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In our concluding paper something will be said about the temperature influence, which will manifest itself by continual diminution of $b_k - b_0$, at first slowly, then more rapidly, as the absolute zero is approached. Descending from high to low temperatures one can therefore pass through all the types. If the critical region of a substance lies in the region of low temperatures, the critical quantities, and also the isotherms in the neighbourhood of the critical point, will present, as far as the course of b is concerned, the little variable type with slight $b_k - b_0$ (γ in the neighbourhood of 0,5). But these same substances will of course show the same variability of b as the "ordinary" substances at high temperatures. Reversely the ordinary substances, considered at low temperatures, will assume the Argon-, Hydrogen- or Helium-type, with respect to the slight variability of b at these temperatures. Etc. Etc.

In this concluding paper I shall also communicate the b-values for Argon I have calculated; besides I shall venture to give some theoretical considerations concerning the diminution of the factor 4 in $b_q = 4m$ with fall of the temperature.

Fontanivent sur Clarens, February 1914.

Mathematics. — *"The envelope of the osculating ellipses, which are described by the representative point of a vibrating mechanism having two degrees of freedom of nearly equal frequencies."* By H. J. E. BETH. (Communicated by Professor D. J. KORTEWEG).

(Communicated in the meeting of February 28, 1914).

§ 1. In my paper on the small oscillations of mechanisms with two degrees of freedom [1]), LISSAJOUS curves with their envelopes were discussed, which envelopes form the boundaries of the domain of motion. In a summarizing treatment of a more general problem [2]) my further inquiries as to these envelopes have also been included. These inquiries were extended over a system of LISSAJOUS curves, more general than the system which is of importance for the dynamical problem. However the envelopes were considered exclusively from a dynamical point of view, so that purely geometrical properties together with the shape of the curves outside the domain of motion remained unknown. Moreover what came to light about the shape of the envelope remained for the greater part restricted to simple cases, e.g. in the case, formerly indicated by $S=2$, to the symmetrical case, as quoted, indicated by $p+q=0$, $l=0$.

1) These Proceedings pp. 619—635 and 735—750 (1910).

2) Phil. Mag., sixth series Number 152 (1913).

In what follows for $S=2$, the case of the equality of frequencies, the envelope will be treated anew and from a more geometrical point of view.

We shall therefore have to occupy ourselves with the envelope (L) of the system of ellipses

$$x=\sqrt{\zeta}\cos t \quad , \quad y=\sqrt{1-\zeta}\cos(t-\varphi),$$

in which ζ and φ are two variable quantities in the most general case connected by the relation:

$$\sqrt{\zeta(1-\zeta)}\cos\varphi=-\frac{1}{2}l\pm\sqrt{p\zeta^2+q\zeta+r+\frac{1}{4}l^2}.$$

By elimination of t we find for the equation of the ellipses

$$(A) \qquad (1-\zeta)x^2-2\sqrt{\zeta(1-\zeta)}\cos\varphi\,.\,xy+\zeta y^2=\zeta(1-\zeta)\sin^2\varphi.$$

Let us now determine the reciprocal polar curves of these ellipses with respect to the circle:

$$(C) \qquad x^2+y^2=1,$$

in the circumference of which circle the vertices are situated of the rectangles, which are circumscribed to the ellipses (A), and have their sides parallel with the axes. The envelope (L') of this new system of ellipses will be the reciprocal polar curve of the envelope (L) wanted. (Cf. note p. 943).

The new system of ellipses appears to be given by:

$$(A') \qquad \zeta x^2+2\sqrt{\zeta(1-\zeta)}\cos\varphi\,.\,xy+(1-\zeta)y^2=1.$$

By elimination of φ between this equation and the given relation between ζ and φ we find:

$$4\left(p\zeta^2+q\zeta+r+\frac{1}{4}l^2\right)x^2y^2=\{1-\zeta x^2+lxy-(1-\zeta)y^2\}^2.$$

This equation contains ζ to a no higher order than two. The equation of the envelope of (A') may consequently be written down at once. After some reduction this equation of (L') becomes:

$$-(4r+l^2)(y^2-x^2)^2-4p(1+lxy-y^2)^2+4p(4r+l^2)x^2y^2=$$
$$=4q^2x^2y^2-4q(1+lxy-y^2)(y^2-x^2).$$

As (L') is now apparently of the fourth order, *the envelope (L) wanted is of the fourth class.*

As (L') in general as we shall see, has no double points or cusps, it has been determined by this, that (L) *is of the twelfth order.*

(L), like (L'), has the origin as centre.

If we multiply the equation found for (L') by p, (the cases $p=0$ and $p=\infty$ we shall consider separately in § 9) it appears that it may be written thus:

(L') $\{qx^2+2plxy-(2p+q)y^2+2p\}^2 = s\{x^4-2(2p+1)x^2y^2+y^4\}$,
where

$$s \equiv q^2 - p\,(4r + l^2).$$

§ 2. It is evident from the equation just found that out of the origin 4 bitangents may be drawn to (L'), given by

$$x^4 - 2\,(2p+1)\,x^2y^2 + y^4 = 0.$$

The 8 points of contact are lying on the conic

$$(K) \qquad qx^2 + 2plxy - (2p+q)y^2 + 2p = 0.$$

The bitangents are real or imaginary, according to p being positive or negative. They form two pairs of perpendicular lines, lying symmetrically with regard to the axes and with regard to the straight lines that bisect the angles of the axes.

If (K) has its axes along the axes of coordinates or along the bisectrices, then (L') and consequently (L) as well will have those lines as lines of symmetry. The first occurs for $l = 0$, the second for $p + q = 0$. These two suppositions consequently give rise to the same simplification in the shape of (L). In the formerly amply discussed case that $l = 0$ as well as $p + q = 0$, (K) becomes a circle with $\sqrt{2}$ as radius.

§ 3. *Nodes of* (L). Let us write the equation of (L') found in § 1 in the shape

$$U^2 = MN,$$

in which

$$U = \frac{qx^2 + 2plxy - (2p+q)y^2 + 2p}{\sqrt{s}}$$

and M and N are expressions of the second order, obtained by separation of the expression $x^4 - 2(2p+1)x^2y^2 + y^4$, then we see, that (L') is touched in 4 points by each conic of the system,

$$\lambda^2 M + 2\lambda U + N = 0,$$

in which λ represents a parameter.

The separation of the expression mentioned, may be executed in the following ways:

$$\begin{aligned} x^4 - 2(2p+1)x^2y^2 + y^4 &= (x^2 + 2\sqrt{p}\,xy - y^2)(x^2 - 2\sqrt{p}\,xy - y^2) \\ &= (x^2 + 2\sqrt{p+1}\,xy + y^2)(x^2 - 2\sqrt{p+1}\,xy + y^2) \\ &= \{x^2 - (\sqrt{2p+1} - 2\sqrt{p(p+1)})y^2\}\{x^2 - \\ &\quad -(\sqrt{2p+1} + 2\sqrt{p(p+1)})y^2\}. \end{aligned}$$

The first way of separation leads to the following system of inscribed conics

$$(\lambda^2 + 2\frac{q}{\sqrt{s}}\lambda + 1)x^2 + 2(\lambda^2 + l\sqrt{\frac{p}{s}}\lambda - 1)xy - (\lambda^2 + \\ + 2\frac{2p+q}{\sqrt{s}}\lambda + 1)y^2 = -\frac{4p}{\sqrt{s}}\lambda.$$

The values of λ, other than 0 or ∞, for which this equation represents a degeneration, viz. a degeneration into two parallel straight lines, are determined by the equation:

$$(\lambda^2 + l\sqrt{\frac{p}{s}}\lambda - 1)^2 + (\lambda^2 + 2\frac{q}{\sqrt{s}}\lambda + 1)(\lambda^2 + 2\frac{2p+q}{\sqrt{s}}\lambda + 1) = 0.$$

Each of the straight lines of a degeneration touches (L') in two points, is therefore a bitangent of (L'). If we write the equation of such a straight line in the shape

$$ax + by = 1,$$

then we see easily that we have

$$a^2 + b^2 = 1,$$

i.e. the 4 pairs of parallel bitangents touch (C).

We may observe that the system of conics to which we have arrived is the system of ellipses (A') itself, which is apparent, if we replace the parameter ζ by λ, in such a way that:

$$4p\zeta^2 + 4q\zeta + 4r + l^2 = \frac{q^2 - 4pr - pl^2}{4p}\left(\lambda - \frac{1}{\lambda}\right)^2.$$

Let us proceed now to the second way of separation. The equation of the second system of inscribed conics and the equation determining the degenerations may be written down. So we come again to 4 pairs of parallel bitangents of (L'); they appear to touch the hyperbola:

$$x^2 - y^2 = -\frac{p}{p+q}.$$

In the same way the third method of separating leads to 4 pairs of parallel bitangents of (L'), which touch the hyperbola

$$xy = -\frac{1}{2l}.$$

Hence:

Of the 28 bitangents which the envelope (L'), possesses 4 pass through O; the remaining ones are pairs of parallel lines; 8 of them touch the circle $x^2 + y^2 = 1$, 8 the hyperbola $x^2 - y^2 = -\dfrac{p}{p+q}$, and 8 the hyperbola $xy = -\dfrac{1}{2l}$.

We now transfer what we have found to (L):

61*

Of the 28 *nodes of the envelope* (L), 4 *are lying at infinity*, 8 *on the circumference of the circle* $x^2+y^2=1$, 8 *on the hyperbola* $x^2-y^2=-\frac{p+q}{p}$ *and* 8 *on the hyperbola* $xy=-2l$.

The 4 pairs of parallel asymptotes of (L), which correspond with the bitangents of (L') passing through O, touch the conic (K'), which is the reciprocal polar curve of (K).

The nodes of (L) lying on (C), if they are real, are for the dynamical problem under discussion the vertices of the quadrangular figures, which as appeared before, may serve as envelopes; the branches intersecting in those points meet perpendicularly, as was proved for a more general case [1]).

§ 4. *Asymptotes of* (L). Besides the 4 pairs of parallel asymptotes, (L) has moreover generally speaking 4 asymptotes passing through O, which are perpendicular to the asymptotes of (L').

Of (L') two asymptotical directions may coincide.

In this case the corresponding asymptotes do not pass through O, but they are removed from O at equal distances. In that case on the straight lines passing through O (L) has two cusps in which the straight line is a tangent. The said straight line is to be considered to belong to (L); consequently (L) is degenerate.

Various shapes of (L').

§ 5. The equation of (L') reads (§ 1):

$$\{qx^2+2plxy-(2p+q)y^2+2p\}^2=s\{x^4-2(2p+1)x^2y^2+y^4\},$$

where

$$s\equiv q^2-p(4r+l^2).$$

Its shape will in the first place be dependent on the nature of the bitangents drawn from O, viz. whether they are imaginary $(p<0)$, or real $(p>0)$ and touch the curve in real points or are isolated.

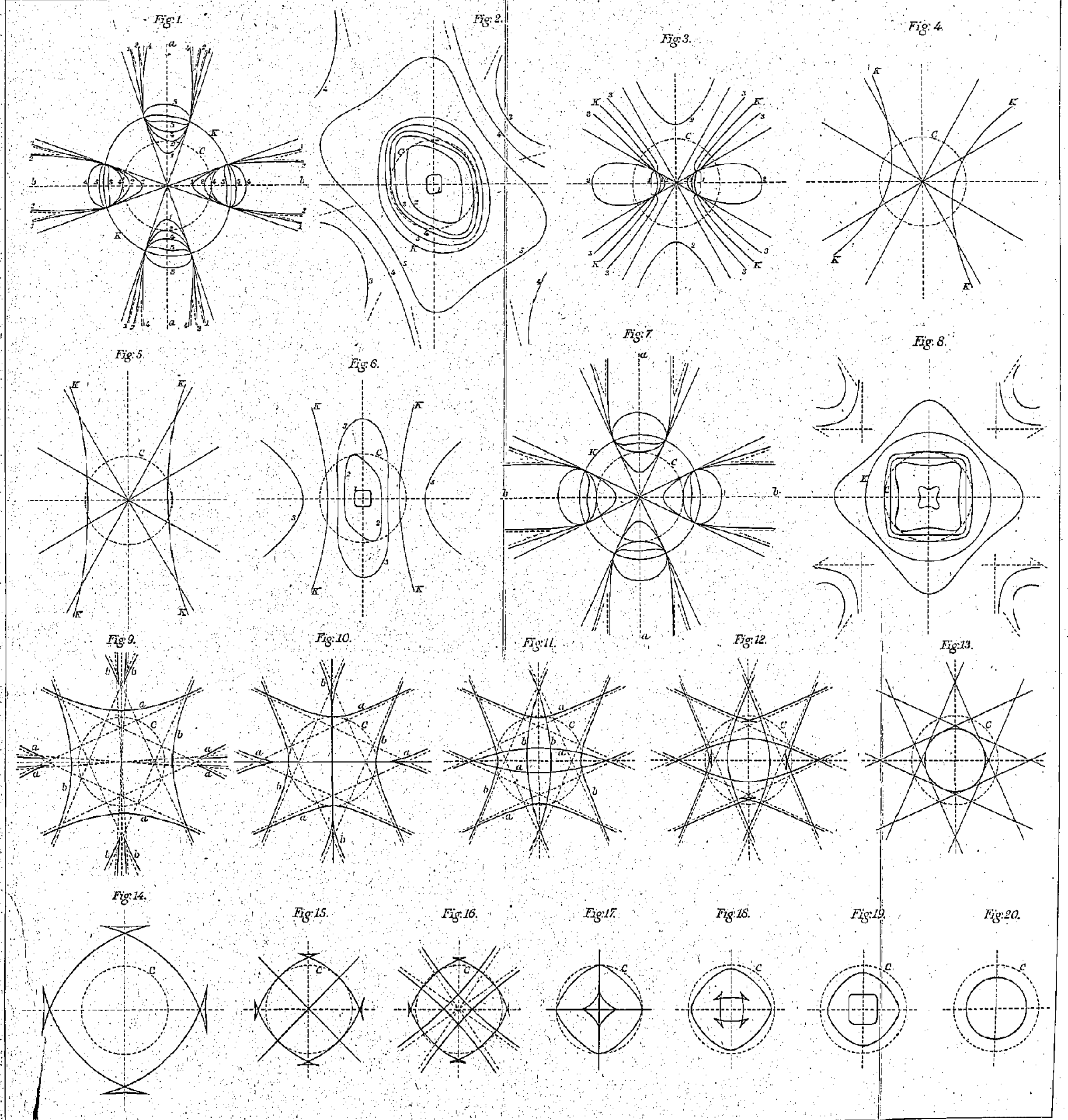
Further on the nature of the conic (K) which may be an ellipsis, an hyperbola or a degeneration.

Finally on the reality of the asymptotes.

We can prove now, that (L') *has as many real asymptotical directions as it has pairs of real points of intersection with* (C).

Let $(\cos\alpha, \sin\alpha)$ be the point of (C) lying on (L'), then we have:

1) Phil. Mag. l c., p. 297.



J. Bijtel lith. P. J. Mulder & Zn. impr. Leiden.

$$\{q \cos^2 \alpha + 2pl \cos \alpha \sin \alpha - (2p + q) \sin^2 \alpha + 2p\}^2 =$$
$$= s \{\cos^4 \alpha - 2(2p + 1) \cos^2 \alpha \sin^2 \alpha + \sin^4 \alpha\}.$$

If we write this in the form:

$$\{q \sin^2 \alpha - 2pl \cos \alpha \sin \alpha - (2p + q) \cos^2 \alpha\}^2 =$$
$$= s \{\sin^4 \alpha - 2(2p + 1) \cos^2 \alpha \sin^2 \alpha + \cos^4 \alpha\}$$

then it is evident, that

$$y = - x \operatorname{cotg} \alpha$$

in an asymptotical direction of (L').

If (L') touches (C), two asymptotical directions coincide, they are perpendicular to the line that connects O with the points of contact.

§ 6. *(K) is an ellipsis.*

1°. $p > 0$, consequently the bitangents from O are real. They cut (K) in real points, in which points they touch (L').

The bitangents divide the plane into 8 angles, in which

$$H_4 \equiv x^4 - 2(2p + 1) x^2 y^2 + y^4$$

is alternately positive and negative. (L') lies for positive values of

$$s \equiv q^2 - p(4r + l^2)$$

in the angles, where H_4 is positive.

Let us call the branches of (L'), which are lying in the one pair of opposite angles, a, those which are situated in the other pair, b.

Let us begin by giving positive values to s and let us first consider a exclusively.

For $s = \infty$ degeneration in two bitangents. For large values of s, a consists of one branch with two asymptotes and four points of inflexion. For decreasing values of s the angle between the asymptotes becomes smaller, the apices are removed from each other and the points of inflexion move towards infinity. For a definite value of s the asymptotes are parallel. If there is a further decrease in s, a will consist of two closed branches in which for another special value of s points of osculation occur in the sides turned towards O. Then two points of inflexion appear in each branch and the branches contract, till we have for $s = 0$ degeneration in the ellipsis (K) [1]).

[1]) The case $s \equiv q^2 - p(4r + l^2) = 0$ must be inquired into separately. For $s = 0$ is the condition that in the second part of the relation between ζ and φ (p. 939) the root may be drawn. In this case (A') represents two *pencils* of ellipses. Consequently the required envelope (L) has now degenerated into 8 straight lines, which are the polar lines of the base points of those pencils, and in (K'), which is the polar curve of (K).

If we allow s to change from ∞ into 0, b passes through an equal change of shape. If we consider a and b, however, together, then the general and special values of s, for which two asymptotical directions coincide, and those for which points of osculation occur, will not be the same for a and b.

If we take into consideration what has been observed in § 5 with respect to the asymptotical directions of (L') and its points of intersection with (C), it is evident that we have to distinguish the following cases, which are represented in fig. 1 (with the exception of the 3rd):

1. a and b both cut (C); they have each two intersecting asymptotes.

2. a touches (C), b cuts (C), a has two intersecting, b two parallel asymptotes.

3. a lies outside (C), b cuts (C); a has two intersecting asymptotes, b consists of closed branches.

4. a lies outside (C), b touches (C); a has two parallel asymptotes, b consists of closed branches.

5. a and b lie both outside (C); both consist of closed branches.

In this we have not yet paid attention to the presence or absence of the points of inflexion in the closed branches; the number of cases would be increased by this.

It is evident that a value of s exists, below which points of inflexion occur both in the closed branches a and b. In that case all the 28 bitangents of (L') are real.

We have now allotted to s all positive values, for negative values of s (L') lies in the other four angles. If we revolve the system of axes 45°, we shall get the same cases again.

The value of p determines the situation of the bitangents drawn from O. For increasing values of p they move towards the axes, for decreasing values of p towards the lines that bisect the axes-angles. We shall have to consider the limit-cases separately.

§ 7. 2°. $p < 0$, consequently the bitangents from O are imaginary.

For a very great value of s (which we have always to take positive here) (L') consists of a small closed branch, given by

$$x^4 - 2\,(2p + 1)\,x^2y^2 + y^4 = \frac{4p^2}{s},$$

symmetrical with regard to the axes and the bisectrices. It possesses

8 points of inflexion or none, according to p being $< -\frac{1}{2}$ or $> -\frac{1}{2}$.

We shall suppose $p > -1$. This is sufficient, for it is easy to prove that (L') for a value of $p < -1$ by revolving the system of axes 45° passes into a curve answering to a value of $p > -1$.

If s decreases, the closed branch will increase while the symmetry is lost. For a certain value of s it touches (C) in two points. Then it cuts (C) in four points, in consequence of which according to the observations made in § 5, infinite branches occur. For a smaller value of s the closed branch which we shall call a, again touches (C) internally in two points. Then a cuts (C) in 8 points while new infinite branches appear. If s decreases further, then a touches (C) externally in two points; two asymptotes of b become parallel. Further a cuts (C) moreover in 4 points while two asymptotes of b have become imaginary. After this external touching occurs again, after which a has quite passed outside (C). At the same time b has become a closed branch. All the time a has remained inside (K), b outside (K), for (L') cannot cut (K) now as H_4 cannot become zero. It is evident, that, if (L') has assumed the form of a ring, a must have lost its points of inflexion if it possessed them. They will have disappeared with four at a time. After the falling together of two asymptotical directions, points of inflexion will occur in b so that the closed branch b may possess 8 points of inflexion. On further decrease of s these points of inflexion will disappear by four at a time, while the branches a and b approach each other, in order to coincide with (K) for $s = 0$.

In Fig. 2 (L') is represented for a certain value of $p < 0$ (viz. $< -\frac{1}{2}$) for some values of s.

From the equation of (L') appears at once that for $p = -1$, (L') has degenerated into two conics; at the same time (L) has degenerated into two conics.

In the figures (K) and (C) have not been drawn as intersecting; it is easily shown that they cannot intersect each other if (K) is an ellipsis.

§ 8. *(K) is an hyperbola.*

1°. $p > 0$, so the bitangents from O are real.

From the equation of (K) we deduce easily that the angle of the asymptotes is always greater than 90°. Hence (K) will cut at least

2 of the bitangents from O. Of the 4 bitangents 0, 1 or 2 are consequently isolated.

Fig. 3 refers to the case that two of the bitangents are isolated. For a few positive and negative values of s, (L') has been drawn.

Fig. 4 refers to the case that 1 bitangent is isolated.

Fig. 5 to the case that none of the bitangents is isolated; (L') therefore touches the 4 bitangents drawn from O in real points.

2°. $p < 0$, so the bitangents from O are imaginary.

Fig. 6 gives a representation of this (p is supposed $> -\frac{1}{2}$).

(In the figures (K) and (C) are represented as intersecting; this is indeed always the case if (K) is an hyperbola).

(K) is a degeneration.

As $p \neq 0$ is supposed, we have only to consider the case of degeneration in two parallel lines that touch (C). Generally speaking we can say that substantially everything is as when (K) is an hyperbola. If the bitangents are real they will generally touch (L') in real points.

§ 9. *Special cases $p = 0$ and $p = \infty$* These cases had to be considered separately (§ 1).

For $p = 0$ and $q \neq 0$ the first equation which we have found in § 1 for (L') passes into:

$$+ (4r + l^2)(y^2 - x^2)^2 + 4q^2 x^2 y^2 + 4q(1 + lxy - y^2)(y^2 - x^2) = 0.$$

If we write:

$$\frac{4r + l^2}{4q} = t,$$

then the equation becomes:

$$\{tx^2 + lxy - (t+1)y^2 + 1\}(y^2 - x^2) = qx^2y^2,$$

(L') has now a node in O. For the rest various cases may occur also here, which we are not going to consider separately.

If $p = 0$ and besides $q = 0$, then we have to consider the problem separately (cf. note p. 143). It is evident then that (L) consists of two rectangles [1]).

For $p = \infty$ and $q \neq \infty$, the first equation of (L') found in § 1 represents two hyperbolae, intersecting in the points $(0, \pm 1)$; $p = \infty$ involves, according to the relation between ζ and φ (§ 1), $\zeta = 0$. There is therefore no question of an envelope (L'). For $p = \infty$ and at the same time $q = \infty$ the envelope must be found again. It appears that (L) consists of 2 rectangles [2]).

[1]) Phil. Mag. p. 315.
[2]) Phil. Mag. p. 315.

Various shapes of the envelope (L).

§ 10. The number of various shapes which (L') and consequently also (L) may assume is, as we have deduced in what precedes, very great. In order to facilitate the survey of those various forms, we shall begin with the case that $p+q=0$ and at the same time $l=0$. The equation of (L') runs:

$$q^2(x^2+y^2-2)^2 = s\{x^4-2(1-2q)x^2y^2+y^4\} \quad (s=q^2+4qr).$$

The equations of the 4th order in λ as mentioned in § 3 are now of a quadratic form. The situation of the double points of (L) may therefore be determined by means of quadratic equations; of the double points 8 are lying on the axes, 8 on the bisectrices. The cases $q=0$ and $q=\infty$ have been considered separately (§ 9).

For an arbitrary value of q we have besides the values $s=0$ and $s=\infty$, for which (L') degenerates, two more special values of s, viz. a value for which the asymptotical directions coincide in pairs and one for which the points of inflexion coincide in pairs.

The asymptotical directions are determined by:

$$(q^2-s)(x^2-y^2)^2+4q(q-s)x^2y^2=0.$$

They are real if q^2-s and $q(q-s)$ have different signs.

They coincide in pairs:

for $s=q^2$ $(r=0)$ with the directions of the axes,

for $s=q\left(r=\frac{1}{4}(1-q)\right)$ with the directions of the bisectrices.

For $s=q^2$ the asymptotes are removed at a distance $\sqrt{\frac{1}{1-q}}$ from O, for $s=q$ at a distance $\frac{1}{2}\sqrt{\frac{-2q}{1-q}}$.

For $s=q^2$ (L') touches (C) in 4 points, lying on the axes, for $s=q$ in 4 points on the bisectrices (§ 5).

If the points of inflexion coincide in pairs those points are situated either on the axes or on the bisectrices.

If they are lying on the axes at a distance a from O, then the equation should run:

$$(x^2+y^2-a^2)^2 = s'(x^2-a^2)(y^2-a^2).$$

From this we deduce:

$$s=\frac{q^2}{(1-2q)^2}\,;\left(r=q^3\frac{(1-q)}{(1-2q)^2}\right)\,;\ a^2=\frac{2q-1}{q-1}.$$

The points of inflexion coincide in pairs on the bisectrices for:

$$s=\frac{q^2}{(q-2)^2}\;;\;\left(r=\frac{q(q-1)\left(1-\frac{1}{4}q\right)}{(q-2)^2}\right)\;;\;a^2=\frac{2-q}{1-q}$$

From what was observed in § 7 follows that we have to consider for q negative values only, and positive ones smaller than unity.

The asymptotes, parallel to the axes, are real for all these values of q.

The asymptotes, parallel to the bisectrices, are real for negative values of q, imaginary for positive ones, smaller than unity.

The points where the points of inflexion coincide on the bisectrices, are always real.

The points where the points of inflexion coincide on the axes are real for all negative values of q, and further for positive values of q, smaller than $\frac{1}{2}$. For values of q between $\frac{1}{2}$ and 1 they are imaginary. Further we observe that the value of s, for which these points occur, is between ∞ and q, if q lies between $\frac{1}{4}$ and $\frac{1}{2}$; s lies between q and q^2, if q lies between 0 and $\frac{1}{4}$.

After the deductions made in § 6 and § 7 and this § it will be superfluous to give an explanation of fig. 7, where (L') is represented for a negative value of q and some various values of s, and fig. 8, where (L') is represented for a positive value of $q\left(<\frac{1}{4}\right)$.

§ 11. From the shape of (L') that of (L) as reciprocal polar curve may be at once deduced.

Let in the first place q be negative. There are 4 pairs of parallel asymptotes, touching at the circle $x^2+y^2=\frac{1}{2}$. They are parallel with the bitangents of (L'), passing through O. Let us now consider various values of s.

$s>q^2$. $(r<0)$. Fig. 9. Besides the 8 asymptotes just mentioned there are 4 more, which pass through O. The entire curve (L) lies outside (C) and can therefore not be of any consequence as an envelope. For on (C) the velocity of the moving point is 0; outside (C) the vis viva would be negative. In fact q^2 is the greatest value that s can have in the dynamical problem.

$s=q^2$. $(r=0)$. Fig. 10. The cusps have coincided in pairs in the axes, with which the four asymptotes passing through O have now coincided in pairs. (L) touches in 4 points at (C).

The only forms of motion which the dynamical problem allows of are an X-vibration, and a Y-vibration.

$\frac{q^2}{(1-2q)^2} < s < q^2 \left(q^2 \frac{(1-q)}{(1-2q)^2} > r > 0 \right)$. Fig. 11. (L) delimits two quadrilateral domains of motion with vertices on (C) [1].

$s = \frac{q^2}{(1-2q)^2} \cdot \left(r = q^2 \frac{(1-q)}{(1-2q)^2} \right)$. On the axes 4 pairs of cusps have coincided. (L) deviates only a little from the shape indicated in Fig. 11.

$0 < s < \frac{q^2}{(1-2q)^2} \cdot \left(-\frac{1}{4} q > r > q^2 \frac{(1-q)}{(1-2q)^2} \right)$. Fig. 12.[2]) 8 cusps occur. (The "stirrups" lying within the domains of motion contribute indeed to the envelope).

$s = 0 \left(r = -\frac{1}{4} q \right)$. Fig. 13. Degeneration in 8 asymptotes.

Two domains of motion each bounded by a square.

We now get to the negative values of s. No figures have been drawn for them as they are of exactly the same nature as those for the positive values of s; we have only to revolve the figures 45°. Consequently:

$\frac{q^3}{(q-2)^2} < s < 0 \quad \left(\frac{q(q-1)\left(1-\frac{1}{4}q\right)}{(q-2)^2} > r > -\frac{1}{4}q \right)$. Fig. 12, having revolved 45°.

$s = \frac{q^3}{(q-2)^2} \quad \left(r = \frac{q(q-1)\left(1-\frac{1}{4}q\right)}{(q-2)^2} \right)$ Here we have to take into consideration that the distance of the special points to O is another one than for

$$s = \frac{q^2}{(1-2q)^2}.$$

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1) One domain of motion is bounded by two opposite branches a as far as they are lying inside (C), and the branches b which pass through the points of intersection of the just mentioned branches a with (C).

2) This Fig. and Fig. 18 we also find in a treatise of F. KLEIN: "Über den Verlauf der ABEL'schen Integrale bei den Curven vierten Grades". (Math. Ann. 10. Bd, 1876).

$q < s < \frac{q^3}{(q-2)^2} \cdot \left(\frac{1}{4}(1-q) > r > \frac{q(q-1)\left(1-\frac{1}{4}q\right)}{(q-2)^2}\right)$. Fig. 11, having revolved 45°.

$s = q \cdot \left(r = \frac{1}{4}(1-q)\ .\right)$. Fig. 10, having revolved 45°. The distance of the cusps to O has changed however.

$s < q \cdot \left(r > \frac{1}{4}(1-q)\right)$. Fig. 9, having revolved 45°.

Let us now suppose that q *is positive and* < 1.

$s < q \cdot \left(r > \frac{1}{4}(1-q)\right)$. Fig. 14. ($L$) has no dynamical meaning for the same reasons as in Fig. 9.

$s = q\left(r = \frac{1}{4}(1-q)\right)$. Fig. 15. The dynamical problem allows of two simple vibrations only.

$\frac{q^2}{(1-2q)^2} < s < q \cdot \left(q^2 \frac{(1-q)}{(1-2q)^2} < r < \frac{1}{4}(1-q)\right)$. Fig. 16. Two domains of motion. [1])

$s = \frac{q^2}{(1-2q)^2}\left(r = q^2 \frac{(1-q)}{(1-2q)^2}\right)$. The cusps of the preceding Fig. have coincided in pairs now.

$q^2 < s < \frac{q^2}{(1-2q)^2}\left(0 < r < q^2 \frac{(1-q)}{(1-2q)^2}\right)$. Fig. 16, from which the cusps have disappeared.

$s = q^2$. ($r = 0$). Fig. 17. (L) has 4 points of contact with (C). In the dynamical problem we are concerned with an asymptotical approach to the X- or Y-vibration. This case should be considered as the *transition* between two domains of motion and a single domain of motion.

$\frac{q^3}{(q-1)^2} < s < q^2 \cdot \left(\frac{q(q-1)\left(1-\frac{1}{4}q\right)}{(q-2)^2} < r < 0\right)$. Fig. 18. The "stirrups" contribute to the "envelope". [2])

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[1]) Of the closed branch of (L) 4 parts lie inside (C). Each of the domains of motion is bounded by 2 opposite parts and by the infinite branches that pass through their final points.

[2]) The inner branch serves partly as exterior, partly as interior envelope. The parts which, seen from the centre, are hollow, touch internally, the rest externally.

$s = \frac{q^3}{(q-2)^2} \cdot \left(r = \frac{q(q-1)\left(1 - \frac{1}{4} q\right)}{(q-2)^2} \right)$. The cusps of Fig. 18 have incided in pairs.

$0 < s < \frac{q^3}{(q-2)^2} \cdot \left(-\frac{1}{4} q < r < \frac{q(q-1)\left(1 - \frac{1}{4} q\right)}{(q-2)^2} \right)$. Fig. 19 [1])

$s = 0. \left(r = -\frac{1}{4} q \right)$ Fig. 20. Degeneration in the circle $x^2 + y^2 = \frac{1}{2}$.

We have now supposed, that q lies between 0 and $\frac{1}{4}$. If q lies tween $\frac{1}{4}$ and $\frac{1}{2}$, we have a little change. Then the 8 cusps of g. 14 would already have disappeared for $s = q$.

For q between $\frac{1}{2}$ and 2 the forms of the envelope, indicated by g 16, do not exist.

For q *positive and* > 1 no figures have been drawn for reasons ıted already.

§ 12. Let us now consider the shape of (L) in general, first in se (K) is an ellipsis.

The symmetry with regard to the axes and the bisectrices does ıt exist anymore now. The nodes, which for $l = 0$ lie on the es, lie for positive values of l in the second and the fourth adrant (§ 3); those which lie for $p + q = 0$ on the bisectrices ve been removed for positive values of $\frac{p+q}{p}$ into the direction of e Y-axis (§ 3). The changes in form which (L) undergoes in conquence of this are easily understood.

Other forms of (L) are, however, possible.

Let us first suppose $p > 0$. We have to start now from the 5 ses mentioned in § 6.

In case 1, (L) has mainly the shape which has been represented Fig. 9, in which we have to take into consideration the observaıns just mentioned.

In case 5, (L) has, with due observation of these remarks, the neral shape of Fig. 11, or of Fig. 12, or it is a combination of

[1]) For $q = 1$ (S) consists of two circles; we have then the well known case the conical pendulum.

those two forms, i.e., the envelope of one system of osculating ellipses has 4 cusps, the envelope of the other has none.

Case 2 is to be considered as a combination of Fig. 9 and Fig. 10. a touches (C) in two points, b has two cusps on the line which connects O with the points of contact of a with (C). The dynamical problem allows of a single simple vibration.

Case 3 gives rise to a combination of Fig. 10 and Fig. 11 (or Fig. 12). There is *one* system of osculating ellipses.

Case 4 to a combination of Fig. 10 and Fig. 11 (or Fig. 12). There is *one* system of osculating ellipses. Moreover the dynamical problem allows of a simple vibration.

In the case $p < 0$ we have again in the first place envelopes corresponding in the main with those represented in the Fig. 14—20. We should, however, bear in mind, that in general the cusps do not disappear by 8 but by 4 at a time. There is for instance a transitional form possible between Fig. 18 and Fig. 19 in which 4 cusps occur, and in Fig. 14 and Fig. 15 4 cusps may have fallen out. In order to obtain the other forms of the envelope we must make use of the observation about (L') in § 7.

If the branch of (L) lying outside (C) touches (C) in two points, then the dynamical problem allows of *one* simple vibration. If (L) cuts (C) in 4 points, then we get one of the two domains of motion of Fig. 16, etc.

Is (K) an hyperbola or a degeneration then the various shapes of (L) may be deduced in the same way from the Fig. 3—6.

Physiology. — "*On the reflectorical influence of the thoracal autonomical nervous system on the rigor mortis in cold-blooded animals.*" [1]). By S. DE BOER. (Communicated by Prof. C. A. PEKELHARING.)

(Communicated in the meeting of January 31, 1914).

The rigor mortis that is caused by hardening and shortening of the muscles begins in warm- and cold-blooded animals after the circulation of the blood has stopped for some time, in warm-blooded ones 5—8 hours, in cold-blooded ones 1—2 days. If with a muscle that has been removed, we make provision for a sufficient supply of oxygen, it mortifies without stiffening. A special chemical state

[1]) According to experiments made in the physiological laboratory of the University of Amsterdam.