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8. Let (φ^1) be a pencil belonging to the net $[\varphi^3]$, which is produced by the intersection of the net $[\Phi^3]$ with the plane φ. The locus of the points which have the same polar line with regard to a curve γ^p and the curves of a pencil (φ^n), is a curve ψ of order $2n+p-3$ [1]), hence a curve of order 9, if for γ^p the curve of coincidences γ^6 is taken. In the points S_k ψ^9 like γ^6, has nodes and there the same tangents as γ^6, so the two curves have 30 points in common in S_k. Further both of them pass through the 12 nodes of the pencil $[\varphi^3]$. In each of the remaining 12 common points D, γ^6 is touched by ψ^9, which means that there the curves of a pencil belonging to $[\varphi^3]$ have three-point contact. In (Q^4) occur therefore *twelve* groups, in which every time *three* points have coincided.

In each of the 12 points D, γ^6 is touched by the *complementary curve* γ^{12}, into which γ^6 is transformed by (Q,Q'); the latter is the locus of the pairs of points which complete the coincidences of (Q^4) into quadruples. The figure of order 48, into which γ^6 is transformed, consists of γ^6 itself, of the 5 curves σ_k^3, each counted twice, and the complementary curve; the latter is consequently indeed of order 12. With τ^2 it has four points in common, arising from the 4 coincidences of the I^3 lying on u; the remaining 20 lie in the points S_k. From this it ensues that γ^{12} has *quadruple* points in the 5 singular points S.

In S_k, γ^{12}, and γ^6 have therefore $5 \times 4 \times 2 = 40$ points in common, they further touch in the 12 points D. The remaining 8 intersections arise from quadruples of which twice two points have coincided; so (Q^4) contains *four* groups, which consist each of *two* coincidences.

Mathematics. — "*On* HERMITE's *functions.*" By Prof. W. KAPTEYN.

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1. The n^{th} derivative of e^{-x^2} may be put in this form, first given by HERMITE

$$\frac{d^n}{dx^n}(e^{-x^2}) = (-1)^n e^{-x^2} H_n(x)$$

where

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \ldots \quad (1)$$

These polynomia satisfy the following relations [2])

[1]) See e.g. CREMONA-CURTZE, *Einleitung in eine geometrische Theorie der ebenen Curven*, p. 121.

[2]) Exerc. de Tisserand, 1877, p. 26, 27 and 140.

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{d^2 H_n}{dx^2} - 2x \frac{dH_n}{dx} + 2n H_n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{dH_n}{dx} - 2n H_{n-1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$H_n - 2x H_{n-1} + 2(n-1) H_{n-2} = 0 \quad . \quad . \quad . \quad (5)$$

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = 0 \qquad m \neq n \quad . \quad . \quad . \quad (6)$$

$$\int_{-\infty}^{\infty} H_n^2(x) e^{-x^2} dx = 2^n . n! \sqrt{\pi} \quad . \quad . \quad . \quad . \quad (7)$$

The object of this paper is to examine these polynomia and the series connected with these, which also satisfy the differential equation (3).

2. To integrate the differential equation (3) by means of definite integrals, put

$$H_n = e^{x^2} z$$

then we have

$$\frac{d^2 z}{dx^2} + 2x \frac{dz}{dx} + 2(n+1) z = 0$$

To solve this, we assume

$$z = \int_P^Q e^{-xt} \, T \, dt$$

where T is a function of t, and P and Q are constants. The result of this substitution is

$$2 (t \, T \, e^{-xt})_P^Q + \int_P^Q e^{-xt} \left[-2t \frac{dT}{dt} + (t^2 + 2n) T \right] dt = 0 .$$

Now this equation will be satisfied, if we make

$$T = t^n e^{\frac{t^2}{4}}$$

and

$$P = 0 \qquad Q = \pm i \infty .$$

Hence the general integral is

$$z = c_1 \int_0^{i\infty} e^{-xt + \frac{t^2}{4}} t^n \, dt + c_2 \int_0^{-i\infty} e^{-xt + \frac{t^2}{4}} t^n \, dt$$

c_2 and c_1 being arbitrary constants.

Putting

$$t = iu$$

this takes the form

$$z = \int_0^\infty e^{-\frac{u^2}{4}} u^n (A_n \cos xu + B_n \sin xu)\, du.$$

A_n and B_n being again arbitrary constants.

The general integral of

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

therefore may be written

$$y = e^{x^2} \int_0^\infty e^{-\frac{u^2}{4}} u^n (A_n \cos xu + B_n \sin xu)\, du.$$

Choosing

$$A_n = \frac{(-1)^n}{\sqrt{\pi}} \cos\frac{n\pi}{2} \qquad B_n = \frac{(-1)^{n+1}}{\sqrt{\pi}} \sin\frac{n\pi}{2}$$

we get the particular integral

$$y = \frac{1}{\sqrt{\pi}} e^{x^2} \int_0^\infty e^{-\frac{u^2}{4}} u^n \cos\left(xu - \frac{n\pi}{2}\right) du$$

which for $x = 0$, reduces to

$$y_{x=0} = \frac{\cos\frac{n\pi}{2}}{\sqrt{\pi}} \int_0^\infty e^{-\frac{u^2}{4}} u^n du$$

where

$$\int_0^\infty e^{-\frac{u^2}{4}} u^n du = \begin{cases} \dfrac{n!}{\frac{n}{2}!} \sqrt{\pi} \ (n \text{ even}) \\ \dfrac{1}{2}\left(\dfrac{n-1}{2}\right)! \ (n \text{ odd}) \end{cases}$$

Now we know that

$$H_n(0) = \begin{cases} (-1)^{\frac{n}{2}} \dfrac{n!}{\frac{n}{2}!} \ (n \text{ even}) \\ 0 \quad (n \text{ odd}) \end{cases} \qquad \text{. . . . . . (8)}$$

therefore this particular integral is $H_n(x)$ and we have

$$H_n(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{u^2}{4}} u^n \cos\left(xu - \frac{n\pi}{2}\right) du \quad . \quad . \quad . \quad (9)$$

Choosing again

$$A_n = \frac{(-1)^n}{\sqrt{\pi}} \sin\frac{n\pi}{2} \qquad B_n = \frac{(-1)^n}{\sqrt{\pi}} \cos\frac{n\pi}{2}$$

the second particular integral may be written

$$L_n(x) = \frac{1}{\sqrt{\pi}} e^{x^2} \int_0^\infty e^{-\frac{u^2}{4}} u^n \sin\left(xu - \frac{n\pi}{2}\right) du \quad . \quad . \quad (10)$$

3. This second integral satisfies also the relations (4) and (5). For, differentiating, we have

$$L'_n(x) = 2xL_n(x) + \frac{1}{\sqrt{\pi}} e^{x^2} \int_0^\infty e^{-\frac{u^2}{4}} u^{n+1} \cos\left(xu - \frac{n\pi}{2}\right) du$$

$$= 2xL_n(x) - \frac{1}{\sqrt{\pi}} e^{x^2} \int_0^\infty e^{-\frac{u^2}{4}} u^{n+1} \sin\left(xu - \frac{(n+1)\pi}{2}\right) du$$

or

$$L'_n(x) = 2xL_n(x) - L_{n+1}(x) \, . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Differentiating again, and remarking that $L_n(x)$ satisfies the differential equation (3), we find

$$(2n+2)L_n - 2x L_{n+1} + L_{n+2} = 0$$

or, changing n in $n-2$

$$L_n - 2x L_{n-1} + 2(n-1) L_{n-2} = 0 \, . \quad . \quad . \quad . \quad . \quad (12)$$

which is in accordance with (5).

If now we substitute the value

$$L_{n+1} = 2x L_n - 2n L_{n-1}$$

from (12) in (11), we get

$$L_n' = 2n L_{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which is in accordance with (4).

4. The function $L_n(x)$ may be expanded in series of ascending powers of x.

If n is even, we have

$$L_n(x) = \frac{(-1)^{\frac{n}{2}}}{\sqrt{\pi}} e^{x^2} \int_0^\infty e^{-\frac{u^2}{4}} u^n \sin xu \, du$$

$$= \frac{(-1)^{\frac{n}{2}}}{\sqrt{\pi}} e^{x^2} \sum_0^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \int_0^\infty e^{-\frac{u^2}{4}} u^{2k+n+1} \, du$$

$$= \frac{(-1)^{\frac{n}{2}}}{\sqrt{\pi}} e^{x^2} \cdot 2^n \sum_0^\infty (-1)^k \left(\frac{n+2k}{2}\right)! \frac{(2x)^{2k+1}}{(2k+1)!}$$

or

$$L_{2m}(x) = \frac{(-1)^m}{\sqrt{\pi}} 2^{2m} \cdot m! \, e^{x^2} \left[2x - (m+1)\frac{(2x)^3}{3!} + (m+1)(m+2)\frac{(2x)^5}{5!} - \dots \right] \quad (14)$$

Proceeding in the same manner if n is odd, we get

$$L_{2m+1}(x) = \frac{(-1)^{m+1}}{\sqrt{\pi}} 2^{2m+1} m! e^{x^2} \left[1 - (m+1)\frac{(2x)^2}{2!} + (m+1)(m+2)\frac{(2x)^4}{4!} - \dots \right] \quad (15)$$

Both these series are converging for all finite values of the variable, and show that

$$L_n(0) = \begin{cases} 0 & (n \text{ even}) \\ \dfrac{(-1)^{\frac{n+1}{2}} 2^n \left(\frac{n-1}{2}\right)!}{\sqrt{\pi}} & (n \text{ odd}) \end{cases} \quad \dots \quad (16)$$

5. Te investigate the value of $L_n(x)$ for large values of x, take the differential equations

$$\frac{d^2 H_n}{dx^2} - 2x \frac{dH_n}{dx} + 2n H_n = 0$$

$$\frac{d^2 L_n}{dx^3} - 2x \frac{dL_n}{dx} + 2n L_n = 0.$$

Multiply the former by L_n the latter by H_n and subtract, so

$$H_n \frac{d^2 L_n}{dx^2} - L_n \frac{d^2 H_n}{dx^2} - 2x \left(H_n \frac{dL_n}{dx} - L_n \frac{dH_n}{dx} \right) = 0$$

or integrating

$$H_n \frac{dL_n}{dx} - L_n \frac{dH_n}{dx} = C e^{x^2}$$

C, being the arbitrary constant.

Introducing the relations (4) and (13) this may be written

$$2n \left[H_n L_{n-1} - L_n H_{n-1} \right] = C e^{x^2}.$$

If $x = 0$ we have

$$2n\, H_n(0)\, L_{n-1}(0) = C \ (n \text{ even})$$
$$-\,2n\, L_n(0)\, H_{n-1}(0) = C$$

therefore in both cases

$$C = \frac{2^{n+1}\, n!}{\sqrt{\pi}}$$

thus finally

$$H_n(x)\, L_{n-1}(x) - L_n(x)\, H_{n-1}(x) = \frac{2^n\,(n-1)!}{\sqrt{\pi}}\, e^{x^2} \quad . \quad . \quad . \quad (17)$$

Now x having a large value, we may write approximately

$$H_n(x) \equiv (2x)^n \qquad H_{n-1}(x) \equiv (2x)^{n-1}$$

$$L_n(x) \equiv \frac{B_n}{x^{n+1}}\, e^{x^2}$$

and therefore

$$L_n(x) \equiv \frac{n!}{\sqrt{\pi}}\, \frac{e^{x^2}}{x^{n+1}} \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

6. Summation of some series containing the functions $H_n(x)$. Let

$$\sum_0^\infty (-1)^k \frac{H_{2k}(x)\, H_{2k}(a)}{(2k)!} = P \quad \text{and} \quad \sum_0^\infty (-1)^k \frac{H_{2k+1}(x)\, H_{2k+1}(a)}{(2k+1)!} = Q$$

and write H_{2k} and H_{2k+1} as definite integrals by means of (9), then we have

$$P = \frac{1}{\pi}\, e^{x^2+a^2} \int_0^\infty e^{-\frac{v^2}{4}} \cos av\,dv \int_0^\infty e^{-\frac{u^2}{4}} \cos xu\,du \sum_0^\infty (-1)^k \frac{(uv)^{2k}}{(2k)!}$$

$$Q = \frac{1}{\pi}\, e^{x^2+a^2} \int_0^\infty e^{-\frac{v^2}{4}} \sin av\,dv \int_0^\infty e^{-\frac{u^2}{4}} \sin xu\,du \sum_0^\infty (-1)^k \frac{(uv)^{2k+1}}{(2k+1)!}$$

where

$$\sum_0^\infty (-1)^k \frac{(uv)^{2k}}{(2k)!} = \cos uv \qquad \sum_0^\infty (-1)^k \frac{(uv)^{2k+1}}{(2k+1)!} = \sin uv.$$

Now

$$\int_0^\infty e^{-\frac{u^2}{4}} \cos xu \cos uv\,du = \tfrac{1}{2} \int_0^\infty e^{-\frac{u^2}{4}} \left[\cos(x+v)\,u + \cos(x-v)\,u\right] du$$

and

$$\int_0^\infty e^{-\frac{u^2}{4}} \sin xu \sin uv du = \tfrac{1}{2}\int_0^\infty e^{-\frac{u^2}{4}} [\cos (x-v)\, u - \cos (x+v)\, u]\, du$$

which may be reduced by means of the relation

$$\int_0^\infty e^{-p^2u^2} \cos \lambda pu du = \frac{\sqrt{\pi}}{2p} e^{-\frac{\lambda^2}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

In this way we get

$$\int_0^\infty e^{-\frac{u^2}{4}} \cos xu \cos uv du = \frac{\sqrt{\pi}}{2} e^{-x^2-v^2} (e^{2xv} + e^{-2xv})$$

$$\int_0^\infty e^{-\frac{u^2}{4}} \sin xu \sin uv du = \frac{\sqrt{\pi}}{2} e^{-x^2-v^2} (e^{2xv} - e^{-2xv})$$

and

$$P = \frac{1}{2\sqrt{\pi}} e^{x^2} \int_0^\infty e^{-\frac{5v^2}{4}} \cos av\, (e^{2xv} + e^{-2xv})\, dv$$

$$Q = \frac{1}{2\sqrt{\pi}} e^{x^2} \int_0^\infty e^{-\frac{5v^2}{4}} \sin av\, (e^{2xv} - e^{-2xv})\, dv.$$

To evaluate these integrals we may remark that the relation (a) holds for complex values of λ. Putting therefore $\lambda = a + ib$ and equating the real and imaginary part in both members of the equation, we obtain

$$\left.\begin{aligned} \int_0^\infty e^{-p^2u^2} \cos apu\, (e^{bpu} + e^{-bpu})\, du &= \frac{\sqrt{\pi}}{p} e^{-\frac{a^2}{4}+\frac{b^2}{4}} \cos \frac{ab}{2} \\ \int_0^\infty e^{-p^2u^2} \sin apu\, (e^{bpu} - e^{-bpu})\, du &= \frac{\sqrt{\pi}}{p} e^{-\frac{a^2}{4}+\frac{b^2}{4}} \sin \frac{ab}{2} \end{aligned}\right\} \quad . \quad . \quad (b)$$

which reduce the values of P and Q to

$$P = \frac{1}{\sqrt{5}} e^{\frac{4(a^2+x^2)}{5}} \cos \frac{4ax}{5} = \sum_0^\infty (1)^k \frac{H_{2k}(x) H_{2k}(a)}{(2k)!} \quad . \quad . \quad . \quad (19)$$

$$Q = \frac{1}{\sqrt{5}} e^{\frac{4(a^2+x^2)}{5}} \sin \frac{4ax}{5} = \sum_0^\infty (-1)^k \frac{H_{2k+1}(x) H_{2k+1}(a)}{(2k+1)!} \quad (20)$$

Investigating in the same way a second series

$$S = \sum_0^\infty \frac{\theta^n H_n(n) H_n(\alpha)}{2^n \cdot n!}$$

where θ represents a value between 0 and 1, we get

$$\frac{\theta^n H_n(x) H_n(x)}{2^n \cdot n!} = \frac{\theta^n e^{x^2+\alpha^2}}{\pi \cdot 2^n \cdot n!} \int_0^\infty \int_0^\infty e^{-\frac{u^2+v^2}{4}} u^n v^n \cos\left(xu - \frac{n\pi}{2}\right) \cos\left(\alpha v - \frac{n\pi}{2}\right) du dv$$

and

$$\sum_0^\infty \frac{\theta^n H_n(x) H_n(\alpha)}{2^n \cdot n!} = \frac{1}{\pi} e^{x^2+\alpha^2} \int_0^\infty \int_0^\infty e^{-\frac{u^2+v^2}{4}} \cos xu \cos \alpha v \sum_0^\infty \frac{\left(\frac{\theta uv}{2}\right)^{2k}}{(2k)!} du dv$$

$$+ \frac{1}{\pi} e^{x^2+\alpha^2} \int_0^\infty \int_0^\infty e^{-\frac{u^2+v^2}{4}} \sin xu \sin \alpha v \sum_0^\infty \frac{\left(\frac{\theta uv}{2}\right)^{2k+1}}{(2k+1)!} du dv$$

$$= \frac{1}{2\pi} e^{x^2+\alpha^2} \int_0^\infty \int_0^\infty e^{-\frac{u^2+v^2}{4}} \cos xu \cos \alpha v \left(e^{\frac{\theta uv}{2}} + e^{-\frac{\theta uv}{2}}\right) du dv$$

$$+ \frac{1}{2\pi} e^{u^2+\alpha^2} \int_0^\infty \int_0^\infty e^{-\frac{u^2+v^2}{4}} \sin xu \sin \alpha v \left(e^{\frac{\theta uv}{2}} - e^{-\frac{\theta uv}{2}}\right) du dv.$$

Now, by means of (b), we may write

$$\int_0^\infty e^{-\frac{u^2}{4}} \cos xu \left(e^{\frac{\theta uv}{4}} + e^{-\frac{\theta uv}{2}}\right) du = 2\sqrt{\pi}\, e^{-x^2+\frac{\theta^2 v^2}{4}} \cos \theta xv$$

$$\int_0^\infty e^{-\frac{u^2}{4}} \sin xu \left(e^{\frac{\theta uv}{2}} - e^{-\frac{\theta uv}{2}}\right) du = 2\sqrt{\pi}\, e^{-x^2+\frac{\theta^2 v^2}{4}} \sin \theta xv$$

therefore

$$S = \frac{1}{\sqrt{\pi}} e^{\alpha^2} \int_0^\infty e^{-\frac{(1-\theta^2)v^2}{4}} \cos(\alpha - \theta x)\, v dv \quad . \quad . \quad . \quad . \quad (21)$$

or

$$S = \frac{1}{\sqrt{1-\theta^2}} e^{\alpha^2 - \frac{(\alpha-\theta x)^2}{1-\theta^2}} = \sum_0^\infty \frac{\theta^n H_n(x) H_n(\alpha)}{2^n \cdot n!} \quad , \quad . \quad . \quad (22)$$

This result shows that the series is diverging when $\theta = 1$.

7. We shall next determine whether any function whatever of a real variable can be expressed in a series of this form.

$$f(x) = A_0 H_0(x) + A_1 H_1(x) + A_2 H_2(x) + \dots$$

Supposing this expansion to be possible the coefficients A_n may be found by means of the relations (6) and (7)

$$A_n = \frac{1}{2^n \cdot n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2} f(\alpha)\, H_n(\alpha)\, d\alpha.$$

With these values the second member reduces to

$$S = \lim_{\theta=1} \sum_0^{\infty} \theta^n A_n H_n(x) \text{ [1]}$$

where

$$\sum_0^{\infty} \theta^n A_n H_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2} f(\alpha)\, d\alpha \sum_0^{\infty} \frac{\theta^n H_n(x) H_n(\alpha)}{2^n \cdot n!}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(\alpha)\, d\alpha \int_0^{\infty} e^{-\frac{(1-\theta^2)\beta^2}{4}} \cos(\alpha - \theta x)\beta\, d\beta.$$

Hence

$$S = \lim_{\theta=1} \int_0^{\infty} e^{-\frac{(1-\theta^2)\beta^2}{4}} d\beta \int_{-\infty}^{\infty} f(\alpha) \cos(\alpha - \theta x)\beta\, d\alpha$$

or

$$S = \frac{1}{\pi} \int_0^{\infty} d\beta \int_{-\infty}^{\infty} f(\alpha) \cos(\alpha - x)\beta\, d\alpha.$$

Now the second member of this equation represents $f(x)$, when this function satisfies the conditions of DIRICHLET between the limits $-\infty$ and $+\infty$. Every function of this kind may therefore be expanded in a series of the functions H.

8. We now proceed to give some examples of this expansion.

I Let $f(x) = x^p$, then we have

$$x^p = A_0 H_0 + A_1 H_1 + A_2 H_2 + \dots$$

where

$$A_n = \frac{1}{2^n \cdot n!} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^p H_n e^{-x^2} dx.$$

Evidently this integral is zero when $x^p H_n$ is an uneven function

[1]) The idea of introducing θ was suggested to me by Prof. P. DEBYE.

or if $n+p$ is an uneven number, the integral vanishes also when $p<n$. Supposing therefore $p+n$ even and $n \leqq p$, we have

$$\int_{-\infty}^{\infty} x^p H_n e^{-x^2} dx = (-1)^n \int_{-\infty}^{\infty} x^p \frac{d^n}{dx^n}(e^{-x^2})\, dx$$

$$= (-1)^n \int_{-\infty}^{\infty} x^p \frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) dx$$

$$= (-1)^{n-1} p \int_{-\infty}^{\infty} x^{p-1} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} dx$$

or

$$\int_{-\infty}^{\infty} x^p H_n e^{-x^2} dx = p \int_{-\infty}^{\infty} x^{p-1} H_{n-1} e^{-x^2} dx .$$

Hence

$$\int_{-\infty}^{\infty} x^p H_n e^{-x^2} dx = \frac{p!}{2^{p-n} \frac{p-n}{2}!} \sqrt{\pi}$$

and

$$A_n = \frac{p!}{2^p \frac{p-n}{2}!\, n!}$$

which gives

$$(2x)^p = H_p + \frac{p(p-1)}{1!} H_{p-2} + \frac{p(p-1)(p-2)(p-3)}{2!} H_{p-4} + \ldots$$

II. In the second place expanding

$$e^{2\beta x - \beta^2} = A_0 H_0 + A_1 H_1 + A_2 H_2 + \ldots$$

the coefficients are given by

$$A_n = \frac{1}{2^n \cdot n!} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\alpha-\beta)^2} H_n(\alpha)\, d\alpha .$$

or, putting $\alpha = y + \beta$, by

$$A_n = \frac{1}{2^n \cdot n!} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} H_n(y+\beta)\, dy .$$

Now, expanding $H_n(y+\beta)$ by MACLAURIN's theorem we have

$$H_n(y+\beta) = H_n(y) + \beta H_n'(y) + \ldots \frac{\beta^n}{n!} H_n^{(n)}(y)$$

where according to (4)

$$H_n'(y) = 2n\, H_{n-1}(y)$$
$$H_n''(y) = 2^2\, n\,(n-1)\, H_{n-2}(y)$$
$$\dots\dots\dots\dots$$
$$H_n^{(n)}(y) = 2^n \,.\, n!\, H_0(y)$$

thus

$$H_n(y+\beta) = H_n(y) + 2n\beta\, H_{n-1}(y) + 2^2 n(n-1)\frac{\beta^2}{2!}H_{n-2}(y) + \,.\, + 2^n . n! \frac{\beta^n}{n!} H_0(y).$$

Introducing this value, we get immediately

$$A_n = \frac{\beta^n}{n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy = \frac{\beta^n}{n!}$$

and

$$e^{2\beta x - \beta^2} = 1 + \frac{\beta}{1!} H_1(x) + \frac{\beta^2}{2!} H_2(x) + \frac{\beta^3}{3!} H_3(x) + \dots$$

From this equation several others may be deduced, for instance

$$e^{-2\beta x - \beta^2} = 1 - \frac{\beta}{1!} H_1(x) + \frac{\beta^2}{2!} H_2(x) - \frac{\beta^3}{3!} H_3(x) + \dots$$

$$e^{-\beta^2} \frac{e^{2\beta x} + e^{-2\beta x}}{2} = 1 + \frac{\beta^2}{2!} H_2 + \frac{\beta^4}{4!} H_4 + \dots$$

$$e^{-\beta^2} \frac{e^{2\beta x} - e^{-2\beta x}}{2} = \frac{\beta}{1!} H_1 + \frac{\beta^3}{3!} H_3 + \frac{\beta^5}{5!} H_5 + \dots$$

$$e^{\gamma^2} \cos 2\gamma x = \sum_0^{\infty} (-1)^k \frac{\gamma^{2k}}{(2k)!} H_{2k}(x)$$

$$e^{\gamma^2} \sin 2\gamma x = \sum_0^{\infty} (-1)^k \frac{\gamma^{2k+1}}{(2k+1)!} H_{2k+1}(x)$$

III. As a third example we will expand a discontinuous function. Supposing $f(x) = 1$ from $x = 0$ to $x = 1$ and $f(x) = 0$ for $1 < x < 0$, we have

$$f(x) = A_0 H_0 + A_1 H_1 + A_2 H_2 + \dots$$

where

$$A_n = \frac{1}{2^n \,.\, n!} \frac{1}{\sqrt{\pi}} \int_0^1 e^{-\alpha^2} H_n(\alpha)\, d\alpha.$$

This coefficient may be determined in the following way.

Let

$$I_n = \int_0^1 e^{-\alpha^2} H_n(\alpha)\, d\alpha = \int_0^1 (2\alpha H_{n-1} - 2(n-1) H_{n-2})\, e^{-\alpha^2} d\alpha$$

then

$$\int_0^1 2\alpha e^{-\alpha^2} H_{n-1} d\alpha = -\int_0^1 H_{n-1} d(e^{-\alpha^2}) = -(e^{-\alpha^2} H_{n-1})_0^1 + \int_0^1 e^{-\alpha^2} H'_{n-1} d\alpha$$

$$= H_{n-1}(0) - e^{-1} H_{n-1}(1) + 2(n-1) \int_0^1 e^{-\alpha^2} H_{n-2} d\alpha$$

and

$$I_n = H_{n-1}(0) - e^{-1} H_{n-1}(1) \qquad (n > 0)$$

Now $H_{n-1}(0)$ vanishes for odd values of n, therefore

$$I_{2k} = -e^{-1} H_{2k-1}(1) \qquad (k > 0)$$

$$I_{2k+1} = -e^{-1} H_{2k}(1) + H_{2k}(0) \qquad (k \geq 0)$$

The following relations hold between three successive values of I:

$$I_{2k+1} - 2I_{2k} + 2(2k-1) I_{2k-1}(0) = 0 \qquad (k > 0).$$

$$I_{2k} - 2 I_{2k-1} + 2(2k-2) I_{2k-2}(0) = (-1)^k 2 \cdot \frac{(2k-2)!}{(k-1)!} \qquad (k > 1).$$

For

$$I_{2k+1} - 2I_{2k} + 2(2k-1) I_{2k-1} = H_{2k}(0) + 2(2k-1) H_{2k-2}(0) -$$

$$- e^{-1} [H_{2k}(1) - 2H_{2k-1}(1) + 2(2k-1) H_{2k-2}(1)]$$

where the second member vanishes according to (5).

In the same way the second relation may be proved.

From this it is evident that all values of I depend upon the values of I_1 and I_2, and these may be obtained directly for

$$I_2 = \int_0^1 e^{-\alpha^2} (4\alpha^2 - 2)\, d\alpha = -2e^{-1}$$

$$I_1 = \int_0^1 e^{-\alpha^2} 2\alpha\, d\alpha = 1 - e^{-1}.$$

If $x = 0$ or $x = 1$ the expansion does not hold. For these values however we may easily verify that the second member reduces to the value $\frac{1}{2}$.

Taking $x = 0$, the second member reduces to

$$\lim_{\theta=1} \frac{1}{\sqrt{\pi}} \int_0^1 e^{-\alpha^2} d\alpha \sum_0^\infty \frac{\theta^n H_n(0) H_n(\alpha)}{2^n \cdot n!}$$

or, according to (22), to

$$\lim_{\theta=1} \frac{1}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{1-\theta^2}} e^{-\frac{\alpha^2}{1-\theta^2}} d\alpha \cdot$$

Assuming

$$\frac{\alpha}{\sqrt{1-\theta^2}}=\beta,$$

we have

$$\lim_{\theta=1} \frac{1}{\sqrt{\pi}}\int_0^{\frac{1}{\sqrt{1-\theta^2}}} e^{-\beta^2}\, d\beta=\frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-\beta^2}\, d\beta=\tfrac{1}{2}\,.$$

In the same way the value for $x=1$ may be found.

SECOND SECTION.

9. Considering the functions

$$\varphi_n(x)=C_n\, e^{-\frac{x^2}{2}} H_n(x)$$

and determining the value of the constant C_n so that

$$\int_{-\infty}^{\infty}\varphi_n^2(x)\, dx=1$$

we easily get

$$C_n=\frac{1}{2^{\frac{n}{2}}\sqrt{n!}\,\sqrt[4]{\pi}}$$

and

$$\varphi_n(x)=\frac{1}{2^{\frac{n}{2}}\sqrt{n!}\,\sqrt[4]{\pi}}\, e^{-\frac{x^2}{2}} H_n(x)$$

Putting these values in the integral equation

$$\varphi_n(x)=\lambda_n\int_{-\infty}^{\infty}\varphi_n(\alpha)\, K(x,\alpha)\, d\alpha$$

we shall now determine the unknown function $K(x,\alpha)$ and the unknown constant λ_n, which verify this equation.

The expansion II from Art. 8 gave

$$A_n=\frac{1}{2^n\,.\,n!}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(\alpha-\beta)^2} H_n(\alpha)\, d\alpha=\frac{\beta^n}{n!}$$

thus, changing β into u,

$$(2u)^n=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(\alpha-u)^2} H_n(\alpha)\, d\alpha.$$

Substituting this value in (9) we have

$$H_n(x) = \frac{1}{2^n\pi} e^{x^2} \int_0^\infty e^{-\frac{u^2}{4}} \cos\left(xu - \frac{n\pi}{2}\right) du \int_{-\infty}^\infty e^{-(\alpha - u)^2} H_n(\alpha)\, d\alpha$$

$$= \frac{1}{2^n\pi} e^{x^2} \int_{-\infty}^\infty e^{-\alpha^2} H_n(\alpha)\, d\alpha \int_0^\infty e^{2\alpha u - \frac{5}{4}u^2} \cos\left(xu - \frac{n\pi}{2}\right) du.$$

Changing α into $-\alpha$, this gives

$$H_n(x) = \frac{(-1)^n}{2^n\pi} e^{x^2} \int_{-\infty}^\infty e^{-\alpha^2} H_n(\alpha)\, d\alpha \int_0^\infty e^{-2\alpha u - \frac{5}{4}u^2} \cos\left(xu - \frac{n\pi}{u}\right) du \quad (c)$$

and putting $-u$ instead of u, the same equation leads to

$$H_n(x) = \frac{1}{2^n\pi} e^{x^2} \int_{-\infty}^\infty e^{-\alpha^2} H_n(\alpha)\, d\alpha \int_{-\infty}^0 e^{-2\alpha u - \frac{5}{4}u^2} \cos\left(xu + \frac{n\pi}{2}\right) du$$

which, by the relation

$$\cos\left(xu - \frac{n\pi}{2}\right) = (-1)^n \cos\left(xu + \frac{n\pi}{2}\right)$$

is equivalent with

$$H_n(x) = \frac{(-1)^n}{2^n\pi} e^{x^2} \int_{-\infty}^\infty e^{-\alpha^2} H_n(\alpha)\, d\alpha \int_{-\infty}^0 e^{-2\alpha u - \frac{5}{4}u^2} \cos\left(xu - \frac{n\pi}{2}\right) du \,.\, (d) \,.$$

Now, adding the equations (c) and (d) we find

$$H_n(x) = \frac{(-1)^n}{2^{n+1}\pi} e^{x^2} \int_{-\infty}^\infty e^{-\alpha^2} H_n(\alpha)\, d\alpha \int_{-\infty}^\infty e^{-2\alpha u - \frac{5}{4}u^2} \cos\left(xu - \frac{n\pi}{2}\right) du$$

where, putting $u = v - \frac{4}{5}\alpha$

$$\int_{-\infty}^\infty e^{-2\alpha u - \frac{5}{4}u^2} \cos\left(xu - \frac{n\pi}{2}\right) du = e^{\frac{4}{5}\alpha^2} \int_{-\infty}^\infty e^{-\frac{5}{4}v^2} \cos\left(xv - \frac{4}{5}\alpha x - \frac{n\pi}{2}\right) dv =$$

$$= e^{\frac{4}{5}\alpha^2} \cos\left(\frac{4}{5}\alpha x + \frac{n\pi}{2}\right) \int_{-\infty}^\infty e^{-\frac{5}{4}v^2} \cos xv\, dv \,.$$

According to formula (a) Art. 6, we obtain therefore

$$\int_{-\infty}^\infty e^{-2\alpha u - \frac{5}{4}u^2} \cos\left(xu - \frac{n\pi}{2}\right) du = \frac{2\sqrt{\pi}}{\sqrt{5}} e^{\frac{4}{5}\alpha^2 - \frac{1}{5}x^2} \cos\left(\frac{4}{5}\alpha x + \frac{n\pi}{2}\right)$$

and finally

$$H_n(x) = \frac{1}{2^n\sqrt{5\pi}}\, e^{\frac{4}{5}x^2} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{5}} H_n(\alpha) \cos\left(\frac{4}{5}\alpha x - \frac{n\pi}{2}\right) d\alpha\,.$$

Multiplying this equation by $C_n e^{-\frac{x^2}{2}}$ we have

$$\varphi_n(x) = \frac{1}{2^n\sqrt{5\pi}} \int_{-\infty}^{\infty} \varphi_n(\alpha)\, e^{\frac{3(\alpha^2+x^2)}{10}} \cos\left(\frac{4}{5}\alpha x - \frac{n\pi}{2}\right) d\alpha$$

thus

$$\lambda_n = \frac{1}{2^n} \qquad K(x,\alpha) = \frac{1}{\sqrt{5\pi}}\, e^{\frac{3(\alpha^2+x^2)}{10}} \cos\left(\frac{4}{5}\alpha x - \frac{n\pi}{2}\right).$$

To make $K(x,\alpha)$ independent of n, we distinguish two cases·

1. n even $= 2m$, then

$$\varphi_{2m}(x) = \frac{(-1)^m}{2^{2m}\sqrt{5\pi}} \int_{-\infty}^{\infty} \varphi_{2m}(\alpha)\, e^{\frac{3(\alpha^2+x^2)}{10}} \cos\frac{4}{5}\alpha x\, dx$$

$$\lambda_{2m} = \frac{(-1)^m}{2^{2m}} \qquad K_1(x\,.\,\alpha) = \frac{1}{\sqrt{5\pi}}\, e^{\frac{3(\alpha^2+x^2)}{10}} \cos\frac{4}{5}\alpha x\,.$$

2. n odd $= 2m+1$, then

$$\varphi_{2m+1}(x) = \frac{(-1)^m}{2^{2m+1}\sqrt{5\pi}} \int_{-\infty}^{\infty} \varphi_{2m+1}(\alpha)\, e^{\frac{3(\alpha^2+x^2)}{10}} \sin\frac{4}{5}\alpha x\, d\alpha$$

$$\lambda_{2m+1} = \frac{(-1)^m}{2^{2m+1}} \qquad K_2(x\,,\alpha) = \frac{1}{\sqrt{5\pi}}\, e^{\frac{3(\alpha^2-x^2)}{10}} \sin\frac{4}{5}\alpha x.$$

According to the theory of integral equations we know that

$$K_1(x\,.\,\alpha) = \sum_0^{\infty} \frac{\varphi_{2m}(x)\,\varphi_{2m}(\alpha)}{\lambda_{2m}}$$

$$K_2(x\ \ \alpha) = \sum_0^{\infty} \frac{\varphi_{2m+1}(x)\,\varphi_{2m+1}(\alpha)}{\lambda_{2m+1}}$$

or

$$K_1(x\,,\alpha) = \frac{1}{\sqrt{\pi}}\, e^{-\frac{x^2+\alpha^2}{2}} \sum_0^{\infty} (-1)^m \frac{H_{2m}(x)\,H_{2m}(\alpha)}{(2m)!}$$

$$K_2(x\,,\alpha) = \frac{1}{\sqrt{\pi}}\, e^{-\frac{x^2+\alpha^2}{2}} \sum_0^{\infty} (-1)^m \frac{H_{2m+1}(x)\,H_{2m+1}(\alpha)}{(2m+1)!}$$

which may be verified by the equations (19) and (20).

10. We shall now show that the function

$$\sigma = \frac{H_{n+1}(x) + k L_{n+1}(x)}{H_n(x) + k L_n(x)}$$

where k is an arbitrary constant, may be developed in a continuous fraction.

Differentiating and eliminating k, the differential equation for σ, takes the form

$$(y_2 z_1 - y_1 z_2) \frac{d\sigma}{dx} + \left(z_1 \frac{dz_2}{dx} - z_2 \frac{dz_1}{dx} \right) \sigma^2 +$$

$$+ \left(z_2 \frac{dy_1}{dx} - y_1 \frac{dz_2}{dx} + y_2 \frac{dz_1}{dx} - z_1 \frac{dy_2}{dx} \right) \sigma + \left(y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} \right) = 0$$

where

$$y_1 = H_{n+1}(x) \qquad y_2 = L_{n+1}(x)$$
$$z_1 = H_n(x) \qquad z_2 = L_n(x)$$

According to (17) the coefficients of this equation may be written

$$y_2 z_1 - y_1 z_2 = -2^{n+1} n!\, e^{x^2}$$

$$z_1 \frac{dz_2}{dx} - z_2 \frac{dz_1}{dx} = 2n (H_n L_{n+1} - H_{n-1} L_n) = 2^{n+1} n!\, e^{x^2}$$

$$z_2 \frac{dy_1}{dx} - y_1 \frac{dz_2}{dx} + y_2 \frac{dz_1}{dx} - z_1 \frac{dy_2}{dx} = 2n (H_{n-1} L_{n+1} - H_{n+1} L_{n-1}) = -2^{n+2} n!\, x e^{x^2}$$

$$y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} = 2(n+1)(H_{n+1} L_n - H_n L_{n+1}) = 2^{n+2} (n+1)!\, e^{x^2}$$

thus

$$\frac{d\sigma}{dx} = \sigma^2 - 2x\sigma + 2(n+1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Substituting

$$\sigma = 2x - \frac{2n}{\sigma_1}$$

the function σ_1 satisfies an equation of the same kind viz.

$$\frac{d\sigma_1}{dx} = \sigma_1^2 - 2x\sigma_1 + 2n.$$

Substituting again

$$\sigma_1 = 2x - \frac{2(n-1)}{\sigma_2}$$

the transformed equation is

$$\frac{d\sigma_2}{dx} = \sigma_2^2 - 2x\sigma_2 + 2(n-1).$$

and so on, until

$$\frac{d\sigma_n}{dx} = \sigma_n^2 - 2x\sigma_n + 2.$$

Putting now

$$\sigma_n = 2x + \frac{1}{\lambda}$$

we have

$$\frac{d\lambda}{dx} + 2x\lambda = -1$$

thus

$$\lambda = e^{-x^2}\left(C - \int_0^x e^{x^2}dx\right)$$

C being an arbitrary constant.

Hence

$$\sigma = 2x - \cfrac{2n}{2x - \cfrac{2(n-1)}{2x - \cfrac{2(n-2)}{2x - \ddots - \cfrac{2}{2x + \cfrac{e^{x^2}}{C-I}}}}} \tag{24}$$

where

$$I = \int_0^x e^{x^2} dx.$$

Thus for

$$n = 1 \qquad \sigma^{(1)} = 2x - \cfrac{2}{2x + \cfrac{e^{x^2}}{C-I}} = \frac{H_2\,(C-I) + 2xe^{x^2}}{H_1\,(C-I) + e^{x^2}}$$

$$n = 2 \qquad \sigma^{(2)} = \quad . \; . \; . \; . \; . \; . \quad = \frac{H_3\,(C-I) + (4x^2-4)\,e^{x^2}}{H_2\,(C-I) + 2xe^{x^2}}$$

$$n = 3 \qquad \sigma^{(3)} = \quad . \; . \; . \; . \; . \quad = \frac{H_4\,(C-I) + (8x^3-20x)\,e^{x^2}}{H_3\,(C-I) + (4x^2-4)\,e^{x^2}}$$

$$n = n \qquad \sigma^{(n)} = \quad . \; . \; . \; . \; . \; . \quad = \frac{H_{n+1}\,(C-I) + T_n\,e^{x^2}}{H_n\,(C-I) + T_{n-1}\,e^{x^2}}.$$

The following relation holds between three successive functions T:

$$T_n = 2xT_{n-1} - 2nT_{n-2}.$$

as appears from the substitution of the values of $\sigma^{(n)}$ and $\sigma^{(n-1)}$ in

$$\sigma^{(n)} = 2x - \frac{2n}{\sigma^{(n-1)}}.$$

Putting $x = 0$ in this relation, we obtain

$$\left. \begin{aligned} T_n(0) &= (-1)^{\frac{n}{2}}\, 2^n \left(\frac{n}{2}\right)! \quad (n \text{ even}) \\ T_n(0) &= 0 \qquad\qquad\qquad (n \text{ odd}) \end{aligned} \right\} \quad \ldots \quad (25)$$

If now we compare the two forms

$$\sigma = \frac{H_{n+1}(C-I) + T_n e^{x^2}}{H_n(C-I) + T_{n-1} e^{x^2}} = \frac{H_{n+1} + kL_{n+1}}{H_n + kL_n} \quad \ldots \quad (e)$$

we may determine the relation which exists between C and k. For putting $x = 0$, we have $I = 0$ and

$$\frac{T_n(0)}{C} = kL_{n+1}(0) \quad (n \text{ even})$$

$$\frac{C}{T_{n-1}(0)} = \frac{1}{kL_n(0)} \quad (n \text{ odd})$$

thus

$$k = -\frac{\sqrt{\pi}}{2C}.$$

Therefore if $C = \infty$ the continued fraction (24) represents $\frac{H_{n+1}}{H_n}$ and if $C = 0$ the value of this fraction is $\frac{L_{n+1}}{L_n}$.

From (e) we may deduce a new form for $L_0(x)$. For introducing C instead of k, we have

$$2C\, e^{x^2}\,(H_n T_n - H_{n+1} T_{n-1}) - \sqrt{\pi}\,(C-I)\,(H_{n+1} L_n - H_n L_{n+1}) - \\ - \sqrt{\pi}\, e^{x^2} (L_n T_n - L_{n+1} T_{n-1}) = 0.$$

Now the relations

$$\begin{aligned} T_n &= 2x T_{n-1} - 2x T_{n-2} \\ H_{n+1} &= 2x H_n - 2n H_{n-1} \\ L_{n+1} &= 2x L_n - 2n L_{n-1} \end{aligned}$$

lead easily to

$$\begin{aligned} H_n T_n - H_{n+1} T_{n-1} &= 2^n \,.\, n! \\ L_n T_n - L_{n+1} T_{n-1} &= 2^n \,.\, n!\, L_0. \end{aligned}$$

With these values, and (17) therefore, we find

$$2C \,.\, 2^n \,.\, n! - (C-I)\, 2^{n+1} n! - \sqrt{\pi} \,.\, 2^n \,.\, n!\, L_0 = 0$$

or

$$L_0 = \frac{2}{\sqrt{\pi}} I = \frac{2}{\sqrt{\pi}} \int_0^x e^{x^2} dx \quad \ldots \ldots \quad (26)$$

This result leads also to a new form for all the functions $L_n(x)$, for

$$L_1 = -\frac{e^{x^2}}{\sqrt{\pi}}\int_0^\infty e^{-\frac{u^2}{4}} u \cos xu du = 2xL_0 - \frac{2e^{x^2}}{\sqrt{\pi}}$$

and, according to (12)

$$L_n = H_n L_0 - \frac{2}{\sqrt{\pi}} e^{x^2} T_{n-1} \quad \ldots \ldots \quad (27)$$

where

$$T_{n-1} = H_{n-1} - 2(n-2) H_{n-3} + 2^2(n-3)(n-4) H_{n-5} - \ldots \quad (28)$$

11. Applying the preceding expansion, the problem of the momenta may be solved.

Let

$$a_n = \int_0^\infty f(y)\, y^n dy$$

the question is to determine the function $f(y)$ when a_n is given for all positive integral values of n.

Putting

$$f(y) = e^{-y^2}\left[b_0 H_0(y) + b_1 H_1(y) + b_2 H_2(y) + \ldots\right]$$

we have to determine the coefficients b from

$$a_n = \sum_0^\infty b_p \int_{-\infty}^\infty e^{-y^2} y^n H_p(y) dy$$

Here $p+n$ is an even number, for the integral vanishes for $p+n$ odd. Moreover the integral vanishes if $p>n$ therefore

$$a_n = \sum_0^n b_p \int_{-\infty}^\infty e^{-y^2} y^n H_p(y) dy$$

or, according to the expansion I Art. 8

$$a_n = n!\sqrt{\pi}\sum_0^n \frac{b_p}{2^{n-p}\frac{n-p}{2}!}.$$

which may be written

$$\frac{a_n}{n!\sqrt{\pi}} = A_n = \sum_0^n \frac{b_p}{2^{n-p}\frac{n-p}{2}!}.$$

Solving these linear equations, we get immediately

$$b_p = \sum_0^{\leq \frac{p}{2}} (-1)^k \frac{A_{p-2k}}{2^{2k} \cdot k!}$$

and accordingly

$$f(y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{u^2}{4}} du \sum_0^\infty b_p \, u^p \cos\left(yu - \frac{p\pi}{2}\right)$$

where b_p has the preceding values.

Writing this

$$f(y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{u^2}{4}} du \left[\cos yu \;\, S_1 + \sin yu \, S_2\right]$$

we have

$$S_1 = \sum_{0.2}^\infty (-1)^{\frac{p}{2}} b_p u^p$$

$$S_2 = \sum_{1.3}^\infty (-1)^{\frac{p-1}{2}} b_p u^p$$

or, expressing b_p in function of the values A

$$S_1 = A_0 - u^2\left(A_2 - \frac{A_0}{2^2 1!}\right) + u^4\left(A_4 - \frac{A_2}{2^2 1!} + \frac{A_0}{2^4 \cdot 2!}\right) \dots$$

$$= e^{\frac{u^2}{4}} (A_0 - A_2 u^2 + A_4 u^4 - \dots) = e^{\frac{u^2}{4}} \sum_0^\infty (-1)^k A_{2k} u^{2k}$$

$$= \frac{1}{\sqrt{\pi}} e^{\frac{u^2}{4}} \sum_0^\infty (-1)^k \frac{\alpha_{2k}}{(2k)!} u^{2k}$$

and in the same way

$$S_2 = \frac{1}{\sqrt{\pi}} e^{\frac{u^2}{4}} \sum_0^\infty (-1)^k \frac{\alpha_{2k+k1}}{(2k+1)!} u^{2k+1} .$$

therefore

$$f(y) = \frac{1}{\pi} \int_0^\infty du \left[\cos yu \sum_0^\infty (-1)^k \frac{\alpha_{2k}}{(2k)!} u^{2k} + \sin yu \sum_0^\infty (-1)^k \frac{\alpha_{2k+1}}{(2k+1)!} u^{2k+1}\right]$$

or finally

$$f(y) = \frac{1}{\pi} \int_0^\infty \sum_0^\infty (-1)^p \frac{\alpha_p u^p}{p!} \cos\left(yu + \frac{p\pi}{2}\right) du.$$

Of course this is only a formal solution, which holds when the values α_p are such as to make this integral convergent. This is e.g. the case if

$$a_{2k+1} = 0 \qquad a_{2k} = (-1)^k \frac{H_{2k}(1)}{2^{2k}}$$

for then

$$f(y) = \frac{1}{\pi} \int_0^\infty \cos yu \sum_0^\infty \frac{H_{2k}(1)}{(2k)!} \left(\frac{u}{2}\right)^{2k} du$$

or, according to the expansion II

$$f(y) = \frac{1}{2\pi} \int_0^\infty \cos y\, u\, e^{-\frac{u^2}{4}} (e^u + e^{-u})\, du,$$

which reduced by b Art. 6, gives

$$f(y) = \frac{e}{\sqrt{\pi}} e^{-y^2} \cos 2y .$$

Microbiology. — *"On the nitrate ferment and the formation of physiological species"*. By Prof. Dr. M. W. BEIJERINCK.

(Communicated in the meeting of March 28, 1913).

It is a well-known fact that in soil as well as in liquids containing a great many individuals of the nitrate ferment, large amounts of organic substances may be present without preventing nitratation, which is the oxidation of nitrites to nitrates by that ferment.

On the other hand it is certain, that when only few germs of the ferment are present, so that they must first grow and multiply in order to exert a perceptible influence, extremely small quantities of organic substance are already sufficient to make the experiments fail altogether, the nitrite then remaining unchanged in the culture media.

It is generally supposed, that this latter circumstance must be explained by accepting that the nitrate ferment can only then grow and increase, when soluble organic substances are nearly or wholly absent.

My own experiments, however, have led me to quite another result, namely that the nitrate ferment very easily grows and increases in presence of the most various organic substances. But in this case, that is, *when growing at the expense of organic food*, it soon wholly loses the power of oxidising nitrites to nitrates and then changes into an apparently common saprophytic bacterium.

This change may be called the formation of a physiological species, and the two conditions of the ferment thus resulting, respectively the *oligotrophic* and the *polytrophic* form.