SUMMARY.

1. We determined the progressive change of the acid formation from some aliphatic saturated acid anhydrides in presence of an excess of water at 0° and 25°.
2. In the case of the lower acid anhydrides including the butyric acids this proved to be a unimolecular reaction with a relative small temperature coefficient.
3. As from previous investigations it had appeared that the reaction constant is closely connected with the dissociation constant of the acids forming, it could be deduced, by eliminating this influence, that the hydratation constant decreases as the mass of the saturated group increases, and that the branching of the saturated carbon chain has little influence on this constant.
4. From the fall of the "constant" for the acid formation from *iso*valeric anhydride it was deduced that the formation of acid usually takes place in two phases: *a.* Absorption of water, *b.* splitting of the hydrate; that with the lower acid anhydrides the first reaction occurs very rapidly so that only the last unimolecular reaction gets measured; that in the case of the *iso*valeric anhydride the first reaction no longer takes place infinitely in regard to the second so that we must get the image of a follow-reaction with unequal reaction constants.

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Mathematics. — *"Bilinear congruences and complexes of plane algebraic curves."* By Prof. JAN DE VRIES.

1. We shall consider a doubly infinite system of plane curves of order n, consequently a congruence $[\gamma^n]$. We suppose that through an arbitrary point only *one* curve passes, and that an arbitrary straight line is cut in n points by only *one* curve. The congruence is in that case of the *first order*, and of the *first class*; we shall call it for the sake of brevity a *bilinear congruence.*

As a γ^n of the congruence is determined by a straight line r of its plane φ, all planes φ must pass through a fixed point F, which we shall call the *pole.*

A ray f passing through F (*polar ray*) bears ∞^1 planes φ; the curves γ^n lying in it form a surface Σ of order $(n+1)$, for any point of f lies on only one curve γ^n.

We consider now the surfaces Σ^{n+1}, belonging to the rays f and f'; they have in common the γ^n lying in the plane (ff'), and intersect further along a curve σ of order (n^2+n+1), which passes through F[1]).

Through a point S of σ pass two curves γ^n, the planes of which contain successively the straight lines f and f'. S is therefore a *singular point* and lies consequently in ∞^1 curves γ^n. The planes of these γ^n form the pencil with axis FS, the curves themselves lie on a Σ^{n+1}, which has a node in S, for a straight line passing through S meets Σ^{n+1} in $(n-1)$ points situated outside S.

Let f'' be an arbitrary ray through F, $s = FS$ a bisecant of the curve σ; γ^n in the plane $(f''s)$ passes through S. The surface Σ belonging to f'' contains therefore the curve σ and the latter is base-curve of the net which is formed by the ∞^2 surfaces Σ. The γ^n which is determined by an arbitrary point P, forms with σ the base of a pencil belonging to the net.

A γ^n can meet an arbitrary surface Σ^{n+1}, in singular points S only, consequently it rests in $n(n+1)$ points on the *singular curve* σ^{n^2+n+1}, while its plane cuts σ still in the pole F.

A bilinear congruence $[\gamma^n]$ consists of the curves γ^n, which cut a twisted curve of the order (n^2+n+1) in $(n+1)$ points, and send their planes through a fixed point of that curve[2]).

The curve σ may be represented by

$$\left\| \begin{matrix} a_x^n & b_x^n & c_x^n \\ \alpha_x & \beta_x & \gamma_x \end{matrix} \right\| = 0,$$

hence the $[\Sigma^{n+1}]$ by

$$\left| \begin{matrix} \lambda & \mu & \nu \\ a_x^n & b_x^n & c_x^n \\ \alpha_x & \beta_x & \gamma_x \end{matrix} \right| = 0,$$

and the congruence $[\gamma^n]$ by the relations

$$\varrho a_x^n + \sigma b_x^n + \tau c_x^n = 0, \qquad \varrho \alpha_x + \sigma \beta_x + \tau \gamma_x = 0.$$

2. The surface Σ formed by the γ^n, which rest in a singular

1) σ is of the rank $n(2n^2+n+1)$ and the genus $\frac{1}{2}n(n-1)(2n+1)$; it sends $\frac{1}{2}n^2(n^2+1)$ bisecants through one point.

2) For $n=2$ this has been pointed out by Montesano ("Su di un sistema lineare di coniche nello spazio", *Atti di Torino*, XXVII, p. 660—690). Godeaux arrived at the congruence $[\gamma^n]$ by inquiring into *linear* congruences of γ^n of the genus $\frac{1}{2}(n-1)(n-2)$, which possess *one* singular curve, on which the γ^n rest each in $n(n+1)$ points. ("Sulle congruenze lineari di curve piane dotate di una sola curva singolare", *Rend. di Palermo*, XXXIV, p. 288—300).

point S on σ, is cut in $(n+1)$ points by an arbitrary straigh line l; consequently σ is an $(n+1)$-fold curve on the surface Δ (the curves γ^n, which are cut by l. As two surfaces Δ apart fron σ can only have in common a number of γ^n, which agrees wit the order of Δ, we have for the determination of that order x the relatio

$$x^2 = nx + (n+1)^2 (n^2+n+1);$$

from which ensues $x = (n+1)^2$.

The γ^n resting on a straight line l form a surface of orde $(n+1)^2$ on which the γ^n, of which the plane passes through l, is a n-fold curve; the singular curve is $(n+1)$-fold.

Δ is cut $n(n+1)^2$ times by an arbitrary γ^n of the congruence from this appears again that γ^n rests in $n(n+1)$ points on σ.

Two arbitrary straight lines are cut by $(n+1)^2$ curves of th congruence.

A plane φ passing through l intersects Δ moreover along a curve which is apparently cut $n(n-1)$ times on l by the γ^n, of whicl the plane passes through l; in each of the remaining $(n+1)^2-1-n(n-1) = 3n$ points φ is touched by a γ^n.

The curves γ^n, which touch a given plane have their points o contact on a curve of order $3n$, which possesses (n^2+n+1) double points

The last mentioned observation ensues from the fact that th surface Σ^{n+1}, which has a node in a singular point S, is cut by φ along a curve with node S; φ is therefore touched in S by two γ^n

The curve φ^{3n} found just now is the locus of the coincidences o the involution formed from collinear sets of n points in which φ i cut by $[\gamma^n]$.

3. The surface Δ belonging to an arbitrary straight line, no lying in φ, has apart from the (n^2+n+1) points S $3n(n+1)^2 - 2(n+1)(n^2+n+1) = (n+1)(n^2+n-2) = (n+2)(n^2-1)$ points ir common with φ^{3n}.

There are $(n+2)(n^2-1)$ curves in $[\gamma^n]$, which touch a giver plane, and at the same time cut a given straight line.

We can arrive at the last mentioned result in an other way yet The surface Σ^{n+1}, which contains the γ^n, the planes of whicl pass through a polar ray f, is cut by a straight line l in $(n+1)$ points; so the planes of $(n+1)$ curves γ^n pass through f, whicl curves rest on l. Consequently the planes of the γ^n lying on Δ envelop a *cone of class* $(n+1)$.

A plane φ cuts Σ^{n+1} along a curve φ^{n+1}, which passes through the point of intersection of f, and sends $(n+1)n-2 = (n+2)(n-1)$

tangents through that point. From this follows that the planes of the γ^n, touching φ, envelop a *cone of class* $(n+2)(n-1)$.

Each common tangent plane of the two cones, contains a γ^n, which cuts l and touches φ; for the number of those curves we find therefore again $(n+2)(n^2-1)$.

The two cones of class $(n+2)(n-1)$, which are enveloped by the planes of the γ^n, which touch two given planes have $(n+2)^2(n-1)$ tangent planes in common. *As many curves γ^n consequently touch two given planes.*

4. A surface Σ^{n+1}, belonging to the polar ray f, contains a number of γ^n with a node; such a γ^n is the intersection of Σ with a tangent plane passing through f.

In order to determine the number of those planes, we consider the points which Σ outside f, has in common with the polar surfaces α^n and β^n of two points A and B lying on f. A plane φ passing through f cuts these surfaces along two curves a^{n-1} and b^{n-1}, which cut f in two groups of $(n-1)$ points A_k and B_k. If φ is made to revolve round f, these sets of $(n-1)$ points describe two projective involutions so that a correspondence $(n-1, n-1)$ arises on f. In each coincidence C, f is cut by two curves a^{n-1}, b^{n-1} lying in the same plane φ; there α^n and β^n have therefore the same tangent plane which contains at the same time the tangent of the curve φ of the order (n^2-1), which α^n and β^n have in common, apart from f.

The $2(n-1)$ points C are at the same time the coincidences of the involution of the n^{th} degree, which is determined on f by the curve γ^n, out of which Σ is built up; in each point C, Σ is therefore touched by the plane φ and moreover by the curve φ. Consequently ϱ has on f $4(n-1)$ points in common with Σ, the number of intersecting points of φ and Σ lying outside f amounts therefore to $(n^2-1)(n+1) - 4(n-1) = (n-1)^2(n+3)$. [1]

Through each polar ray f pass consequently the planes of $(n-1)^2(n+3)$ nodal curves γ^n_δ.

The planes of the nodal curves γ^n_δ envelop a *cone of class* $(n-1)^2(n+3)$; the planes of the γ^n, which rest on a straight line l, envelop a cone of class $(n+1)$. *From this follows that the nodal curves γ^n_δ form a surface Δ of order* $(n+3)(n+1)(n-1)^2$.

On a straight line f lie $n(n-1)^2(n+3)$ points of the nodal curves γ^n_δ, of which the planes pass through f; in the pole F the surface Δ is cut by f in $(n+3)(n-1)^2$ points.

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[1]) For $n+1=3$, we duly find the five pairs of lines which rest on a straight line of a cubic surface.

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Let S be a point of the singular curve σ; the ray FS is cut in S by the $(n+3)(n-1)^2$ curves γ_δ^n, of which the planes pass through FS.

In connection with what was mentioned above we may therefore conclude that the singular curve σ is $(n+3)(n-1)^2$-fold on the surface Δ.

5. If all γ^n pass through the pole F, so that the latter is a *fundamental point* of the congruence, then all surfaces Σ^{n+1} have a node in F. Two surfaces have four points in F in common in that case; one of them belongs to the γ^n, which forms part of the intersection, consequently the *singular curve* σ has now a *triple point* in F. In an arbitrary plane φ passing through F the two Σ have $(n+1)^2-4$ points in common, apart from F, $(n-1)$ of those points lie on the common γ^n, the remaining (n^2+n-2) on σ.

In those points σ is cut by the curve of the congruence lying in φ. The curves γ^n consequently pass through the triple point of the singular curve, and rest moreover in $(n+2)(n-1)$ other points on it.

Any plane passing through a tangent t_k in F to σ contains a γ^n, which touches t_k in F. In the plane passing through two of those tangents lies therefore a γ_δ^n, which has a node in F. Each of the three bitangent planes of σ which are determined by the three tangents in F contains therefore a γ_δ^n with node F.

The quadric cones of contact in F of the surfaces of the net $[\Sigma^{n+1}]$ form apparently a net which has as base edges the three tangents of the singular curve σ. To that net belongs the figure consisting of the plane $t_k\, t_l$ with an arbitrary plane passing through t_m; so the net $[\Sigma^{n+1}]$ contains three systems of surfaces, which have a biplanar point in F; the edge of the pair of planes into which the cone of contact degenerates lies in one of the three planes $t_k\, t_l$.

6. We shall now consider a triply infinite system of plane algebraic curves γ^n, which form a *bilinear complex* $\{\gamma^n\}$ [1]). In an arbitrary plane lies therefore *one* γ^n, and the curves γ^n, which pass through a point P, lie in the planes of a *pencil* (cone of the first class); the axis p of that pencil we shall call for the sake of brevity, the *axis of P*.

The curves of $\{\gamma^n\}$, of which the planes pass through an arbitrary straight line r form apparently a surface of order $(n+1)$, which we

[1]) The bilinear complexes of conics have been fully treated by D. Montesano ("I complessi bilineari di coniche nello spazio", *Atti R. Acc. Napoli,* XV, ser. *2a,* n°. 8).

shall indicate by Σ_r^{n+1}. Through a point P of r passes only one γ^n, namely the curve lying in the plane (pr).

The surface Σ_p^{n+1} belonging to an axis p has a node in P; for a line l passing through P cuts the γ^n of the plane (pl) in $(n-1)$ points lying outside P.

If r is made to revolve in a plane φ around a point O then Σ_r^{n+1} describes a *pencil*. In order to determine the surface Σ which passes through an arbitrary point P, we have only to find the ray r, which cuts the axis p of P. The base of this pencil consists of the curve γ^n lying in φ and a twisted curve φ^{n^2+n+1}, which cuts γ^n in $n(n+1)$ points.

Any point P of this curve lies on ∞^1 curves γ^n; its axes p must meet all the rays of the pencil (O, φ), consequently pass through O.

To the net of rays of the straight lines r, lying in φ, corresponds a *net* of surfaces Σ_r^{n+1}. Through two arbitrary points P, P' passes the surface belonging to the straight line r, which cuts the axis p,p'.

7. Let us now consider the surfaces of this net belonging to three straight lines, r, r', r'' of φ, which do not pass through one point. The curve ϱ^{n^2+n+1}, which two of these surfaces have in common, cuts the third surface in $(n+1)(n^2+n+1)$ points. To these points belong $n(n+1)$ points of the γ^n lying in φ.

Let H be one of the remaining $(n+1)(n^2+n+1) - (n+1)n = (n+1)(n^2+1)$ intersections. Through H pass the curves γ^n lying in the three planes which connect H with $r, r', r'_{,}$; these planes do not belong to a pencil, consequently H bears ∞^2 curves γ^n and is therefore a *cardinal point* (fundamental point) of the complex $\{\gamma^n\}$. Any straight line through H is apparently an axis and determines by means of its intersection with φ, a pencil (Σ^{n+1}), consequently a curve ϱ^{n^2+n+1}.

The complex $\{\gamma^n\}$ has $(n+1)(n^2+1)$ cardinal points; they are at the same time cardinal points of the complex of rays $\{p\}$ and of the complex of curves $\{\varrho^{n^2+n+1}\}$.

The cardinal points are apparently base points of the net $\{\Sigma^{n+1}\}$ belonging to the plane φ, or, more exactly expressed, of all the nets which are indicated by the planes φ in space.

8. Let us now consider the curves of $\{\gamma^n\}$ which send their planes through an arbitrary point F. Through a point P passes the γ^n of the plane (Fp); through a straight line r passes the plane (Fr) and this plane contains *one* γ^n. So we have set apart out of the complex a *bilinear congruence* $[\gamma^n]$ which has F as *pole*. Its polar rays are the axes p of the points P of the singular curve σ^{n^2+n+1}; they project this curve out of the pole F lying on it, consequently form a cone of order $n(n+1)$. From this follows that *the axes of* $\{\gamma^n\}$ *form a complex of rays of order* $n(n+1)$.

In any plane passing through a cardinal point H lies a γ^n, which passes through H. The ∞^2 γ^n passing through H form therefore a special congruence $[\gamma^n]$, which has H as fundamental point; the singular curve σ of this congruence has therefore a triple point in H (§ 5); it is the σ^k, which has H_k as pole.

Each point H is triple point of a singular curve σ, which passes through the remaining cardinal points.

This curve is base curve of a net of surfaces Σ, which have all a node in H.

The planes of the nodal curves $\gamma^n{}_\delta$ envelop a surface of class $(n-1)^2(n+3)$, for this is the number of tangent planes of $\Sigma_r{}^{n+1}$, which pass through a straight line r (§ 4).

The curves $\gamma^n{}_\delta$ form apparently a *congruence* of which the *order* and *class* are $(n-1)^2(n+3)$.

9. We now assume a tetrahedron of coordinates and consider the net of surfaces Σ belonging to the straight lines of the plane $x_4 = 0$. This net may then be represented by

$$\alpha \begin{vmatrix} a^n{}_x & d^n{}_x \\ x_1 & x_4 \end{vmatrix} + \beta \begin{vmatrix} b^n{}_x & d^n{}_x \\ x_2 & x_4 \end{vmatrix} + \gamma \begin{vmatrix} c^n{}_x & d^n{}_x \\ x_3 & x_4 \end{vmatrix} = 0.$$

The cardinal points are therefore found from

$$\left\| \begin{matrix} a^n{}_x & b^n{}_x & c^n{}_x & d^n{}_x \\ x_1 & x_2 & x_3 & x_4 \end{matrix} \right\| = 0.$$

From this ensues readily that the curves of the complex may be represented by the relations:

$$\alpha a^n{}_x + \beta b^n{}_x + \gamma c^n{}_x + \delta d^n{}_x = 0 \ , \ \alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 = 0.$$

If we consider here α, β, γ as given, but δ as variable, then there arises by elimination of δ the above mentioned equation of the surface Σ belonging to the straight line $x_4 = 0$, $\alpha x_1 + \beta x_2 + \gamma x_3 = 0$.

For the curves passing through a point Y is

$$\Sigma\, \alpha a^n_y \equiv \alpha a^n_y + \beta b^n_y + \gamma c^n_y + \delta d^n_y = 0 \ \text{ and } \ \Sigma\, \alpha y_1 = 0.$$

By elimination of $\alpha, \beta, \gamma, \delta$ out of these equations and $\Sigma\, \alpha a^n_x = 0$, $\Sigma\, \alpha x_1 = 0$, we find for the surface Σ^{n+1} belonging to Y, the equation

$$| \ y_1 \quad a^n_y \quad x_1 \quad a^n_x \ | = 0.$$

The *axis* of Y is indicated by

$$\| \ y_1 \quad a^n_y \quad x_1 \ \| = 0.$$

In order to determine the surface Σ^{n+1} belonging to the straight line which joins the points Y and Z, one has to eliminate $\alpha, \beta, \gamma, \delta$ out of $\Sigma\, \alpha y_1 = 0$, $\Sigma\, \alpha z_1 = 0$, $\Sigma\, \alpha x_1 = 0$ and $\Sigma\, \alpha a^n_x = 0$; then one finds

$$\left| \; y_1 \quad z_1 \quad x_1 \quad a^n_x \; \right| = 0,$$

while the straight line YZ is indicated by

$$\left\| \; y_k \quad z_k \quad x_k \; \right\| = 0.$$

Through the point X pass the axes of the points Y, for which we have

$$\begin{vmatrix} y_1 & a^n_y & x_1 \\ y_2 & b^n_y & x_2 \\ y_3 & c^n_y & x_3 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} y_2 & b^n_y & x_2 \\ y_3 & c^n_y & x_3 \\ y_4 & d^n_y & x_4 \end{vmatrix} = 0.$$

These surfaces of order $(n+1)$ have the curve

$$\left\| \begin{matrix} y_2 & b^n_y & x_2 \\ y_3 & c^n_y & x_3 \end{matrix} \right\| = 0$$

in common, which is of order n, but is not situated on the two other surfaces of order $(n+1)$, which are indicated by

$$\left\| \; y_1 \quad a^n_y \quad x_1 \; \right\| = 0$$

The last mentioned relations determine therefore a curve of order $(n^2 + n + 1)$ as locus of the points Y. From this ensues again that the axes form a complex of rays of order $n\,(n+1)$.

Mathematics. — "*A bilinear congruence of twisted quartics of the first species.*" By Prof. JAN DE VRIES.

1. As we know, we distinguish with congruences of algebraic twisted curves two characteristic numbers, called *order* and *class.*

The *order* indicates how many curves pass through an arbitrary point, the *class* the number of curves which have an arbitrarily chosen straight line as a bisecant If both numbers are o n e the congruence is called *bilinear*. In volume XVI of the *Rend. del Circ. mat. di Palermo* (p. 210) E. VENERONI has proved that there exist principally two kinds of bilinear congruences of twisted cubics. An analogous inquiry concerning congruences of twisted quartics of the first species, ϱ^4, has not been made till now. [1])

1) The bilinear congruences of conics have been treated by MONTESANO (Atti di Torino XXVII p. 660).