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Mathematics. — *"The theory of* BRAVAIS *(on errors in space) for polydimensional space, with applications to Correlation."* By Prof. M. J. VAN UVEN. (Communicated by Prof. J. C. KAPTEYN).

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In the original treatise of BRAVAIS. "Analyse mathématique sur les probabilités des erreurs de situation d'un point" [1]) as well as in the articles that have afterwards appeared on this subject [2]) the problem of the distribution of errors in space has only been investigated for spaces of two and three dimensions. Only Prof. K. PEARSON has also treated the case of four-dimensional space [3]).

It may be interesting to treat this problem also for a space of an arbitrary number of dimensions, not so much with a view to the geometrical side of the problem, as in connection with the subject of correlation. If we consider the problem from this point of view, it comes to this:

A number (σ) of variables $u_1, u_2, \ldots u_\sigma$, are given, each of which follows GAUSS's exponential law:

$$\delta W_i = \frac{h_i}{\sqrt{\pi}} e^{-h_i^2 u_i^2} \delta u_i$$

and consequently may assume any value between $-\infty$ and $+\infty$.

Further we have a certain number (ϱ) of linear functions $x_1, x_2, \ldots x_\varrho$ of the variables u_i, viz.,

$$x_1 = a_{11}u_1 + a_{12}u_2 + \ldots + a_{1\sigma}u_\sigma,$$
$$x_2 = a_{21}u_1 + a_{22}u_2 + \ldots + a_{2\sigma}u_\sigma,$$
$$\vdots$$
$$x_\varrho = a_{\varrho 1}u_1 + a_{\varrho 2}u_2 + \ldots + a_{\varrho\sigma}u_\sigma.$$

The probability that x_j ranges between ξ_j and $\xi_j + \delta\xi_j$ ($j = 1, 2, \ldots \varrho$) is then expressed by the formula

1) A. BRAVAIS. "Anal. math. etc." Paris: Mémoires presentés par divers savants à l'Académie royale des sciences de l'Institut de France; T. 9 (1846), p. 255.

2) E CZUBER. Theorie der Beobachtungsfehler. Leipzig, 1891, Teubner; p. 350.

M. D'OCAGNE. Sur la composition des lois d'erreurs de situation d'un point; Comptes Rendus T. 118 (1894), p. 512, Bulletin de la Soc. math. de France, T. 23 (1895), p. 65; Annales de la Soc. scientif. de Bruxelles, T. 18 (1894) p. 86.

S. H. BURBURY. On the Law of Error in the case of correlated variations; Report of the British Assoc. (65th m.) (1895), p. 621.

V. REINA. Sulla probabilità degli errori di situazioni di un punto nello spazio; Atti della R. Accad. dei Lincei, serie 5a, T. 6, sem. 1 (1897), p. 107.

3) K. PEARSON. Mathematical contributions to the Theory of Evolution: Regression; Phil. Trans. vol. 187 (1895), p. 253.

$$W = \sqrt{\frac{E}{\pi^\rho}} \cdot e^{-H} \delta\xi_1, \delta\xi_2 \ldots \delta\xi_\rho,$$

in which

$$H = b_{11}\xi_1^2 + 2b_{12}\xi_1\xi_2 + \ldots + b_{\rho\rho}\xi_\rho^2.$$

The aim of this paper is:

1. to express the coefficients b_{jk} of the quadratic expression H and the quantity E in the coefficients α_{ji},

2. to elucidate the notion of a *coefficient of correlation* by means of the expressions found.

The probability of the simultaneous occurrence of the values $u_1, u_2, .. u_\sigma$ is

$$\delta W = \frac{\prod_1^\sigma h_i}{\pi^{\frac{\rho}{2}}} e^{-\sum_1^\sigma h_i^2 u_i^2} \prod_1^\sigma \delta u_i .$$

We begin by writing

$$h_i u_i = v_i \qquad (i = 1,2, \ldots \sigma)$$

and

$$\alpha_{ji} = h_i\, a_{ji} \qquad (j = 1,2, \ldots \varrho;\; i = 1,2, \ldots \sigma).$$

Thus we get

$$\delta W = \frac{1}{\pi^{\frac{\sigma}{2}}} e^{-\sum_1^\sigma v_i^2} \prod_1^\sigma \delta v_i$$

and

$$\begin{aligned} x_1 &= a_{11}v_1 + a_{12}v_2 + \ldots + a_{1\sigma}v_\sigma, \\ x_2 &= a_{21}v_1 + a_{22}v_2 + \ldots + a_{2\sigma}v_\sigma, \\ &\vdots \\ x_\rho &= a_{\rho 1}v_1 + a_{\rho 2}v_2 + \ldots + a_{\rho\sigma}v_\sigma. \end{aligned}$$

For the present we shall continue working with the coefficients α_{ji} only in the final result.

Like BRAVAIS we moreover introduce $\sigma - \varrho$ auxiliary variables, viz,

$$\begin{aligned} x_{\rho+1} &= \sum_1^\sigma a_{\rho+1,i}\, v_i \\ &\vdots \\ x_\sigma &= \sum_1^\sigma a_{\sigma i}\, v_i \end{aligned}$$

The determinant of substitution of $\begin{pmatrix} x_1, x_2, \ldots x_\sigma \\ v_1, v_2, \ldots v_\sigma \end{pmatrix}$ is then

$$\Delta = \begin{vmatrix} a_{11}, a_{12}, \dots a_{1\sigma} \\ a_{21}, a_{22}, \dots a_{2\sigma} \\ \vdots \quad \vdots \qquad \vdots \\ a_{\sigma 1}, a_{\sigma 2}, \dots a_{\sigma\sigma} \end{vmatrix} = |a_{ji}|.$$

The algebraic complement of a_{ji} we call A_{ji}.

By the substitution $\begin{pmatrix} x \\ v \end{pmatrix}$

$$H = \sum_1^\sigma v_i^2$$

becomes

$$H = \sum_1^\sigma b_{jj}\, x_j^2 + 2 \sum_1^\sigma b_{jk}\, x_j\, x_k.$$

The functions $x_1, \dots x_\rho$ are given. We now dispose of the remaining $(x_{\rho+1}, \dots x_\sigma)$ in such a manner that the following relations are satisfied:

$$b_{jk} = 0 \quad \text{for} \quad j = 1,2, \dots \sigma \quad ; \quad k = \rho + 1, \dots \sigma.$$

In this way we attain that the introduced $\sigma - \rho$ auxiliary variables occur only squared.

Solving v_i from the equations of substitution we find

$$v_i = \frac{\sum_{j=1}^{\sigma} A_{ji}\, x_j}{\Delta} \qquad (i = 1,2 \dots \sigma).$$

Consequently we find for H

$$H = \sum_1^\sigma v_i^2 = \frac{\sum_{i=1}^{\sigma} \left(\sum_{j=1}^{\sigma} A_{ji}\, x_j \right)^2}{\Delta^2} = \frac{\sum_{i=1}^{\sigma} (A_{1i}\, x_1 + A_{2i}\, x_2 + \dots + A_{\sigma i}\, x_\sigma)^2}{\Delta^2} =$$

$$= \frac{\sum_{i=1}^{\sigma} (A_{1i}^2\, x_1^2 + 2 A_{1i} A_{2i}\, x_1\, x_2 + \dots + A_{\sigma i}^2\, x_\sigma^2)}{\Delta^2} =$$

$$= \frac{\sum_{i=1}^{\sigma} A_{1i}^2 \,.\, x_1^2 + 2 \sum_{i=1}^{\sigma} A_{1i} A_{2i} \,.\, x_1\, x_2 + \dots + \sum_1^\sigma A_{\sigma i}^2 \,.\, x_\sigma^2}{\Delta^2} =$$

$$= \frac{\sum_{j=1}^{\sigma} \left(\sum_{j=1}^{\sigma} A_{ji}^2 \right) x_j^2 + 2 \sum_{j=1,k=1}^{\sigma} \left(\sum_{i=1}^{\sigma} A_{ji} A_{ki} \right) x_j\, x_k}{\Delta^2},$$

or putting

$$\frac{\sum_{i=1}^{\sigma} A_{ji}^2}{\Delta^2} = b_{jj} \quad ; \quad \frac{\sum_{i=1}^{\sigma} A_{ji} A_{ki}}{\Delta^2} = b_{jk} \, (= b_{kj})$$

$$H = \sum_{j=1}^{\sigma} b_{jj}\, x_j^2 + 2 \sum_{j=1,k=1}^{\sigma} b_{jk}\, x_j\, x_k.$$

We must now try to express the coefficients b_{jj} and b_{jk} for $j=1, 2, \dots \varrho$, $k=1, 2, \dots \varrho$, in terms of the coefficients of the given equations of substitution:

$$x_1 = \Sigma\, a_{1i} v_i, \dots, x_\rho = \Sigma\, a_{\rho i} v_i .$$

The conditions $b_{jh}=0$ for $h=\varrho+1, \dots \sigma$ are equivalent with the conditions

$$\sum_{i=1}^{\sigma} A_{ji} A_{hi} = 0 \quad \text{for} \quad h=\varrho+1, \dots \sigma;$$

but

$$\sum_{i=1}^{\sigma} A_{ji} a_{ji} = \Delta$$

$$\sum_{i=1}^{\sigma} A_{ji}\, a_{li} = 0 \quad \text{for} \quad l \neq j.$$

are also always satisfied.

So we have the following set of equations,

$$\Sigma A_{ji} a_{1i} = 0,\ \Sigma A_{ji} a_{2i} = 0, \dots \Sigma\, A_{ji} a_{j-1,i} = 0,\ \Sigma\, A_{ji} a_{ji} = \Delta,$$
$$\Sigma\, A_{ji} a_{j+1,i} = 0, \dots \Sigma\, A_{ji} a_{\rho i} = 0,\ \Sigma\, A_{ji} A_{ki} = \Delta^2\, b_{jk},$$
$$\Sigma\, A_{ji} A_{\rho+1,i} = 0, \dots, \Sigma\, A_{ji} A_{\sigma i} = 0.$$

Hence

$$\begin{vmatrix} a_{11}, & a_{12}, & \dots a_{1\sigma}, & 0 \\ a_{21}, & a_{22}, & \dots a_{2\sigma}, & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{j-1,1}, & a_{j-1,2}, & \dots a_{j-1,\sigma}, & 0 \\ a_{j1}, & a_{j2}, & \dots a_{j\sigma}, & \Delta \\ a_{j+1,1}, & a_{j+1,2}, & \dots a_{j+1,\sigma}, & 0 \\ \vdots & \vdots & \vdots & \\ a_{\rho 1}, & a_{\rho 2}, & \dots a_{\rho\sigma}, & 0 \\ A_{k1}, & A_{k2}, & \dots A_{k\sigma}, & \Delta^2 b_{jk} \\ A_{\rho+1,1}, & A_{\rho+1,2}, & \dots A_{\rho+1,\sigma}, & 0 \\ \vdots & \vdots & \vdots & \\ A_{\sigma 1}, & A_{\sigma 2}, & \dots A_{\sigma\sigma}, & 0 \end{vmatrix} = 0$$

or

$$\Delta b_{jk} \begin{vmatrix} a_{11}, & a_{12}, & \dots a_{1\sigma} \\ a_{21}, & a_{22}, & \dots a_{2\sigma} \\ \vdots & \vdots & \vdots \\ a_{\rho 1}, & a_{\rho 2}, & \dots a_{\rho\sigma} \\ A_{\rho+1,1}, & A_{\rho+1,2}, & \dots A_{\rho+1,\sigma} \\ \vdots & \vdots & \vdots \\ A_{\sigma 1}, & A_{\sigma 2}, & \dots A_{\sigma\sigma} \end{vmatrix} = (-1)^{\rho+j} \begin{vmatrix} a_{11}, & a_{12}, & \dots a_{1\sigma} \\ a_{21}, & a_{22}, & \dots a_{2\sigma} \\ \vdots & \vdots & \vdots \\ a_{j-1,1}, & a_{j-1,2}, & \dots a_{j-1,\sigma} \\ a_{j+1,1}, & a_{j+1,2}, & \dots a_{j+1,\sigma} \\ \vdots & \vdots & \vdots \\ a_{\rho 1}, & a_{\rho 2}, & \dots a_{\rho\sigma} \\ A_{k1}, & A_{k2}, & \dots A_{k\sigma} \\ A_{\rho+1,1}, & A_{\rho+1,2}, & \dots A_{\rho+1,\sigma} \\ \vdots & \vdots & \vdots \\ A_{\sigma 1}, & A_{\sigma 2}, & \dots A_{\sigma\sigma} \end{vmatrix}$$

or

$$\Delta b_{jk} \,.\, N = (-1)^{\rho+j}\, T_{jk}.$$

Now the following relation holds good:

$$N = \Sigma \begin{vmatrix} a_{1r_1}, & a_{1r_2}, & \dots a_{1r_\rho} \\ a_{2r_1}, & a_{2r_2}, & \dots a_{2r_\rho} \\ \vdots & \vdots & \vdots \\ a_{\rho r_1}, & a_{\rho r_2}, & \dots a_{\rho r_\rho} \end{vmatrix} \times \begin{vmatrix} A_{\rho+1,\, r_{\rho+1}}, & A_{\rho+1,\, r_{\rho+2}}, & \dots A_{\rho+1,\, r_\sigma} \\ A_{\rho+2,\, r_{\rho+1}}, & A_{\rho+2,\, r_{\rho+2}}, & \dots A_{\rho+2,\, r_\sigma} \\ \vdots & \vdots & \vdots \\ A_{\sigma r_{\rho+1}}, & A_{\sigma r_{\rho+2}}, & \dots A_{\sigma r_\sigma} \end{vmatrix}$$

in which $r_1, r_2, \dots r_\rho, r_{\rho+1}, \dots r_\sigma$ represents a permutation of numbers $1, 2, \dots \sigma$ and the summation must be extended over all these permutations.

As

$$\begin{vmatrix} A_{\rho+1,\, r_{\rho+1}}, & \dots A_{\rho+1,\, r_\sigma} \\ \vdots & \vdots \\ A_{\sigma r_{\rho+1}}, & \dots A_{\sigma r_\sigma} \end{vmatrix}$$

is the minor of the reciprocal determinant

$$\overline{\Delta} = \begin{vmatrix} A_{11}, & A_{12}, & \dots A_{1\sigma} \\ A_{21}, & A_{22}, & \dots A_{2\sigma} \\ \vdots & \vdots & \vdots \\ A_{\sigma 1}, & A_{\sigma 2}, & \dots A_{\sigma\sigma} \end{vmatrix}$$

which corresponds to the algebraic complement of

$$\begin{vmatrix} a_{1r_1}, & \dots a_{1r_\rho} \\ \vdots & \vdots \\ a_{\rho r_1}, & \dots a_{\rho r_\rho} \end{vmatrix},$$

we have the relation

$$\begin{vmatrix} A_{\rho+1,\, r_{\rho+1}}, & \dots A_{\rho+1, r_\sigma} \\ \vdots & \vdots \\ A_{\sigma r_{\rho+1}}, & \dots A_{\sigma r_\sigma} \end{vmatrix} = \Delta^{\sigma-\rho-1} \begin{vmatrix} a_{1r_1}, & \dots a_{1r_\rho} \\ \vdots & \vdots \\ a_{\rho r_1}, & \dots a_{\rho r_\rho} \end{vmatrix}.$$

Consequently we find for N

$$N = \Delta^{\sigma-\rho-1} \times \Sigma \begin{vmatrix} a_{1r_1}, & \dots a_{1r_\rho} \\ \vdots & \vdots \\ a_{\rho r_1}, & \dots a_{\rho r_\rho} \end{vmatrix}^2;$$

i.e. N is $\Delta^{\sigma-\rho-1}$ times the sum of the squares of all determinants of the ρ^{th} order of the matrix

$$M = \begin{vmatrix} a_{11}, & a_{12}, & \dots a_{1\sigma} \\ \vdots & & \vdots \\ a_{\rho 1}, & a_{\rho 2}, & \dots a_{\rho\sigma} \end{vmatrix}.$$

which is formed from the coefficients of the given equations of substitution.

If we represent such a determinant of the ρ^{th} order in general by D, we can write

$$N = \Delta^{\sigma-\rho-1} \Sigma D^2$$

The numerator T_{jk} is reduced as follows:

$$T_{jk} = \Sigma \begin{vmatrix} a_{1r_1} & ,\dots a_{1r_{\rho-1}} \\ \vdots & \\ a_{j-1,r_1}, & \dots a_{j-1,r_{\rho-1}} \\ a_{j+1,r_1}, & \dots a_{j+1,r_{\rho-1}} \\ \vdots & \vdots \\ a_{\rho r_1} & ,\dots a_{\rho r_{\rho-1}} \end{vmatrix} \times \begin{vmatrix} A_{kr_\rho}, & A_{kr_{\rho+1}}, & \dots A_{kr_\sigma} \\ A_{\rho+1,r_\rho}, & A_{\rho+1,r_{\rho+1}}, & \dots A_{\rho+1,r_\sigma} \\ \vdots & \vdots & \vdots \\ A_{\sigma r_\rho}, & A_{\sigma r_{\rho+1}}, & \dots A_{\sigma r_\sigma} \end{vmatrix}$$

The determinant

$$D_j = \begin{vmatrix} a_{1i} \\ \vdots \\ a_{j-1,i} \\ a_{j+1,i} \\ \vdots \\ a_{\rho i} \end{vmatrix} \quad (i = r_1, r_2, \dots r_{\rho-1})$$

belongs to the matrix

$$M_j = \begin{vmatrix} a_{11} & ,\dots a_{1\sigma} \\ \vdots & \vdots \\ a_{j-1,1}, & \dots a_{j-1,\sigma} \\ a_{j+1,1}, & \dots a_{j+1,\sigma} \\ \vdots & \vdots \\ a_{\rho 1} & ,\dots a_{\rho\sigma} \end{vmatrix},$$

which is obtained by omitting the row a_{ji} $(i = 1,2, \dots \sigma)$ in the matrix M.

Besides

$$\begin{vmatrix} A_{kr_\rho} & ,\dots A_{kr_\sigma} \\ A_{\rho+1,r_\rho}, & \dots A_{\rho+1,r_\sigma} \\ \vdots & \vdots \\ A_{\sigma r_\rho} & ,\dots A_{\sigma r_\sigma} \end{vmatrix}$$

is the minor of the reciprocal determinant $\overline{\Delta}$, which, apart from the sign, corresponds to the algebraic complement of

$$D_k = \begin{vmatrix} a_{1i} \\ \vdots \\ a_{k-1,i} \\ a_{k+1,i} \\ \vdots \\ a_{\rho i} \end{vmatrix} \quad (i = r_1, r_2 \dots r_{\rho-1})$$

Observing the sign, we have

$$\begin{vmatrix} A_{kr_\rho} & ,\dots A_{kr_\sigma} \\ A_{\rho+1,r_\rho}, & \dots A_{\rho+1,r_\sigma} \\ \vdots & \vdots \\ A_{\sigma r_\rho} & ,\dots A_{\sigma r_\rho} \end{vmatrix} = (-1)^{\rho+k} \Delta^{\rho-k} . D_k,$$

in which D_k is obtained by omitting the row a_{ki} in the matrix M. So we find

$$T_{jk} = (-1)^{\rho+k} \Delta^{\sigma-\rho} \Sigma D_j D_k,$$

in which the summation must be extended over all determinants of the $(\varrho-1)^{\text{th}}$ order, resp. of the matrices M_j and M_k, and this in such a way that the determinants D_j and D_k in the products are built up from *the same* columns of M.

The coefficients b_{jk} $(j=1, 2, \dots \varrho;\ k=1, 2, \dots \varrho)$ are finally found from

$$\Delta b_{jk} \,.\, \Delta^{\sigma-\rho-1}\, \Sigma D^2 = (-1)^{\rho+j}\, (-1)^{\rho+k}\, \Delta^{\sigma-\rho}\, \Sigma D_j D_k,$$

so that

$$b_{jk} = (-1)^{j+k} \frac{\Sigma D_j D_k}{\Sigma D^2} \quad \left(\begin{matrix} j \\ k \end{matrix} \right| = 1, 2, \ \dots \varrho \Big),$$

and in particular

$$b_{jj} = \frac{\Sigma D_j^{\,2}}{\Sigma D^2} \,.\quad (j = 1, 2, \dots \varrho)$$

The determinant of the coefficients b_{jk} $(j, k = 1, 2, \dots \sigma)$ runs

$$| b_{jk} | = \begin{vmatrix} b_{11}, & b_{12}, \dots b_{1\rho}, & 0 & , 0 & , \dots 0 \\ b_{21}, & b_{22}, \dots b_{2\rho}, & 0 & , 0 & , \dots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{\rho 1}, & b_{\rho 2}, \dots b_{\rho\rho}, & 0 & , 0 & , \dots 0 \\ 0 \ , & 0 \ , \dots 0 \ , & b_{\rho+1,\rho+1}, & 0 & , \dots 0 \\ 0 \ , & 0 \ , \dots 0 \ , & 0 & , b_{\rho+2,\rho+2}, & \dots 0 \\ \vdots & \vdots \quad \vdots & \vdots & \vdots & \vdots \\ 0 \ , & 0 \ , \dots 0 \ , & 0 & , 0 & , \dots b_{\sigma\sigma} \end{vmatrix},$$

or, if we write

$$\begin{vmatrix} b_{11}, & b_{12}, \dots b_{1\rho} \\ b_{21}, & b_{22}, \dots b_{2\rho} \\ \vdots & \vdots \quad \vdots \\ b_{\rho 1}, & b_{\rho 2}, \dots b_{\rho\rho} \end{vmatrix} = E$$

$$| b_{jk} | = E \times \prod_{h=\rho+1}^{\sigma} b_{hh};$$

E is the determinant of the quadratic expression H in $x_1, x_2, \dots x_\rho$

As the determinant quadratic expression in $v_1, v_2, \dots v_\sigma$ has the value 1, we have

$$| b_{jk} | = \frac{1}{\Delta^2},$$

hence

$$\Delta = \frac{1}{\sqrt{E \,.\, \prod\limits_{\rho+1}^{\sigma} b_{hh}}}.$$

Further we have

$$\prod_1^\sigma \delta x_j = \Delta \prod_1^\sigma \delta v_i ,$$

therefore

$$\delta W = \frac{1}{\pi^{\frac{\sigma}{2}}} e^{-\sum\limits_1^\sigma v_i^2} \prod_1^\sigma \delta v_i = \frac{1}{\pi^{\frac{\sigma}{2}}} \cdot \frac{e^{-\left(\sum\limits_1^\rho b_{jj} x_j^2 + 2\sum\limits_1^\rho b_{jk} x_j x_k\right)} \times e^{-\sum\limits_{\rho+1}^\sigma b_{hh} x_h^2}}{\Delta} \prod_1^\sigma \delta x_j =$$

$$= \sqrt{\frac{E}{\pi^\rho}} \cdot e^{-\left(\sum\limits_1^\rho b_{jj} x_j^2 + 2\sum\limits_1^\rho b_{jk} x_j x_k\right)} \cdot \prod_1^\rho \delta x_j \times \sqrt{\frac{\prod\limits_{\rho+1}^\sigma b_{hh}}{\pi^{\sigma-\rho}}} \cdot e^{-\sum\limits_{\rho+1}^\sigma b_{hh} x_h^2} \prod_{\rho+1}^\sigma \delta x_h .$$

In order to obtain the total probability W we must integrate over $x_{\rho+1}, \ldots x_\sigma$ from $-\infty$ to $+\infty$, and over $x_1, x_2, \ldots x_\rho$ resp. from $\xi_1, \xi_2, \ldots \xi_\rho$ to $\xi_1 + \delta\xi_1, \xi_2 + \delta\xi_2, \ldots \xi_\rho + \delta\xi_\rho$; i. e. the integration over $x_1, x_2, \ldots x_\rho$ consists in this, that in the integrant $x_1, x_2, \ldots x_\rho$ are replaced by $\xi_1, \xi_2, \ldots \xi_\rho$, while $\delta x_1, \delta x_2, \ldots \delta x_\rho$ are resp. replaced by $\delta\xi_1, \delta\xi_2, \ldots \delta\xi_\rho$.

So we find,

$$W = \sqrt{\frac{E}{\pi^\rho}} \cdot e^{-\left(\sum\limits_1^\rho b_{jj} \xi_j^2 + 2 \sum\limits_1^\rho b_{jk} \xi_j \xi_k\right)} \prod_1^\rho \delta \xi_j \times$$

$$\times \underset{x_h (h=\rho+1, \ldots \sigma)}{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}} \sqrt{\frac{\prod\limits_{\rho+1}^\sigma b_{hh}}{\pi^{\sigma-\rho}}} \cdot e^{-\sum\limits_{\rho+1}^\sigma b_{hh} x_h^2} \cdot \prod_{\rho+1}^\sigma dx_h =$$

$$= \sqrt{\frac{E}{\pi^\rho}} \cdot e^{-\left(\sum\limits_1^\rho b_{jj} \xi_j^2 + 2 \sum\limits_1^\rho b_{jk} \xi_j \xi_k\right)} \prod_1^\rho \delta \xi_j .$$

We have already calculated the coefficients b_{jk}, that is to say, expressed them in terms of the coefficients of the given equations of substitution. Their determinant E is consequently also known. For this latter, however, a simpler expression may be deduced. In order to find it we start from the relation

$$E = \frac{|b_{jk}|}{\prod\limits_{\rho+1}^\sigma b_{hh}} = \frac{1}{\Delta^2 \prod\limits_{\rho+1}^\sigma b_{hh}} .$$

Now $b_{hh}=\frac{1}{\Delta^2}\sum_{i=1}^{\sigma} A_{hi}^2$ and $b_{hk}=\frac{1}{\Delta^2}\sum_{i=1}^{\sigma} A_{hi}A_{ki}=0$ (for $h=\varrho+1,\ldots\sigma$).

Consequently we have

$$\prod_{\rho+1}^{\sigma} b_{hh}=\begin{vmatrix} b_{\rho+1,\rho+1}, & 0 & ,\ldots 0 \\ 0 & , b_{\rho+2,\rho+2}, & \ldots 0 \\ \vdots & \vdots & \\ 0 & , 0 & ,\ldots b_{\sigma\sigma} \end{vmatrix}=\begin{vmatrix} b_{\rho+1,\rho+1}, & b_{\rho+1,\rho+2}, & \ldots b_{\rho+1,\sigma} \\ b_{\rho+2,\rho+1}, & b_{\rho+2,\rho+2}, & \ldots b_{\rho+2,\sigma} \\ \vdots & \vdots & \vdots \\ b_{\sigma,\rho+1} & , b_{\sigma,\rho+2} & ,\ldots b_{\sigma\sigma} \end{vmatrix}=$$

$$=\frac{1}{\Delta^{2(\sigma-\rho)}}\begin{vmatrix} \Sigma A_{\rho+1,i}^2 & , \Sigma A_{\rho+1,i}\,\Sigma A_{\rho+2,i} & ,\ldots \Sigma A_{\rho+1,i}A_{\sigma i} \\ \Sigma A_{\rho+2,i}A_{\rho+1,i} & , \Sigma A_{\rho+2,i}^2 & ,\ldots \Sigma A_{\rho+2,i}A_{\sigma i} \\ \vdots & \vdots & \vdots \\ \Sigma A_{\sigma i}A_{\rho+1,i} & , \Sigma A_{\sigma i}A_{\rho+2,i} & ,\ldots \Sigma A_{\sigma i}^2 \end{vmatrix}=$$

$$=\frac{(-1)^{\sigma\rho}}{\Delta^{2(\sigma-\rho)}}\begin{vmatrix} A_{\rho+1,1}, & A_{\rho+1,2}, & \ldots A_{\rho+1,\sigma}, & 0 & , 0 & ,\ldots 0 \\ A_{\rho+2,1}, & A_{\rho+2,2}, & \ldots A_{\rho+2,\sigma}, & 0 & , 0 & ,\ldots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{\sigma 1} & , A_{\sigma 2} & ,\ldots A_{\sigma\sigma} & , 0 & , 0 & ,\ldots 0 \\ -1 & , 0 & ,\ldots 0 & , A_{\rho+1,1}, & A_{\rho+2,1}, & \ldots A_{\sigma 1} \\ 0 & ,-1 & ,\ldots 0 & , A_{\rho+1,2}, & A_{\rho+2,2}, & \ldots A_{\sigma 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & , 0 & ,\ldots -1 & , A_{\rho+1,\sigma}, & A_{\rho+2,\sigma}, & \ldots A_{\sigma\sigma} \end{vmatrix} \begin{matrix} ^{1)}\ R_1 \\ R_2 \\ \\ R_{\sigma-\rho} \\ R'_1 \\ R'_2 \\ \vdots \\ R'_\sigma \end{matrix}$$

or

$$\prod_{\rho+1}^{\sigma} b_{hh}=\frac{+1}{\Delta^{2(\sigma-\rho)}}\times\Sigma\begin{vmatrix} A_{\rho+1,r_{\rho+1}}, & A_{\rho+1,r_{\rho+2}}, & \ldots A_{\rho+1,r_\sigma} \\ A_{\rho+2,r_{\rho+1}}, & A_{\rho+2,r_{\rho+2}}, & \ldots A_{\rho+2,r_\sigma} \\ \vdots & \vdots & \\ A_{\sigma,r_{\rho+1}} & , A_{\sigma,r_{\rho+2}} & ,\ldots A_{\sigma,r_\sigma} \end{vmatrix}^2$$

But

$$\begin{vmatrix} A_{\rho+1,r_{\rho+1}}, & \ldots A_{\rho+1,r_\sigma} \\ \vdots & \\ A_{\sigma,r_{\rho+1}} & ,\ldots A_{\sigma,r_\sigma} \end{vmatrix}$$

is the minor of the reciprocal determinant $\overline{\Delta}$, which corresponds to the complementary minor of

$$D=\begin{vmatrix} a_{1r_1}, & a_{1r_2}, & \ldots a_{1r_\rho} \\ a_{2r_1}, & a_{2r_2}, & \ldots a_{2r_\rho} \\ \vdots & \vdots & \vdots \\ a_{\rho r_1}, & a_{\rho r_2}, & \ldots a_{\rho r_\rho} \end{vmatrix},$$

1) This reduction is easily controlled by first multiplying the rows $R'_1, R'_2, \ldots R'_\sigma$ resp. by $A_{\rho+1,1}, A_{\rho+1,2}, \ldots A_{\rho+1,\sigma}$ and adding all these products to R_1; then by multiplying the same rows resp. by $A_{\rho+2,1}, A_{\rho+2,2}, \ldots A_{\rho+2,\sigma}$ and adding these products to R_2, etc

if $r_1, r_2, \ldots r_\rho, r_{\rho+1}, \ldots r_\sigma$ represents a permutation of the numbers $1, 2, \ldots \sigma$. Apparently this last minor is again a determinant D of the ϱ^{th} order of the matrix M. Hence we have

$$\begin{vmatrix} A_{\rho+1,r_{\rho+1}}, \ldots A_{\rho+1,r_\sigma} \\ \vdots \qquad\qquad \vdots \\ A_{\sigma,r_{\rho+1}}, \ldots A_{\sigma,r_\sigma} \end{vmatrix} = \Delta^{\sigma-\rho-1} \times D$$

and

$$\prod_{\rho+1}^{\sigma} b_{hh} = \frac{1}{\Delta^{2(\sigma-\rho)}} \cdot \Sigma\,(\Delta^{\sigma-\rho-1} D)^2 = \frac{\Sigma D^2}{\Delta^2},$$

so that

$$E = \frac{1}{\Sigma D^2}.$$

So our result is:

$$W = \sqrt{\frac{E}{\pi^\rho}} \cdot e^{-\left(\sum_1^\rho b_{jj}\xi_j^2 + 2\sum_1^\rho b_{jk}\xi_j\xi_k\right)} \prod_1^\rho d\xi_j,$$

in which

$$b_{jk} = (-1)^{j+k} \frac{\Sigma D_j D_k}{\Sigma D^2} \qquad \left(\begin{matrix} j \\ k \end{matrix}\right\} = 1, 2, \ldots \varrho\Big)$$

and

$$E = \frac{1}{\Sigma D^2},$$

while D represents a determinant of the ϱ^{th} order of the matrix

$$M = \begin{vmatrix} a_{11}, & a_{12}, \ldots & a_{1\sigma} \\ a_{21}, & a_{22}, \ldots & a_{2\sigma} \\ \vdots & \vdots & \vdots \\ a_{\rho 1}, & a_{\rho 2}, \ldots & a_{\rho\sigma} \end{vmatrix}$$

and D_j a determinant of the $(\varrho-1)^{th}$ order of the matrix M_j, which is obtained by omitting the row $a_{j1}, a_{j2}, \ldots a_{j\sigma}$ in M.

Moreover the two determinants D_j and D_k, in the products are built up from the same columns of M.

Returning now to the coefficients α_{ji} we have only to write

$$a_{ji} = \frac{\alpha_{ji}}{h_i}.$$

Denoting by $\overline{D}, \overline{D}_j, \overline{D}_k$ the determinants in the coefficients α_{ji}, corresponding to D, D_j and D_k we have

$$D = \begin{vmatrix} a_{1r_1}, & a_{1r_2}, & \dots & a_{1r_\rho} \\ \vdots & & & \vdots \\ a_{\rho r_1}, & a_{\rho r_2}, & \dots & a_{\rho r_\rho} \end{vmatrix} = \frac{\overline{D}}{h_{r_1} h_{r_2} \dots h_{r_\rho}},$$

$$D_j = \begin{vmatrix} a_{1r_1}, & a_{1r_2}, & \dots & a_{1r_{\rho-1}} \\ \vdots & & & \\ a_{j-1,r_1}, & a_{j-1,r_2}, & \dots & a_{j-1,r_{\rho-1}} \\ a_{j+1,r_1}, & a_{j+1,r_2}, & \dots & a_{j+1,r_{\rho-1}} \\ \vdots & \vdots & & \vdots \\ a_{\rho r_1}, & a_{\rho r_2}, & \dots & a_{\rho r_{\rho-1}} \end{vmatrix} = \frac{\overline{D}_j}{h_{r_1} h_{r_2} \dots h_{r_{\rho-1}}}$$

$$D_k = \frac{\overline{D}_k}{h_{r_1} h_{r_2} \dots h_{r_{\rho-1}}},$$

or, if we introduce the mean errors ε_i by means of the formulae

$$h_i = \frac{1}{\varepsilon_i \sqrt{2}},$$

$$D = 2^{\frac{\rho}{2}} \varepsilon_{r_1} \varepsilon_{r_2} \dots \varepsilon_{r_\rho} \overline{D}\,;\; D_j = 2^{\frac{\rho-1}{2}} \varepsilon_{r_1} \varepsilon_{r_2} \dots \varepsilon_{r_{\rho-1}} \overline{D}_j\,;\; D_k = 2^{\frac{\rho-1}{2}} \varepsilon_{r_1} \varepsilon_{r_2} \dots \varepsilon_{r_{\rho-1}} \overline{D}_k.$$

Before applying these results to questions concerning correlation we shall first deduce simple expressions for the minors B_{jk} of the coefficients b_{jk} in the determinant E.

If we represent the minor of b_{jk} in the determinant

$$|b_{jk}| = \begin{vmatrix} b_{11}, & b_{12}, & \dots & b_{1\sigma} \\ \vdots & \vdots & & \vdots \\ b_{1\sigma}, & b_{2\sigma}, & \dots & b_{\sigma\sigma} \end{vmatrix} = \frac{1}{\Delta^2} = \prod_{\rho+1}^{\sigma} b_{hh} \times E = \frac{\prod_{\rho+1}^{\sigma} b_{hh}}{\Sigma D^2}$$

by β_{jk}, then for $j \leqq \varrho$, $k \leqq \varrho$:

$$B_{jk} = \frac{\beta_{jk}}{\prod_{\rho+1}^{\rho} b_{hh}} = \frac{\Delta^2}{\Sigma D^2} \cdot \beta_{jk}.$$

Now

$$\beta_{jk} = \frac{(-1)^{j+k}}{\Delta^{2(\sigma-1)}} \begin{vmatrix} \Sigma A_{1i}^2, & \Sigma A_{1i} A_{2i} & \dots \Sigma A_{1i} A_{j-1,i}, & \Sigma A_{1i} A_{j+1,i}, & \dots \Sigma A_{1i} A_{\sigma i} \\ \Sigma A_{2i} A_{1i}, & \Sigma A_{2i}^2, & \dots \Sigma A_{2i} A_{j-1,i}, & \Sigma A_{2i} A_{j+1,i}, & \dots \Sigma A_{2i} A_{\sigma i} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Sigma A_{k-1,i} A_{1i}, & \Sigma A_{k-1,i} A_{2i}, & \dots \Sigma A_{k-1,i} A_{j-1,i}, & \Sigma A_{k-1,i} A_{j+1,i}, & \dots \Sigma A_{k-1,i} A_{\sigma i} \\ \Sigma A_{k+1,i} A_{1i}, & \Sigma A_{k+1,i} A_{2i}, & \dots \Sigma A_{k+1,i} A_{j-1,i}, & \Sigma A_{k+1,i} A_{j+1,i}, & \dots \Sigma A_{k+1,i} A_{\sigma i} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Sigma A_{\sigma i} A_{1i}, & \Sigma A_{\sigma i} A_{2i}, & \dots \Sigma A_{\sigma i} A_{j-1,i}, & \Sigma A_{\sigma i} A_{j+1,i}, & \dots \Sigma A_{\sigma i}^2 \end{vmatrix}$$

$$
= \frac{+1}{\Delta^{2(\sigma-1)}} \begin{vmatrix} A_{11} & , A_{12} & ,\ldots A_{1\sigma} & , 0 & , 0 & ,\ldots 0 & , 0 & ,\ldots 0 \\ A_{21} & , A_{22} & ,\ldots A_{2\sigma} & , 0 & , 0 & ,\ldots 0 & , 0 & ,\ldots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{k-1,1} & , A_{k-1,2} & ,\ldots A_{k-1,\sigma} & , 0 & , 0 & ,\ldots 0 & , 0 & ,\ldots 0 \\ A_{k+1,1} & , A_{k+1,2} & ,\ldots A_{k+1,\sigma} & , 0 & , 0 & ,\ldots 0 & , 0 & ,\ldots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{\sigma 1} & , A_{\sigma 2} & ,\ldots A_{\sigma\sigma} & , 0 & , 0 & ,\ldots 0 & , 0 & ,\ldots 0 \\ -1 & , 0 & ,\ldots 0 & , A_{11}, & A_{21}, & \ldots A_{j-1,1} & , A_{j+1,1} & ,\ldots A_{\sigma 1} \\ 0 & , -1 & ,\ldots 0 & , A_{12}, & A_{22}, & \ldots A_{j-1,2} & , A_{j+1,2} & ,\ldots A_{\sigma 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & , 0 & ,\ldots -1 & , A_{1\sigma}, & A_{2\sigma}, & \ldots A_{j-1,\sigma} & , A_{j+1,\sigma} & ,\ldots A_{\sigma\sigma} \end{vmatrix} =
$$

$$
= \frac{(-1)^{j+k}}{\Delta^{2(\sigma-1)}} \sum_{l=1}^{\sigma} \begin{vmatrix} A_{11} & , A_{12} & , A_{1,l-1} & , A_{1,l+1} & ,\ldots A_{1\sigma} \\ A_{21} & , A_{22} & , A_{2,l-1} & , A_{2,l+1} & ,\ldots A_{2\sigma} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{k-1,1} & , A_{k-1,2} & , A_{k-1,l-1} & , A_{k-1,l+1} & ,\ldots A_{k-1,\sigma} \\ A_{k+1,1} & , A_{k+1,2} & , A_{k+1,l-1} & , A_{k+1,l+1} & ,\ldots A_{k+1,\sigma} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{\sigma 1} & , A_{\sigma 2} & , A_{\sigma,l-1} & , A_{\sigma,l+1} & ,\ldots A_{\sigma\sigma} \end{vmatrix} \times
$$

$$
\times \begin{vmatrix} A_{11} & , A_{21} & ,\ldots A_{j-1,1} & , A_{j+1,1} & ,\ldots A_{\sigma 1} \\ A_{12} & , A_{22} & ,\ldots A_{j-1,2} & , A_{j+1,2} & ,\ldots A_{\sigma 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{1,l-1} & , A_{2,l-1} & ,\ldots A_{j-1,l-1} & , A_{j+1,l-1} & ,\ldots A_{\sigma,l-1} \\ A_{1,l+1} & , A_{2,l+1} & ,\ldots A_{j-1,l+1} & , A_{j+1,l+1} & ,\ldots A_{\sigma,l+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{1\sigma} & , A_{2\sigma} & ,\ldots A_{j-1,\sigma} & , A_{j+1,\sigma} & ,\ldots A_{\sigma\sigma} \end{vmatrix} =
$$

$$
= \frac{+1}{\Delta^{2(\sigma-1)}} \cdot \Delta^{2(\sigma-2)} \sum_{l=1}^{\sigma} a_{jl} a_{kl} = \frac{+\sum_{l=1}^{\sigma} a_{jl} a_{kl}}{\Delta^2} .
$$

Consequently we find

$$
B_{jk} = \frac{\Delta^2}{\Sigma D^2} \beta_{jk} = \frac{+\sum_{l=1}^{\sigma} a_{jl} a_{kl}}{\Sigma D^2} ,
$$

and in particular

$$
B_{jj} = \frac{\sum_{l=1}^{\sigma} a_{jl}^2}{\Sigma D^2} .
$$

(*To be continued*).