2 - m_2 - 2\,\xi_2 - 3\,\nu_2)$$

or

$$n_1 n_2 (n_1 + n_2 - 2) = m_1 n_2 + m_2 n_1 + 2\,(n_1 \xi_2 + n_2 \xi_1) + 3\,(n_1 \nu_2 + n_2 \nu_1) \ldots . (D')$$

combining the formulae (D^1) and (B) we get the formula (A).

If S_2 is a plane, n_2 becomes equal to unity and m_2 equal to nought, whilst the curve s becomes a plane section and the rank r of s passes into the class of the plane section. So formula (A) gives for that class

$$r = m_1 - 2\,\sigma - 3\,\chi,$$

which is indeed the class of a section of S_1 with a plane, having with S_1 in σ points an ordinary contact and in χ points a stationary contact.

5. If S_2 is of the second degree and S_1 of the degree n and of the class m, the formula (A) gives for the rank of the curve of intersection

$$r = 2\,(m + n) - 2\,\sigma - 3\,\chi.$$

If S_2 is a quadratic cone K^2 this formula will be proved directly once more as follows for the sake of verification.

The rank of the curve of intersection s is the number of its tangents meeting an arbitrary right line, e.g. a generator l of K^2. Each tangent of s, meeting the generator l has three points in common with the cone K^2, in fact the two consecutive points it has in common with s and its point of intersection with l, unless the latter coincides with the point of contact to s. Each right line having three points in common with K^2 lies entirely on K^2. The only tangents of s meeting l are thus the generating lines of K^2 which are at the same time tangents of s and the tangents to s at its points of intersection with l. The generator l of K^2 meets S_1 and therefore s too n times; through each of these points of intersection pass two consecutive tangents of s. Whence already $2n$ tangents of s meeting l.

Tangents of s, being at the same time generating lines of K^2, pass through the vertex T of K^2 and, being tangents of s, are also tangents of S_1, and therefore situated on the tangent cone K of S_1, having T for its vertex. Conversely every common generator of the two cones K^2 and K is a generator of K^2 having with S_1, thus also with s, two coinciding points in common. A right line having with s two coinciding points in common is either a tangent of s or it passes through a double point of s. So the common generators of the cones K^2 and K are either tangents of s or they pass through double points of s. The order of the tangent cone K, being equal to the class m of S_1, the number of common generators is $2m$. The number of tangents of s meeting l in the vertex T will be $2m$, diminished by a number still to be determined for the common generators passing through a double point of s.

If K^2 has in a point σ an ordinary contact with S_1 the common tangent plane π in σ is a tangent plane of S_1 passing through T. So π is also a tangent plane to the cone K along the line $T\sigma$. So the two cones K^2 and K have along the common generator $T\sigma$ a common tangent plane. The line $T\sigma$ must therefore count for two common generators of the cones K^2 and K. A point σ is a node of s and with the exception of very particular cases the two tangents of s in σ will not coincide with $T\sigma$. So for every point σ the number of tangents of s passing through T must be diminished by two.

The following example proves that for every point χ in which S_1 and K^2 have a stationary contact, the number of generators of K^2 touching s must be diminished by three. Let S_1 also be a quadratic surface and let the curve of intersection s be a not degenerated biquadratic curve R^4 with a cusp χ. Then the line $T\chi$ counts already at least for two common generators of the cones K^2 and K and is again not a tangent in χ to s or R^4. If now $T\chi$ were to count only for two common generators the cones K^2 and K would have two more generators in common. These latter two cannot be two consecutive generators, for in that case R^4 would have two double points and so it would have to break up. Now it is easy to see that these two remaining generators are tangents to R^4 or s at points for which the osculating plane is a stationary plane. So R^4 would have to possess two stationary planes α whilst a R^4 with cusp possesses but one stationary plane α [1]). The right line $T\chi$ must

[1]) E. PASCAL, loc. cit. p. 363.

therefore count for three common generators of K^2 and K. The number of tangents of s meeting the line l, thus the rank of s is consequently

$$r = 2n + 2m - 2\delta - 3\chi .$$

6. The reciprocal polar figure s' of the curve of intersection s of K^2 and S_1 is a developable circumscribed to a conic c^2 and to a surface S' of order m and of class n, whilst the conic c^2 touches δ times the surface S' and osculates it χ times. If we take for the conic c^2 the imaginary circle at infinity the developable s' becomes the developable focal surface of S'. The rank of s' is the same as that of s. So we find the theorem:

The rank of the focal developable of a surface of order m and of class n touching the imaginary circle at infinity δ times and osculating it χ times is

$$r = 2m + 2n - 2\delta - 3\chi.$$

If S_1 is a developable the point of contact of a common tangent plane that is an ordinary plane of S_1 is always a node of s[1]. The developables K^2 and S_1 will only then have a stationary contact in a point χ, when the common tangent plane is a stationary plane α of S_1. The line $T\chi$ counts thus for four lines of intersection of the cone K^2 with the tangent cone K which breaks up into m planes. It is easy to see that now the line $T\chi$ is at the same time tangent to s at the special cusp χ which is a singularity of order two, of rank unity and of class three[2]. So a stationary contact χ gives rise to four lines of intersection of K^2 with K of which only one is an ordinary tangent of s lying on K^2. Each point χ now also diminishes the rank of s by three. The reciprocal polar figure of S_1 is a curve S' of order m and of class n. Each common tangent plane of K^2 and S_1 is transformed in a common point of c^2 and S'. If the common plane is a stationary plane α of S_1 the common point is a cusp on the curve S'. So we find the theorem:

The rank of the focal developable of a plane curve or a twisted curve of the degree m and of the class n and of which δ ordinary points and χ cusps lie on the imaginary circle at infinity is

$$r = 2m + 2n - 2\delta - 3\chi.$$

7. If S'_1 and S'_2 are the reciprocal polar figures of the surfaces S_1 and S_2, then S'_1 and S'_2 are respectively of the degree m_1 and m_2 and of the class n_1 and n_2.

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1) Versluys, Mémoires de Liège, 3me série t. VI. loc. cit.

2) Halphen, Bull. de la Soc. Mat. de France, t. VI, p. 10.

If the surfaces S_1 and S_2 have an ordinary contact in σ points, the common tangent planes in these σ points are ordinary double tangent planes of the developable D circumscribing S_1 and S_2 [1]. The surfaces S'_1 and S'_2 will also have in σ points an ordinary contact.

If the surfaces S_1 and S_2 have in χ points a stationary contact the tangent planes in these χ points are stationary tangent planes of the developable D [2]. The surfaces S'_1 and S'_2 have thus also in χ points a stationary contact.

So the rank of the curve of intersection d' of the surfaces S'_1 and S'_2 is according to formula (A), just as the rank of the curve s,

$$r = m_1 n_2 + m_2 n_1 - 2\sigma - 3\chi.$$

The curve d' being the reciprocal polar figure of the circumscribing developable D, the rank of D is equal to the rank of d'. Whence the theorem:

For two arbitrary algebraic surfaces the rank of the curve of intersection is equal to the rank of the circumscribing developable.

Here we have supposed that the points of contact σ and χ are ordinary points on both surfaces and the tangent planes ordinary tangent planes in these points [3].

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1) VERSLUYS, Mém. de Liège. 3me séiie t. VI. De l'influence d'un contact etc.

2) VERSLUYS, loc. cit.

3) VERSLUYS, loc. cit.

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