however that this rotation, which is incomprehensibly denied by LEGROS does not set in before the mouth has reached its minimal size, consequently in the middle of the period of the metamorphosis. In consequence of this rotation the rostral rim of the mouth of the larva becomes right-rim, whilst at the same time the posterior rim becomes left-rim.

The mouth-opening having become velaropening lies now symmetrically with regard to the median plane, but the nerves, that surround it, indicate that it continues to be an organ of the leftside.

In the higher animals the middle-ear originates from the first gill-pouch, whilst amphioxus lacks the auditive organ entirely. If we wish to express ourselves in a popular way, we may say, as I did already on a former opportunity: Amphioxus cannot hear; he eats however with the left ear, and has consequently lost the mouth.

Mathematics. — *"Applications of* SONINE'*s extension of* ABEL'*s integralequation."* By Dr. J. G. RUTGERS. (Communicated by Prof. W. KAPTEYN).

(Communicated in the meeting of September 27, 1913).

SONINE[1]) has given to ABEL's integralequation an extension which comes to the following.

The unknown function u in the equation

$$f(x)=\int_a^x \psi(x-\xi)\, u(\xi)d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

is determined by

$$u(x)=\int_a^x \sigma(x-\xi) f'(\xi)d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

where we suppose $f(x)$ to be finite and continuous, $f'(x)$ finite, $a \leqq x \leqq b$, and $f(a)=0$. Moreover σ and ψ are connected in the following way:

Suppose

$$\varphi(y)=\sum_0^\infty c_m y_m \quad , \quad \frac{1}{\varphi(y)}=\sum_0^\infty d_n y^n,$$

[1]) Acta Matem. **4**; 1884.

then if

$$a_m = \frac{c_m}{\Gamma(m-\lambda+1)} \quad , \quad b_n = \frac{d_n}{\Gamma(n+\lambda)}$$

we shall find

$$\psi(x) = x^{-\lambda} \sum_0^\infty a_m x^m \text{ and } \sigma(x) = x^{-(1-\lambda)} \sum_0^\infty b_n x^n,$$

and at the same time we find λ bound to the condition $1 > \lambda > 0$.

This rather intricate connection between ψ and σ greatly limits the number of applications with some practical significance. As a matter of fact SONINE gives two, for the third furnishes nothing new as we shall see.

1. ABEL's equation appears when in $(1a)$ we take $\psi(x) = \frac{1}{x^\lambda} (1 > \lambda > 0)$. By this $a_0 = 1$, $a_m = 0 \, (m > 0)$, by which $c_0 = \Gamma(1-\lambda)$, $c_m = 0 \; (m > 0)$ and therefore

$$q(y) = \Gamma(1-\lambda) \quad , \quad \frac{1}{q(y)} = \frac{1}{\Gamma(1-\lambda)};$$

furtheron

$$d_0 = \frac{1}{\Gamma(1-\lambda)} \quad , \quad d_n = 0 \; (n > 0)$$

and therefore

$$b_0 = \frac{1}{\Gamma(1-\lambda)\Gamma(\lambda)} = \frac{\sin \lambda\pi}{\pi} \quad , \quad b_n = 0 \; (n > 0).$$

Finally follows:

$$\sigma(x) = \frac{\sin \lambda\pi}{\pi} \cdot \frac{1}{x^{1-\lambda}}.$$

Substitution of ψ and σ in $(1a)$ and $(1b)$ now gives us:

$$f(x) = \int_a^x \frac{u(\xi)}{(x-\xi)^\lambda} d\xi \quad , \quad u(x) = \frac{\sin \lambda\pi}{\pi} \int_a^x \frac{f'(\xi)}{(x-\xi)^{1-\lambda}} d\xi \quad . \quad . \quad (2)$$

2. For the second application SONINE starts from [1]:

$$q(y) = \Gamma(1-\lambda) e^{-\frac{z^2 y}{4}} = \Gamma(1-\lambda) \sum_0^\infty \frac{(-1)^m \left(\frac{z^2 y}{4}\right)^m}{m!},$$

so that

$$c_m = \Gamma(1-\lambda) \frac{(-1)^m \left(\frac{z^2}{4}\right)^m}{m!}, \text{ thus } a_m = \Gamma(1-\lambda) \cdot \frac{(-1)^m \left(\frac{z}{2}\right)^{2m}}{m! \, \Gamma(m-\lambda+1)}$$

[1]) The factor $\Gamma(1-\lambda)$ is added for practical reasons.

by which

$$\psi(x)=\Gamma(1-\lambda)x^{-\lambda}\sum_0^\infty\frac{(-1)^m\left(\frac{z\sqrt{x}}{2}\right)^{2m}}{m!\,\Gamma(m-\lambda+1)}=\Gamma(1-\lambda)\left(\frac{z}{2}\right)^{\lambda}x^{-\frac{\lambda}{2}}I_{-\lambda}(z\sqrt{x})$$

Further we find

$$\frac{1}{\varphi(y)}=\frac{1}{\Gamma(1-\lambda)}e^{\frac{z^2y}{4}}=\frac{1}{\Gamma(1-\lambda)}\sum_0^\infty\frac{\left(\frac{z^2y}{4}\right)^n}{n!},$$

so that

$$d_n=\frac{1}{\Gamma(1-\lambda)}\cdot\frac{\left(\frac{z}{2}\right)^{2n}}{n!},$$

thus

$$b_n=\frac{1}{\Gamma(1-\lambda)}\cdot\frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma(n+\lambda)}=\frac{\sin\lambda\pi\,\Gamma(\lambda)}{\pi}\cdot\frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma(n+\lambda)},$$

by which

$$\sigma(x)=\frac{\sin\lambda\pi\,\Gamma(\lambda)}{\pi}x^{-(1-\lambda)}\sum_0^\infty\frac{\left(\frac{z\sqrt{x}}{2}\right)^{2n}}{n!\,\Gamma(n+\lambda)}=$$

$$=\frac{i^{1-\lambda}\sin\lambda\pi\,\Gamma(\lambda)}{\pi}\left(\frac{z}{2}\right)^{1-\lambda}x^{-\frac{1-\lambda}{2}}I_{-(1-\lambda)}(iz\sqrt{x}).$$

By substitution of these values of ψ and σ we see that $(1a)$ and $(1b)$ pass into

$$f(x)=\Gamma(1-\lambda)\left(\frac{z}{2}\right)^{\lambda}\int_a^x(x-\xi)^{-\frac{\lambda}{2}}I_{-\lambda}(z\sqrt{x-\xi})\,u(\xi)\,d\xi\ .\quad .\quad (3a)$$

with

$$u(x)=\frac{i^{1-\lambda}\sin\lambda\pi\,\Gamma(\lambda)}{\pi}\left(\frac{z}{2}\right)^{1-\lambda}\int_a^x(x-\xi)^{-\frac{1-\lambda}{2}}I_{-(1-\lambda)}(iz\sqrt{x-\xi})f'(\xi)d\xi.\ (3b)$$

For $\lambda=\frac{1}{2}$ follow from this some important relations as SONINE already noticed. The forms of ABEL appear when we take $z=0$.

3. As third application SONINE gives:

$$\varphi(y)=\Gamma(1-\lambda)(1+zy)^{-(1-\lambda)}=\sum_0^\infty(-1)^m\frac{\Gamma(m-\lambda+1)}{m!}(zy)^m,$$

by which

$$c_m = (-1)^m \cdot \frac{\Gamma(m-\lambda+1)}{m!} z^m \quad , \quad a_m = \frac{(-1)^m z^m}{m!},$$

thus

$$\psi(x) = x^{-\lambda} \sum_0^\infty (-1)^m \frac{(zx)^m}{m!} = \frac{e^{-zx}}{x^\lambda}.$$

Further ensues

$$\frac{1}{\varphi(y)} = \frac{1}{\Gamma(1-\lambda)} (1+zy)^{1-\lambda} = \frac{\lambda-1}{\Gamma(1-\lambda)\Gamma(\lambda)} \sum_0^\infty (-1)^n \frac{\Gamma(n+\lambda-1)}{n!} (zy)^n,$$

so that

$$d_n = \frac{(\lambda-1)\sin\lambda\pi}{\pi} \cdot \frac{(-1)^n\,\Gamma(n+\lambda-1)}{n!} z^n \, , \, b_n = \frac{(\lambda-1)\sin\lambda\pi}{\pi} \frac{(-1)^n z^n}{n!(n+\lambda-1)}$$

and therefore

$$\sigma(x) = \frac{(\lambda-1)\sin\lambda\pi}{\pi} x^{\lambda-1} \sum_0^\infty \frac{(-1)^n (zx)^n}{n!(n+\lambda-1)},$$

to which SONINE gives another form, which is, however, not correct. It would be better to write for it:

$$\sigma(x) = \frac{\sin\lambda\pi}{\pi} \left[\frac{1}{x^{1-\lambda}} - (1-\lambda) \sum_1^\infty (-1)^n \frac{z^n x^{n+\lambda-1}}{n!(n+\lambda-1)} \right],$$

for indeed it is now again evident that for $z=0$ we find ψ and σ assuming the form as in § 1.

Substitution of ψ and σ in (1a) and (1b) (SONINE leaves this out) now gives:

$$f(x) = \int_a^x \frac{e^{-z(x-\xi)}}{(x-\xi)^\lambda} u(\xi)\, d\xi, \quad \ldots \ldots \quad (4a)$$

with

$$u(x) = \frac{\sin\lambda\pi}{\pi} \int_a^x \frac{f'(\xi)}{(x-\xi)^{1-\lambda}} d\xi - \frac{(1-\lambda)\sin\lambda\pi}{\pi} \int_a^x f'(\xi) \left\{ \sum_1^\infty \frac{(-1)^n z^n (x-\xi)^{n+\lambda-1}}{n!(n+\lambda-1)} \right\} d\xi.$$

As $1 > \lambda > 0$ and $f(a)=0$, we find that by means of partial integration the last integral passes into:

$$\int_a^x f(\xi) \left\{ \sum_1^\infty (-1)^n \frac{z^n (x-\xi)^{n+\lambda-2}}{n!} \right\} d\xi,$$

so that

$$u(x) = \frac{\sin\lambda\pi}{\pi} \int_a^x \frac{f'(\xi)}{(x-\xi)^{1-\lambda}} d\xi - \frac{(1-\lambda)\sin\lambda\pi}{\pi} \int_a^x \frac{e^{-z(x-\xi)}-1}{(x-\xi)^{2-\lambda}} f(\xi)\, d\xi \quad . \quad (4b)$$

That (4a) and (4b) do not stand for anything new, we shall immediately see by substituting

$$f(x) = e^{-zx} f_1(x) \quad \text{and} \quad u(x) = e^{-zx} u_1(x)\,,$$

where ($4a$) takes at once the form of ABEL's equation and ($4b$) as its solution can easily be reduced to its ordinary form.

In the following paragraphs we shall be led to really new applications.

4. Let in the first place

$$\varphi(y) = \Gamma(1-\lambda)\,(1+z^2y^2)^{-\frac{1-\lambda}{2}} = \frac{\Gamma(1-\lambda)}{\Gamma\left(\frac{1-\lambda}{2}\right)} \sum_0^\infty (-1)^m \frac{\Gamma\left(m+\frac{1-\lambda}{2}\right)}{m!} (zy)^{2m},$$

where, by application of

$$\Gamma(\alpha)\,\Gamma(\alpha+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2\alpha-1}} \cdot \Gamma(2\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we find that

$$c_{2m} = \frac{\Gamma\left(1-\frac{\lambda}{2}\right)}{2^\lambda \sqrt{\pi}} \cdot (-1)^m \frac{\Gamma\left(m+\frac{1-\lambda}{2}\right)}{m!} z^{2m}, \quad c_{2m+1} = 0$$

and thus, again because of (5):

$$a_{2m} = \frac{\Gamma\left(1-\frac{\lambda}{2}\right)}{2^\lambda \sqrt{\pi}} \cdot (-1)^m \frac{\Gamma\left(m+\frac{1-\lambda}{2}\right)}{m!\,\Gamma(2m-\lambda+1)} z^{2m} =$$

$$= \Gamma\left(1-\frac{\lambda}{2}\right) (-1)^m \frac{\left(\frac{z}{2}\right)^{2m}}{m!\,\Gamma\left(m-\frac{\lambda}{2}+1\right)}\,, \quad a_{2m+1} = 0\,,$$

so that:

$$\psi(x) = x^{-\lambda}\,\Gamma\left(1-\frac{\lambda}{2}\right) \sum_0^\infty \frac{(-1)^m \left(\frac{zx}{2}\right)^{2m}}{m!\,\Gamma\left(m-\frac{\lambda}{2}+1\right)} = \Gamma\left(1-\frac{\lambda}{2}\right)\left(\frac{z}{2x}\right)^{\frac{\lambda}{2}} I_{-\frac{\lambda}{2}}(zx)$$

Furthermore we find

$$\frac{1}{\varphi(y)} = \frac{1}{\Gamma(1-\lambda)} \cdot (1+z^2y^2)^{\frac{1-\lambda}{2}} = \frac{1}{\Gamma(1-\lambda)\,\Gamma\left(\frac{\lambda-1}{2}\right)} \cdot \sum_0^\infty (-1)^n \frac{\Gamma\left(n-\frac{1-\lambda}{2}\right)}{n!} (zy)^{2n}\,,$$

by which, on account of (5):

$$d_{2n} = \frac{(\lambda-1)\,\Gamma\left(\frac{\lambda}{2}\right)}{2^{2-\lambda}\sqrt{\pi}} \cdot \frac{\sin\lambda\pi}{\pi} \cdot (-1)^n \, \frac{\Gamma\left(n - \frac{1-\lambda}{2}\right)}{n!} \, z^{2n} \, , \quad d_{2n+1} = 0$$

and thus, again on account of (5):

$$b_{2n} = \frac{(\lambda-1)\,\Gamma\left(\frac{\lambda}{2}\right)}{2^{2-\lambda}\sqrt{\pi}} \; \frac{\sin\lambda\pi}{\pi} \cdot (-1)^n \, \frac{\Gamma\left(n - \frac{1-\lambda}{2}\right)}{n!\,\Gamma(2n+\lambda)} \, z^{2n} =$$

$$= (\lambda-1)\,\Gamma\left(\frac{\lambda}{2}\right) \frac{\sin\lambda\pi}{\pi} \cdot (-1)^n \, \frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma\left(n+\frac{\lambda}{2}\right)(2n+\lambda-1)} \, , \quad b_{2n+1} = 0,$$

so that

$$\sigma(x) = (\lambda-1)\,\Gamma\left(\frac{\lambda}{2}\right) \frac{\sin\lambda\pi}{\pi} \sum_0^\infty (-1)^n \frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma\left(n+\frac{\lambda}{2}\right)} \cdot \frac{x^{2n+\lambda-1}}{2n+\lambda-1} =$$

$$= \frac{\sin\lambda\pi}{\pi} \cdot \frac{1}{x^{1-\lambda}} - (1-\lambda)\,\Gamma\left(\frac{\lambda}{2}\right) \frac{\sin\lambda\pi}{\pi} \sum_1^\infty (-1)^n \frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma\left(n+\frac{\lambda}{2}\right)} \, \frac{x^{2n+\lambda-1}}{2n+\lambda-1} .$$

For $z = 0$ we are evidently again in the special case of ABEL's problem (§ 1).

Let us now substitute ψ and σ now found in (1a) and (1b), we then arrive at the integralequation

$$f(x) = \Gamma\left(1-\frac{\lambda}{2}\right)\left(\frac{z}{2}\right)^{\frac{\lambda}{2}} \int_a^x (x-\xi)^{-\frac{\lambda}{2}} \, I_{-\frac{\lambda}{2}}\{z(x-\xi)\} \, u(\xi)\, d\xi, \qquad (6a)$$

to which belongs as solution:

$$u(x) = \frac{\sin\lambda\pi}{\pi} \int_a^x \frac{f'(\xi)}{(x-\xi)^{1-\lambda}}\, d\xi \; -$$

$$-\frac{\sin\lambda\pi\,(1-\lambda)\,\Gamma\left(\frac{\lambda}{2}\right)}{\pi} \int_a^x f'(\xi)\, d\xi \left\{ \sum_1^\infty (-1)^n \frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma\left(n+\frac{\lambda}{2}\right)} \cdot \frac{(x-\xi)^{2n+\lambda-1}}{2n+\lambda-1} \right\}$$

As $1 > \lambda > 0$, $f(a) = 0$ and $f(x)$ is finite the last integral passes by means of partial integration into

$$\int_a^x f(\xi)\,d\xi \left\{ \sum_1^\infty (-1)^n \frac{\left(\frac{z}{2}\right)^{2n} (x-\xi)^{2n+\lambda-2}}{n!\,\Gamma\left(n+\frac{\lambda}{2}\right)} \right\},$$

so that we find

$$\left.\begin{aligned} u(x) &= \frac{\sin\lambda\pi}{\pi}\int_a^x \frac{f'(\xi)}{(x-\xi)^{1-\lambda}}\,d\xi - \\ &-\frac{\sin\lambda\pi\,(1-\lambda)}{\pi}\int_a^x \left[\Gamma\left(\frac{\lambda}{2}\right)\left(\frac{z}{2}\right)^{1-\frac{\lambda}{2}} I_{-\left(1-\frac{\lambda}{2}\right)}\{z(x-\xi)\} - \frac{1}{(x-\xi)^{2-\lambda}}\right] f(\xi)\,d\xi \end{aligned}\right\} \quad (6b)$$

5. In a similar way we find by starting from

$$\varphi(y) = \Gamma(1-\lambda)\,(1+z^2y^2)^{-1+\frac{\lambda}{2}} = \frac{\Gamma(1-\lambda)}{\Gamma\left(1-\frac{\lambda}{2}\right)} \sum_0^\infty (-1)^m \frac{\Gamma\left(m+1-\frac{\lambda}{2}\right)}{m!} (zy)^{2m}$$

and

$$\frac{1}{\varphi(y)} = \frac{1}{\Gamma(1-\lambda)} (1+z^2y^2)^{1+\frac{\lambda}{2}} = \frac{1}{\Gamma(1-\lambda)\,\Gamma\left(\frac{\lambda}{2}-1\right)} \sum_0^\infty (-1)^n \frac{\Gamma\left(n+\frac{\lambda}{2}-1\right)}{n!} (zy)^{2n}$$

successively

$$\psi(x) = \Gamma\left(\frac{1-\lambda}{2}\right)\frac{z}{2}\cdot\left(\frac{2x}{z}\right)^{\frac{1-\lambda}{2}} I_{-\frac{1+\lambda}{2}}(zx),$$

$$\sigma(x) = \frac{\sin\lambda\pi}{\pi}\cdot\frac{1}{x^{1-\lambda}} - (2-\lambda)\,\Gamma\left(\frac{\lambda+1}{2}\right)\frac{\sin\lambda\pi}{\pi}\sum_1^\infty (-1)^n \frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma\left(n+\frac{\lambda+1}{2}\right)}\cdot\frac{x^{2n+\lambda-1}}{2n+\lambda-2}$$

We can again notice here that for $z=0$ the special forms appear as with ABEL's problem.

Substitution in $(1a)$ and $(1b)$ furnishes the integralequation

$$f(x) = \Gamma\left(\frac{1-\lambda}{2}\right)\left(\frac{z}{2}\right)^{\frac{1+\lambda}{2}} \int_a^x (x-\xi)^{\frac{1-\lambda}{2}} I_{-\frac{1+\lambda}{2}}\{z(x-\xi)\}\,u(\xi)\,d\xi, \quad (7a)$$

with its solution;

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$$u(x)=\frac{\sin\lambda\pi}{\pi}\int_a^x\frac{f'(\xi)}{(x-\xi)^{1-\lambda}}\,d\xi-$$

$$-\frac{\sin\lambda\pi\,(2-\lambda)}{\pi}\,\Gamma\left(\frac{\lambda+1}{2}\right)\int_a^x f'(\xi)d\xi\,.\left\{\sum_1^\infty(-1)^n\frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma\left(n+\frac{\lambda+1}{2}\right)}\cdot\frac{(x-\xi)^{2n+\lambda-1}}{2n+\lambda-2}\right\},$$

of which the last integral can be brought by partial integrating into the form:

$$\int_a^x f(\xi)\,d\xi\,.\left\{\sum_1^\infty(-1)^n\frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma\left((n+\frac{\lambda+1}{2}\right)}\cdot\frac{(2n+\lambda-1)(x-\xi)^{2n+\lambda-2}}{2n+\lambda-2}\right\}=$$

$$=\int_a^x f(\xi)\,d\xi\,.\left\{\sum_1^\infty(-1)^n\frac{\left(\frac{z}{2}\right)^{2n}(x-\xi)^{2n+\lambda-2}}{n!\,\Gamma\left(n+\frac{\lambda+1}{2}\right)}\right\}+$$

$$+\int_a^x f(\xi)\,d\xi\,.\left\{\sum_1^\infty(-1)^n\frac{\left(\frac{z}{2}\right)^{2n}}{n!\,\Gamma\left(n+\frac{\lambda+1}{2}\right)}\cdot\frac{(x-\xi)^{2n+\lambda-2}}{2n+\lambda-2}\right\};$$

by partial integrating the last part we find for it, if we put: $f^{(-1)}(x)=\int f(\xi)\,d\xi$:

$$\int_a^x f^{(-1)}(\xi)\,d\xi\,.\left\{\sum_1^\infty(-1)^n\frac{\left(\frac{z}{2}\right)^{2n}(x-\xi)^{2n+\lambda\ 3}}{n!\,\Gamma\left(n+\frac{\lambda+1}{2}\right)}\right\}.$$

Summarizing we arrive at the following form for the solution of (7*a*):

$$\left.\begin{aligned}u(x)=&\frac{\sin\lambda\pi}{\pi}\int_a^x\frac{f'(\xi)}{(x-\xi)^{1-\lambda}}d\xi-\\&-\frac{\sin\lambda\pi\,(2-\lambda)}{\pi}\int_a^x\left[\Gamma\left(\frac{\lambda+1}{2}\right)\left(\frac{z}{2}\right)^{\frac{1-\lambda}{2}}(x-\xi)^{-\frac{3-\lambda}{2}}I_{-\frac{1-\lambda}{2}}\{z(x-\xi)\}-\right.\\&\left.-\frac{1}{(x-\xi)^{2-\lambda}}\right]\left\{f(\xi)+\frac{f^{(-1)}(\xi)}{x-\xi}\right\}d\xi\end{aligned}\right\}(7b)$$