In order to determine the surface Σ^{n+1} belonging to the straight line which joins the points Y and Z, one has to eliminate $\alpha, \beta, \gamma, \delta$ out of $\Sigma\, \alpha y_1 = 0$, $\Sigma\, \alpha z_1 = 0$, $\Sigma\, \alpha x_1 = 0$ and $\Sigma\, \alpha a^n_x = 0$; then one finds

$$\left| \; y_1 \quad z_1 \quad x_1 \quad a^n_x \; \right| = 0,$$

while the straight line YZ is indicated by

$$\left\| \; y_k \quad z_k \quad x_k \; \right\| = 0.$$

Through the point X pass the axes of the points Y, for which we have

$$\begin{vmatrix} y_1 & a^n_y & x_1 \\ y_2 & b^n_y & x_2 \\ y_3 & c^n_y & x_3 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} y_2 & b^n_y & x_2 \\ y_3 & c^n_y & x_3 \\ y_4 & d^n_y & x_4 \end{vmatrix} = 0.$$

These surfaces of order $(n+1)$ have the curve

$$\left\| \begin{matrix} y_2 & b^n_y & x_2 \\ y_3 & c^n_y & x_3 \end{matrix} \right\| = 0$$

in common, which is of order n, but is not situated on the two other surfaces of order $(n+1)$, which are indicated by

$$\left\| \; y_1 \quad a^n_y \quad x_1 \; \right\| = 0$$

The last mentioned relations determine therefore a curve of order $(n^2 + n + 1)$ as locus of the points Y. From this ensues again that the axes form a complex of rays of order $n\,(n+1)$.

Mathematics. — "*A bilinear congruence of twisted quartics of the first species.*" By Prof. JAN DE VRIES.

1. As we know, we distinguish with congruences of algebraic twisted curves two characteristic numbers, called *order* and *class*.

The *order* indicates how many curves pass through an arbitrary point, the *class* the number of curves which have an arbitrarily chosen straight line as a bisecant If both numbers are o n e the congruence is called *bilinear*. In volume XVI of the *Rend. del Circ. mat. di Palermo* (p. 210) E. VENERONI has proved that there exist principally two kinds of bilinear congruences of twisted cubics. An analogous inquiry concerning congruences of twisted quartics of the first species, ϱ^4, has not been made till now. [1])

[1]) The bilinear congruences of conics have been treated by MONTESANO (Atti di Torino XXVII p. 660).

In a communication which appeared in Volume XIV of these *Proceedings*, I have (p. 255) considered the bilinear congruence $[\varrho^4]$, which arises if the quadrics of two pencils are made to intersect. [1])

It is not difficult to understand that no bilinear congruences of curves of a higher order can be produced by two pencils of surfaces. For, if these pencils are of the degrees m and n, they intersect an arbitrary line in two involutions of the degrees m and n and these have in common $k=(m-1)(n-1)$ pairs; so we find a congruence $[\varrho^{mn}]$ of the first order, and the class $(m-1)(n-1)$; only for $m=n=2$ we find $k=1$.

2. In order to arrive at another group of bilinear congruences, I consider a net of cubic surfaces $[\Phi^3]$. Through an arbitrary point P pass ∞^1 surfaces Φ^3, which form a pencil included in the net, of which pencil the base curve in the general case will be a twisted curve ϱ^9 of genus 10. All the curves ϱ^9 included in the net consequently form a congruence of order *one*. On an arbitrary line the net determines a cubic involution of the second rank; the latter possesses as we know a neutral pair N_1, N_2; all the ϱ^9 through N_1 pass through N_2 as well, consequently the congruence is also of the first *class*, therefore *bilinear*.

If all the Φ^3 have a curve in common, the curves ϱ^9 degenerate into an invariable and a variable part, and a bilinear congruence of curves of a lower order is found. We shall now consider the case in which we have to do with a congruence $[\varrho^4]$.

3. Let ϱ^5 be a twisted curve of order five, and let the genus be 2, so the remaining section of a Φ^3 and a Φ^2, which have a straight line in common. Any surface Φ^3 passing through 14 points of ϱ^5 contains this curve [2]); consequently the Φ^3 passing through ϱ^5 and three arbitrarily chosen points H_1, H_2, H_3, form a net. Two of these surfaces have besides ϱ^5, a ϱ^4 of the 1st species in common, which intersects ϱ^5 in *eight* points [3]). With a third Φ^3, ϱ^4 has 12 points in common, of which 8 lie on ϱ^5, the other four, and to them belong of course H_1, H_2 and H_3 lie apparently on all Φ^3, therefore on all ϱ^4.

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[1]) If the bases of the two pencils have a straight line in common, one of the two congruences $[\varrho^3]$ found by VENERONI arises.

[2]) R. STURM, *Synthetische Untersuchungen über Flächen dritter Ordnung* (1867, p 234). P. H. SCHOUTE, *La courbe d'intersection de deux surfaces cubiques et ses dégénerations* (Archives Teyler 1901, t. VII, p. 219). M. STUYVAERT, *Cinq études de géométrie analytique* (Mem. Soc. Liége, 1907, t. VII, p. 40).

[3]) SCHOUTE, (l. c. p. 241), STUYVAERT, (l. c. p. 41).

Here we have consequently *a bilinear congruence* $[\varrho^4]$ with *four cardinal points* H_k and a *singular curve* ϱ^5; i.e. all ϱ^4 pass through the four cardinal points and rest in 8 points on ϱ^5 [1]).

4. Let t be a trisecant of ϱ^5; the pencil of net surfaces determined by a point of t has for base the complex of ϱ^5, t and a plane cubic γ^3, which has a point T with t in common, and 5 points with ϱ^5. This γ^3 must contain the four cardinal points H; *consequently the cardinal points are situated in a plane* φ.

Any curve γ^3 connects the 4 cardinal points and the 5 points R_k, in which ϱ^5 cuts the plane φ, with the intersecting point T of the trisecant belonging to it. As the trisecants form the quadratic ruled surface Φ^2, on which ϱ^5 lies, the points R, together with T may be connected by a conic τ^2.

The curves γ^3 form a pencil with base (R_k, H_k); any γ^3 intersects τ^2 in the point T, through which the straight line t passes, which, considered together with γ^3 belongs to the congruence $[\varrho^4]$ [2]).

The locus of the degenerate figures $(\gamma^3 + t)$ is apparently the complex of Φ^2 and φ, and consequently belongs to the net $[\Phi^3]$.

5. Let b be one of the four bisecants of ϱ^5, which pass through the cardinal point H_k. All the Φ^3 which contain b, have moreover a ϱ^3 in common, which has b as bisecant and rests in 6 points on ϱ^5.

Consequently there are *sixteen figures* $(\varrho^3 + b)$ in $[\varrho^4]$.

A third group of complex figures is formed by pairs of conics (α^2, β^2). Let α^2 be a conic passing through H_1, H_2, intersecting ϱ^5 in 4 points, the Φ^3 passing through α^2 and ϱ^5 have an other conic β^2 in common, which intersects α^2 in 2 points, ϱ^5 in 4 points and passes through H_3, H_4

The number of α^2 we deduce using the law of permanency of the number. We replace ϱ^5 by the complex of a σ^3 and a σ^2, which have three points in common; through a point P pass consequently 3 straight lines, which rest on σ^3 and σ^2; with the bisecant of σ^3 they form the 4 straight lines which replace the 4 bisecants of ϱ^5; consequently $(\sigma^3 + \sigma^2)$ is to be considered as a degeneration of ϱ^5. In any plane passing through H_1 and H_2 lies a conic φ^2 connecting these points with 3 points of σ^3; as the straight line H_1H_2 cannot

1) If the base of the net consists of a curve ϱ^6, of genus 3, and a cardinal point H, the second bilinear congruence $[\varrho^3]$ is formed.

2) That the figure $(\gamma^3 + t)$ is a special case of a ϱ^4, appears from the fact that through an arbitrarily chosen point P, two straight lines may be drawn which intersect γ^3 and t; they replace the bisecants which ϱ^4 sends out through P.

apparently be a part of a degenerate φ^2, the φ^2 form a quadric. This is cut by σ^2 in 4 points; among them are the 3 common points of σ^3 and σ^2; through the fourth intersecting point passes a φ^2, which has four points in common with the figure $(\sigma^3 + \sigma^2)$.

From this we conclude that *one* conic α^2 can be drawn through H_1 and H_2. As each α^2 is coupled with a β^2 (which passes in that case through H_3 and H_4), $[\varrho^4]$ contains *three figures* $(\alpha^2 + \beta^2)$.

6. Through a point S of the singular curve ϱ^5 pass ∞^1 curves ϱ^4. They cut the plane φ in the points H. To this system of ϱ^4 belongs, however, also the figure consisting of the trisecant t passing through S and a γ^3 lying in φ. From this ensues that the locus of the ϱ^4 meeting in S, is a cubic surface Σ^3, passing through ϱ^5 and the points H, and consequently belongs to the net $[\Phi^3]$.

An arbitrary line passing through S, is a bisecant of *one* ϱ^4, and so intersects Σ^3, apart from S in *one* point. Consequently Σ^3 has a double point in S. Through S pass 6 straight lines of Σ^3, one of them is of course the t mentioned before; each of the remaining 5 is a bisecant p of ∞^1 curves ϱ^4, so a *singular bisecant.*

All the ϱ^4 intersecting p twice pass through S; so they determine on p a parabolic involution, of which all pairs have the point S in common; we shall call p a *singular bisecant of the first species.*

Through each point of ϱ^5 pass therefore five singular bisecants of the first species.

Any line h passing through a cardinal point H is as well a singular bisecant of the first species.

The monoids Σ^3 having two points of ϱ^5 as double points, intersect apart from ϱ^5 in a ϱ^4. Through any two points S passes therefore only *one* curve of the congruence.

7. Let q be a bisecant of a ϱ^5, and at the same time a secant of ϱ^4. The surface Φ^3 passing through ϱ^5 and ϱ^4 and a point of q contains q, and belongs to the net $[\Phi^3]$. Consequently all Φ^3 passing through a point Q of q will cut this straight line moreover in a second point Q'. Consequently q is a bisecant of ∞^1 curves ϱ^4, and the pairs of the intersections Q, Q' form an involution. We call q a *singular bisecant of the second species.*

In order to find the number of lines q that pass through a point P, we consider the cubic cone k^3, which out of P projects the ϱ^4 containing P, and the cone k^5 which has P as vertex and ϱ^5 as curve of direction. To the 15 common generatrices belong the lines drawn to the eight intersecting points of ϱ^4 and ϱ^5. The remaining

7 are bisecants of ϱ^4 intersecting ϱ^5, therefore lines q. Consequently the lines q form a *congruence of order seven.*

We can also arrive at this result in another way. A straight line passing through P is generally speaking, a bisecant of *one* ϱ^4; we call R, R' its intersections with ϱ^4 and consider the surface π, which is the locus of the pairs R, R'. On any generatrix of the cone k^3 one of those points lies in P, hence π has in P a triple point with k^3 as tangent cone; π is consequently a surface of order 5. It passes through ϱ^5, and has nodes in the four cardinal points. For an arbitrary ϱ^4 has in common with π the intersections with the bisecants which it sends through P, and in 8 points of ϱ^5, so twice in each point H.

Now π^5 and k^3 have in common the ϱ^4 which passes through P; further they can, by reason of the definition of π, only have lines in common which contain ∞^1 pairs R, R' each. Therefore *eleven singular bisecants* pass through P. To these the four straight lines $h_k = PH_k$ belong; for through any point of PH_k passes a ϱ^4, which meets this straight line again in the cardinal point H_k, so that PH_k is a singular straight line of the first species (which, however, does not rest on ϱ^5, and consequently may not be interchanged with a straight line p). The remaining 7 singular bisecants passing through P are therefore straight lines q.

For a point S of ϱ^5 the surface π^5 degenerates, and consists of the monoid Φ^3 with node S and a quadratic cone, formed by the straight lines q, which intersect ϱ^5 in S.

In an arbitrary plane lie five points of ϱ^5, consequently 10 straight lines q; they belong therefore to a *congruence of rays of class ten.*

The singular bisecants of the second species form a congruence (7, 10), *which has ϱ^5 as a singular curve.*

The section of π^5 with a plane passing through P is a curve with a triple point, consequently of class 14, of its tangents 8 pass through P. Therefore the tangents of the curves ϱ^4 form a *complex of order eight.*

8. The ϱ^4 which intersect a given line l, form a surface Λ, of which we intend to determine the order $\varkappa$. Any monoid Φ^3 contains three ϱ^4, which intersect l, and rest in the vertex S on ϱ^5; consequently ϱ^5 is a triple curve of Λ.

The surfaces Λ, Λ' belonging to two lines l, l' have, besides the threefold curve ϱ^5 only the $\varkappa$ curves ϱ^4 in common, resting on l and l'. So we have the relation $\varkappa^2 = 4\varkappa + 3^2.5$, hence $\varkappa = 9$.

On Λ^9 lies one trisecant t; for the curve γ^3, which intersects l, determines on τ^2 the point T of the trisecant with which it forms a degenerate ϱ^4 (§ 4).

The curve $\varrho^4{}_l$, which has l as a bisecant belongs to two points of l, and is consequently a twofold curve of Λ^9.

The locus of the ϱ^4 intersected by l is therefore a *surface of order nine* with a *twofold curve* $\varrho^4{}_l$, a *triple curve* ϱ^5 and two straight lines l and t.

9. A plane through l intersects Λ^9 in a curve λ^8; the latter has the two intersections of $\varrho^4{}_l$ and six points R in common with l; in each point R, λ is touched by a ϱ^4.

The points in which a plane is touched by curves ϱ^4 lie therefore on a curve γ^6; it is the *curve of coincidences* of the *quadruple involution* Q^4, in which the plane λ is intersected by the congruences $[\varrho^4]$.

The five intersections S_k of ϱ^5 with λ are apparently *singular points* of Q^4; to S_k are namely conjugated ∞^1 triplets of points, lying on the cubic curve $\sigma^3{}_k$, with double point S_k, in which the monoid Φ^3 (with vertex S_k) is intersected by λ. In S_k λ is therefore touched by two ϱ^4; the *curve of coincidences* γ^6 has consequently nodes in each of the five points S_k, and in S_k the same tangents as $\sigma_k{}^3$.

Any point D of the conic δ^2 through S_k is the intersection of a trisecant t, consequently determines a quadruple, of which the remaining three points are produced by the intersection of the curve γ^3 coupled with t. On the section f of φ we have therefore a cubic involution F^3, of which the groups are completed into quadruples of Q^4 by the points D. It is evident that Q^4, as long as λ remains an arbitrary plane, cannot possess any other collinear triplets.

In each of the points of intersection T_1, T_2 of f with τ^2 (§ 4) a t is cut by a γ^3, consequently these points are coincidences of the Q^4. The remaining coincidences, lying on f, belong to the involution F^3, from this appears again that the order of the curve of coincidences is *six*.

As the singular point S_1 lies on δ^2 and therefore may be considered as a point D, the curve $\sigma_1{}^3$ is intersected by f in a triplet of the cubic involution $I_1{}^3$, of which the groups are completed into quadruples of Q^4 by S_1. As $I_1{}^3$ cannot possess a second collinear triplet, it is not a central involution; so it can be determined in ∞^1 ways by a pencil of conics of which the base points are S_1, an arbitrary point of $\sigma_1{}^3$, and moreover two points of the line f.

10. Any coincidence of the Q^4 is completed into a quadruple by two *complementary* points. The locus σ of those points which we shall call the *complementary curve* has apparently *quadruple points* in S_k; for I_k^3 has four coincidences. Of the four coincidences of F^3, four of the complementary points lie on σ^2; with this conic the curve σ has therefore $4 + 5 \times 4 = 24$ points in common. Consequently the *complementary curve* is of order 12.

The curves ϱ^4, which touch the plane λ in the points of the curve of coincidences γ^6, intersect λ moreover on the complementary curve σ^{12}; so they form a surface of order 24, which passes eight times through the curve ϱ^5.

This surface is intersected by a plane λ' along a curve of order 24 with 5 octuple points S_k. As the curve of coincidences γ'^6 lying in λ' has double points in S_k the two curves outside S_k have $24 \times 6 - 5 \times 8 \times 2 = 64$ points in common. Consequently there are 64 curves ϱ^4, *touching two given planes.*

The surface Λ^9 belonging to the straight line l intersects an arbitrary plane φ along a curve φ^9, which has 5 triple points on ϱ^5. As the curve of coincidences φ^6 lying in φ has 5 nodes on ϱ^5, it intersects φ^9 moreover in $9 \times 6 - 5 \times 3 \times 2 = 24$ points. From this appears once more that the curves ϱ^4, which touch a given plane, form a *surface of order* 24. At the same time, the fact that the complementary curve is of order 12, is confirmed.

Chemistry. — *"Equilibria in ternary systems".* XII. By Prof. SCHREINEMAKERS.

We have seen in the previous communication that the saturation-curve under its own vapour-pressure of the temperature T_H (the point of maximumtemperature of the binary system $F + L + G$) is either a point [fig. 5 (XI)] or a curve [fig. 6 (XI)]. We shall now examine this case more in detail.

If we calculate $\frac{dy}{dx}$ for this curve in the point H from (6) and (7) (XI), then we find an infinitely great value. The curve going through H in fig. 6 (XI) and the curve disappearing in H of figure 5 (XI) come in contact, therefore, in H with the side BC. Now we take a temperature somewhat lower than T_H. The saturationcurve under its own vapour-pressure terminates then in two points n and h situated on different sides of and very close to H. [n and h in fig. 4—6 (XI) may be imagined very close to H.] As the saturationcurve