determined by JOLIBOIS are indicated, who, as also follows from the table, has only been able to continue his research up to 312°.

In a subsequent communication we shall give some theoretical considerations in connection with the results stated here, and also discuss the vapour pressure line of the solid modification, which we determined accurately already some time ago.

Anorg. Chem. Laboratory of the University.

Amsterdam, March 27, 1914.

Mathematics. — *"A bilinear congruence of rational twisted quartics."* By Professor JAN DE VRIES.

(Communicated in the meeting of March 28, 1914).

1. The base-curves of the pencils of cubic surfaces contained in a net $[\Phi^3]$ form a bilinear congruence. [1])

If all the surfaces of the net have a twisted curve ϱ^5 of genus *one* in common, and moreover pass through two fixed points H_1, H_2, every two Φ^3 cut each other moreover along a rational curve ϱ^4, which rests on ϱ^5 in 10 points. [2])

A third Φ^3 cuts ϱ^4 in 12 points, of which 10 lie on ϱ^5; the remaining 2 are H_1 and H_2. Through an arbitrary point P passes *one* ϱ^4; if P is chosen on a trisecant t of ϱ^5, then all Φ^3 passing through P contain the line t, and ϱ^4 is replaced by the figure composed of t and a τ^3, which cuts it, and meets ϱ^5 in 7 points.

2. In order to determine the order of the ruled surface of the trisecants t, we observe that each point of ϱ^5 bears two trisecants, so that ϱ^5 is nodal curve of the ruled surface (t). We can now prove that a bisecant b, outside ϱ^5, cuts only *one* trisecant, from which it ensues that (t) must be of order five.

The bisecants b, which rest on the bisecant b_0, form a ruled surface (b) of order 7, on which b_0 is a quadruple line. In a plane passing through b_0 lie three bisecants; as to each of those three lines the point of intersection of the other two may be associated, by which a correspondence (1,1) is brought about between the lines b and the points of ϱ^5, (b) is of genus *one*. A plane section of (b) has

[1]). See my communication in these *Proceedings*, volume XVI, p. 733. There I have considered the case that all Φ^3 have in common a twisted curve ϱ^5 of genus *two*, so that a bilinear congruence of *elliptic* quartics is formed.

[2]). See e.g STURM, *Synthetische Untersuchungen uber Flachen dritter Ordnung* (p p 215 and 233).

therefore **14** nodes. Of these 5 lie on ϱ^5, 6 in the quadruple point lying on b_0; the remaining 3 are represented by a triple point originating from a trisecant resting on b_0. As b_0 in each of its points of intersection with ϱ^5 meets two trisecants, (t) is consequently a ruled surface of order *five.* [1])

3. A ϱ^4 cutting ϱ^5 in S forms with it the base of a pencil (Φ^3) the surfaces of which touch in S. We shall now consider two pencils (Ψ) and (Ω) in the net $[\Phi^3]$, and associate to each surface Ψ^3 the surface Ω^3, by which it is touched in S. The pencils having become projective in consequence, produce a figure of order 6, which is composed of the surface Φ^3 common to both pencils and a surface Σ^3. On a line l passing through S a correspondence (2,2) is determined by (Ψ^3) and (Ω), one of the coincidences lies in S, because l is touched in S by two corresponding surfaces. The remaining three are intersections of l with the figure of order 6, mentioned above; the latter has consequently a triple point in S, from which it ensues that S is a node of Σ^3. The curves ϱ^4, which meet ϱ^5 in S, form therefore a cubic surface passing through ϱ^5, which possesses a node in S; ϱ^5 is therefore a *singular curve of order three* for the congruence $[\varrho^4]$.

Through S pass 6 lines of Σ^3, to them belong the two trisecants t, meeting in S; the remaining four are *singular bisecants* of the congruence. Such a line p is cut by ∞^1 curves ϱ^4 in two points, of which one coincides with S, (singular bisecant of the *first kind*).

The ∞^2 rays h, which may be drawn through the *cardinal points* H_1, H_2 possess the same property.

4. An arbitrary line r passing through a point P is cut by *one* ϱ^4 in a pair of points R, R'; the locus of those points is a surface Π of order 5 with triple point P.

If P lies on ϱ^5, then Π^5 consists of the surface Σ^3 belonging to $S \equiv P$ and a quadratic cone, of which the generatrices are *singular bisecants* q. Each line q is bisecant of ∞^1 curves of the $[\varrho^4]$.

If, on the other hand, q is bisecant of a ϱ^4 and at the same time secant of ϱ^5, then the cubic surface passing through ϱ^4, ϱ^5 and q belongs to $[\Phi^3]$, consequently q is cut by the surfaces of this net in the pairs of points of an I^2, is therefore bisecant of ∞^1 curves ϱ^4 (singular bisecant of the *second kind*).

The lines q meeting in a point P, belong to the common gene-

[1]) Other properties of the ϱ^5 of genus 1 are to be found in my communication "On twisted quintics of genus unity" (volume II, p. 374 of these *Proceedings*).

rators of two cones, which have as curves of direction the ϱ^4 passing through P, and the singular curve ϱ^5. These cones pass through the 10 intersections of ϱ^4 and ϱ^5; of the 15 common generators 5 are lying in lines q. As a plane contains 5 points S, consequently 10 lines q, the *singular bisecants of the second kind* form a congruence (5, 10), which has ϱ^5 as singular curve of the second order.

The cubic cone k^3, which projects a ϱ^4 out of one of its points P, has a nodal line in the trisecant u of ϱ^4, which trisecant passes through P. The latter is at the same time nodal line of the surface Π^5. To the section of Π^5 and k^3 belongs in the first place the curve ϱ^4; further the singular bisecants h_1, h_2, which connect H_1 and H_2 with P, while u represents four common lines, the rest of the section consists of the 5 lines q, which meet in P.

As u with ϱ^4 and ϱ^5 determines a Φ^3, it is cut by the net $[\Phi^3]$ in the triplets of an involution I^3, and is therefore *singular trisecant* of the congruence (common bisecant of ∞^1 curves ϱ^4).

5. Let us now consider the *quadruple involution* (Q^4) in a plane φ, which is determined by the congruence $[\varrho^4]$. It has *five singular points of the third order* in the five intersections S_k of the singular curve ϱ^5. The monoid $\Sigma^3{}_k$ cuts φ along the nodal curve $\sigma^3{}_k$, the points of which are arranged in the triplets of an I^3, which form with S_k quadruples of (Q^4); σ_k also contains the remaining points S (§ 3).

If the point Q describes a line l, the remaining three points Q' of its quadruple describe a curve λ, which passes three times through each of the points S_k. The curves λ and λ^* belonging to l and l^*, have, besides the 45 intersections lying in the points S_k, the three points in common, which form a quadruple with l^*; moreover as many pairs of points as the order of λ indicates. For, if l^* is cut by λ in L^*, then l contains a point L of the quadruple determined by L^*, and the remaining two points belonging to it are intersections of λ and λ^*. The order x of those curves is consequently found from $x^2 = 2x + 48$; hence $x = 8$.

The coincidences of the I^3 on the singular curve $\sigma_1{}^3$ are at the same time coincidences of the (Q^4). Each point S produces two coincidences, the locus γ of the coincidences has therefore in S 6 points in common with $\sigma_1{}^3$; further two in each of the remaining 4 points S and 4 in the coincidences of the I^3. From this it ensues, that the *curve of coincidences* γ is of order six.

(Q^4) consists of the quadruples of base-points of the pencils of cubic curves belonging to a net with the *fixed* base-points S_k. Each point of γ^6 is node of a curve belonging to the net.

6. The transformation (Q,Q') changes a conic into a curve of order 16, with sextuple points in S_k. For the conic τ^2 passing through the five points S, this figure degenerates into the five curves σ^3 and a line u, which contains the triplets of points Q', corresponding to the points Q of τ^2; consequently u is a singular trisecant of $[\varrho^4]$. On the other hand a bisecant u lying in φ is transformed into a figure of order 8, to which u itself belongs twice; as the completing figure must be counted three times and must contain the points S_k, it is the conic τ^2. Consequently φ bears only *one* line u, and the *singular trisecants* of $[\varrho^4]$ form *a congruence* (1,1).

The surface of trisecants of ϱ^5 cuts φ in a curve τ^5 with 5 nodes in S_k. With u, τ^5 has five points T_k in common; each of these points determines a quadruple (Q^4), of which *one* point lies on τ^2, while the remaining two are situated on u. By means of the transformation (Q,Q') τ^5 is therefore changed into a curve of order 10, τ^{10}. The latter is apparently the intersection of φ with the surface formed by the twisted cubics τ^3, which with the trisecants t are associated into degenerate curves of $[\varrho^4]$.

With σ_1^3, τ^5 has, apart from the singular points S, three points in common; for in S_1 lie 4 intersections and in each of the remaining S, two; therefore S_1 is a triple point on the curve τ^{10}.

The curves τ^3 form therefore a *surface of order ten* with threefold curve ϱ^5.

Of the points of intersection of τ^3 with γ^6, $5 \times 2 \times 2 = 20$ lie in the points S; in each of the remaining 10, a trisecant t is cut by the corresponding cubic curve τ^3. From this it ensues that the locus of the points (t, τ^3) is a *twisted curve of order ten.*

7. The pairs of points Q', Q', which are collinear with a point P, lie (§ 4) on a curve π^5, which passes through the points S_k. If Q describes the line l, QQ' will envelop a curve of class 5. The points Q' describe then (§ 5) a curve λ^8, which passes three times through the points S, consequently has still 25 points in common with π^5; 5 of them connect a point Q' of λ^8 with a point Q of l; the rest form 10 pairs Q', Q''; so that $Q'Q''$ passes through P. From this it ensues that the triplets of the involution $(Q')^3$ lying on λ^8 form triangles which are circumscribed to a curve (*curve of involution*) of class *ten*, $(q)_{10}$.

For a point S_k π^5 degenerates into the curve σ_k^3 and *two singular lines* s_k and s_k^* (§ 4); such a line bears an involution I^2 of pairs Q,Q'. A pair is formed by S_k and the intersection of s_k with u;

as the remaining two points [1]) of the quadruple lie on u, the pairs Q'',Q''' which complete the pairs Q,Q'' into groups of (Q^4), will lie on a *conic* σ_k^3. As s_k with the curve σ_k^3, apart from S_k, has two points in common, σ_k^2 passes through the four points S_l. In the transformation (Q,Q') s_k corresponds to the figure of order 8, which is composed of s_k itself, σ_k^3 and σ_k^2 counted twice; this figure, as it ought to do, passes three times through the points S.

Every singular line s_k is bitangent of the curve of involution $(q)_{10}$, mentioned above, for it bears two pairs Q',Q'', for which the point Q is intersection of σ_k^2 with l. The singular line u is septuple-tangent of $(q)_{10}$, for first l cuts the conic τ^2 in two points, which each determine a triplet of the I^3 lying on u, on account of which u is six times characterized as tangent; but u contains moreover the pair of points Q',Q'' indicated by the intersection Q of u with l.

The curves $(q)_{10}$ and $(q)_{10}$* belonging to l and l^* have therefore in common the line u, which represents 49 common tangents and the 10 lines s, which each represent four of those tangents; the remaining 11 we find in the 3 lines indicated by the point ll^* and the 8 which are determined by the intersections of l^* with λ^8 (cf. § 5).

The curves σ_1^3 and σ_2^3 have the points S_3,S_4,S_5 in common and intersect twice in S_1 and S_2, the remaining two intersections V_{12} and V_{12}' form with S_1 and S_2 a quadruple. From this it ensues that through each two points of ϱ^5 passes only *one* curve of $[\varrho^4]$.

The triangles of involution $Q'Q''Q'''$ described in σ_1^3 envelop a curve of class four (for S_1 belongs to two of those triangles); this curve of involution has u as threefold tangent, for u bears a triplet of points forming with S_1 a group of the (Q^4). So u represents nine common tangents of the curves of involution belonging to S_1 and S_2; the line $V_{12}V_{12}'$ is also a common tangent; the remaining six are apparently singular lines s and form three pairs, which respectively pass through S_3,S_4,S_5.

The singular line s_k^* is cut by the conic σ^2_k in two points, which form a quadruple with two points of s_k; so they lie on σ^{*2}_k. Consequently s_k and s_k^* are opposite sides of *one* quadrangle of involution, which has S_1 as adjacent vertex. The two coincidences of the (Q^4) lying in S_1 also determine quadruples, for which S_1 is adjacent vertex. It is easy to see that there are no other quadruples of which two opposite sides intersect in S_1. From this it is evident that an arbitrary point is *adjacent vertex* of *three quadrangles.*

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[1]). One of those points lies on s_k* and forms with S_k a pair of the I^2 lying on that line.

8. Let (φ^1) be a pencil belonging to the net $[\varphi^3]$, which is produced by the intersection of the net $[\Phi^3]$ with the plane φ. The locus of the points which have the same polar line with regard to a curve γ^p and the curves of a pencil (φ^n), is a curve ψ of order $2n+p-3$ [1]), hence a curve of order 9, if for γ^p the curve of coincidences γ^6 is taken. In the points S_k ψ^9 like γ^6, has nodes and there the same tangents as γ^6, so the two curves have 30 points in common in S_k Further both of them pass through the 12 nodes of the pencil $[\varphi^3]$. In each of the remaining 12 common points D, γ^6 is touched by ψ^9, which means that there the curves of a pencil belonging to $[\varphi^3]$ have three-point contact. In (Q^4) occur therefore *twelve* groups, in which every time *three* points have coincided.

In each of the 12 points D, γ^6 is touched by the *complementary curve* γ^{12}, into which γ^6 is transformed by (Q,Q'); the latter is the locus of the pairs of points which complete the coincidences of (Q^4) into quadruples. The figure of order 48, into which γ^6 is transformed, consists of γ^6 itself, of the 5 curves σ_k^3, each counted twice, and the complementary curve; the latter is consequently indeed of order 12. With τ^2 it has four points in common, arising from the 4 coincidences of the I^3 lying on u; the remaining 20 lie in the points S_k. From this it ensues that γ^{12} has *quadruple* points in the 5 singular points S.

In S_k, γ^{12}, and γ^6 have therefore $5 \times 4 \times 2 = 40$ points in common, they further touch in the 12 points D. The remaining 8 intersections arise from quadruples of which twice two points have coincided; so (Q^4) contains *four* groups, which consist each of *two* coincidences.

Mathematics. — "*On* HERMITE'*s functions.*" By Prof. W. KAPTEYN.

(Communicated in the meetings of March 28 and April 24, 1913).

1. The n^{th} derivative of e^{-x^2} may be put in this form, first given by HERMITE

$$\frac{d^n}{dx^n}(e^{-x^2}) = (-1)^n e^{-x^2} H_n(x)$$

where

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1\,!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\,!}(2x)^{n-4} - \quad . \quad (1)$$

These polynomia satisfy the following relations [2])

[1]) See e.g. CREMONA-CURTZE, *Einleitung in eine geometrische Theorie der ebenen Curven*, p. 121.

[2]) Exerc. de Tisserand, 1877, p. 26, 27 and 140.