If $\frac{1}{n} < 1$ we obtain those cases which we are accustomed to call "adsorption". Analogous to (8) we ought to attribute here the deviation from HENRY's law to "dissociation". But nothing of the kind has been found experimentally.

10. Hence, in the above-mentioned matter, I believe I have demonstrated that HENRY's law (law of division) and the law of PROUST are special instances of the adsorption-isotherm. This is in complete harmony with the results of the investigations recently published by REINDERS [1]) and GEORGIEVICS [2]).

Zwolle, February 1914.

Mathematics. — *"Cubic involutions in the plane"*. By Prof. JAN DE VRIES.

(Communicated in the meeting of February 28, 1914.)

1. The points of a plane form a *cubic involution* (triple involution) if they are to be arranged in groups of three in such a way, that, with the exception of a finite number of points, each point belongs to *one* group only. Suchlike involutions are for instance determined by linear congruences of twisted cubics. The best known is produced by the intersection of the congruence of the twisted cubics, which may be laid through five fixed points; it consists of ∞^2 polar triangles of a definite conic (REYE, *Die Geometrie der Lage*, 3e *Auflage*, 2e *Abtheilung*, p. 225). According to CAPORALI [3]) it may also be determined by the common polar triangles of a conic and a cubic. A quite independent treatment of this involution was given by Dr W. VAN DER WOUDE [4]).

In what follows only cubic involutions will be considered possessing the property that an arbitrary line contains one pair only, and is consequently the side of a single triangle of the involution. The

[1]) Kolloïd. Zeitschr. 13 96 (1913).

[2]) Zeitschr. f physik. Chem. 84 353 (1913).

[3]) *Teoremi sulle curve del terzo ordine* (Transunti R. A. dei Lincei, ser. 3a, vol. 1 (1877) or Memorie di geometria, Napoli 1888, p. 49). If $a^3_x = 0$ and $b^2_x = 0$ are those curves, then the involution is determined by $a_x a_y a_z = 0, b_x b_y = 0, b_y b_z = 0, b_z b_x = 0$.

[4]) *The cubic involution of the first rank in the plane.* (These Proceedings volume XII, p. 751—759).

lines of the plane are then moreover arranged in a cubic involution. It is further supposed that the points of a triplet are never collinear, the lines of a triplet are never concurrent.

2. If each point P is associated to the opposite side p of the *triangle of involution* Δ which is determined by P, *a birational correspondence* (P,p) will arise. Let n be the degree of that correspondence; then the points P of a line r will correspond to the rays p of a system with index n, in other words to the tangents of a rational curve $(p)_n$ of class n; the rays p of a pencil with centre R pass into the points P of a rational curve $(P)^n$ of order n.

Between the points P of r and the points P^*, where r is cut by the lines p, exists a correspondence in which each point P determines *one* point P^* while a point P^* apparently determines n points P. So $(n+1)$ points P lie on the corresponding line $p = P'P''$.

In that case one of the points P' has coincided in a definite direction p with P, while p has joined with p'. The *coincidences* of the involution (P^3) form therefore a curve of order $(n+1)$, which will be indicated by γ^{n+1}. In a similar way it is demonstrated that the *coincidences* of the involution (p^3) envelop a curve of class $(n+1)$.

When P describes the line r, the points P' and P'' describe a curve of order $(n+3)$; for this curve has in common with r the two vertices of the triangle of involution, of which one side falls along r, and the $(n+1)$ coincidences $P \equiv P'$, indicated above; we indicate it by means of the symbol ϱ^{n+3}.

Analogously there belongs to a pencil of rays with its centre in R a curve of class $(n+3)$, which is enveloped by the lines p' and p'' of the triangles Δ, of which one side p passes through R.

3. The two curves $(p)_n$ and $(p)'_n$ belonging to the lines r and r' have the line p, which has been associated to the point of intersection (rr'), as common tangent. Each of the remaining common tangents b is the side of two triangles Δ, of which the opposite vertices are respectively on r and r'; b therefore bears a *quadratic involution* I^2 of pairs (P',P'').

The pairs (p',p''), which form triangles of involution with a *singular straight line* b, envelop a curve (b). If it is of the class μ, then it has b as $(\mu-1)$-fold tangent, for through a point b passes only one line p'. We call b *a singular line of order* μ. The pairs (p',p'') form a *quadratic involution* on the *rational* curve (b). Its *curve of involution* β, i. e. the locus of the point $P \equiv p'p''$, is a curve of order $(\mu-1)$; for it has with b only in common the points in which this line is cut by the $(\mu-1)$ rays p'', with which $b \equiv p'$ forms pairs of the quadratic involution.

As $\beta^{\mu-1}$ has apparently $(\mu-1)$ points in common with r, b is a $(\mu-1)$-fold tangent of the curve $(p)_n$. Hence b, as common tangent of the curves $(p)_n$ and $(p)'_n$ must be taken into account $(\mu-1)^2$ times. The number of singular lines b satisfies therefore the relation.

$$\Sigma (\mu-1)^2 = n^2-1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The singular lines b are apparently *fundamental lines* of the birational correspondence (P,p).

The curves $(P)^n$ belonging to the pencils that have R and R' respectively as centres, pass through the point P, which has been associated to the common ray of those pencils. Each point B, which they have further in common has been associated to two different rays p, is consequently a *singular point* of (P^3) and at the same time a *fundamental point* of $(P.p)$.

The pairs of points (P', P'), forming triangles Δ with B lie on a curve (B), which has B as $(m-1)$-fold point if its order is m; then we call B a *singular point of order m*. On this *rational* curve, the pairs (P', P'') form a *quadratic involution*, in which B belongs to $(m-1)$ pairs; the line $p \equiv P' P''$ envelops therefore a *curve of involution* of class $(m-1)$.

From this ensues that B in the intersection of two curves $(P)^n$ must be counted for $(m-1)^2$ points, so that the number of points B has to satisfy the equation

$$\Sigma (m-1)^2 = n^2-1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

4. The involution (P^3) may also have *singular points* A, for which the pairs of points (P', P'') form an involution I^2 on a line a; the latter is then *singular* for the involution (p^3) and the pairs (p', p'') belong to an involution of rays with A as centre; a and A we call *singular of the first order*. The pairs (A, a) are apparently not fundamental for the correspondence (P, p); we indicate their number by α. If $n=1$, as for the involution of Reye, (cf. § 1), then there are only singular points and lines of the first order; for now $n^2-1=0$.

Let us now consider the curves ϱ^{n+3} and σ^{n+3} belonging to the lines r and s. A point of intersection P' of r with σ determines a triangle of involution of which a second vertex P' lies on s; P'' is therefore a point of intersection of s with ϱ. The third vertex P lies therefore on the two curves ϱ and σ. They have also in common the pair of points that forms a triplet of the (P^3) with the point rs. The remaining points of intersection of ϱ and σ lie in singular points A and B, for they belong each to two triangles of involution, of which one has a vertex on r, the other a vertex on s.

As the singular curve $(B)^m$ cuts each of the lines r, s in m points, ϱ and σ have an m-fold point in B. The numbers m must therefore satisfy the relation $(n+3)^2 = (n+3)+2+\alpha + \Sigma m^2$ or

$$\alpha + \Sigma m^2 = (n+1)(n+4) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In a similar way we find the relation

$$\alpha + \Sigma \mu^2 = (n+1)(n+4) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From the relations [1]) (1), (2), (3), (4) ensues moreover

$$\Sigma m^2 = \Sigma \mu^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\Sigma (m-1)^2 = \Sigma (\mu-1)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

consequently

$$\Sigma (2m-1) = \Sigma (2\mu-1) . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$\alpha + \Sigma (2m-1) = 5(n+1) . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

5. The points P', P'', of which the connecting line p passes through E, lie on a curve ε^4, which has a node in E, and is touched there by the lines EE', EE''.

If E is a singular point B then this locus consists apparently of $(B)^m$ and a curve of order $(4-m)$. Hence m may be *four* at most. If $m=3$, ε^4 degenerates into $(B)^3$ and a *singular line*.

Through E, six tangents pass to ε^4; each of these lines bears a coincidence of the involution (P^3). Such a line belongs to a group of the involution (p^3), in which p'' is connected with p'. The *coincidences* of (p^3) envelop a curve γ_3 of *class three*, reciprocally corresponding to the curve γ^3, which contains the coincidences of (P^3).

By *complementary curve* we shall understand the envelope of the lines p, which form triplets with the coincidences of the (p^3). From what was stated above follows therefore, that the complementary curve of the (p^3) is of the *sixth class*.

Analogously we find a *complementary curve* of the sixth order, $\varkappa^6$, as locus of the points P, which complete the coincidences of the (P^3) into triplets. It has nodes in all the *singular points* of (P^3), for each curve $(B)^m$ and each line a bears two coincidences, which form triplets with the corresponding singular points.

As the curve $(B)^m$ has an $(m-1)$-fold point in B, the curve of

[1]) In my paper "*A quadruple involution in the plane*" (These Proceedings vol. XIII, p. 82) I have considered a (P^3), which possesses a singular point of the fourth order and six singular points of the second order. In correspondence to the formulae mentioned above, $n=4$ was found.

coincidences γ^{n+} passes also $(m-1)$ times through B. Consequently γ^{n+1} and $\varkappa^6$ have yet $6(n+1)-2\,\Sigma(m-1)$ points in common besides the points B, but these points must coincide in pairs in points where the two curves touch, where consequently the three points of a group of the (P^3) have coincided.

Now

$$2\delta = 6(n+1)-2\,\Sigma(m-1) = 6(n+1)-\Sigma(2m-1)+\beta,$$

if β indicates the number of points B.

By means of (8) we find further

$$2\delta = (n+1)+\alpha+\beta.$$

Let σ represent the number of singular points $(\sigma = \alpha+\beta)$, we have found then, that the involution (P^3) is in possession of

$$\delta = \tfrac{1}{2}(n+1+\sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

groups of which the three points P have coincided.

Apparently this is at the same time the number of groups of (p^3), which consist of three coincided lines.

If the number of singular points of the order k is represented by σ_k then it ensues from (2) and (8), as $m \leq 4$,

$$9\,\sigma_4 + 4\,\sigma_3 + \sigma_2 = n^2 - 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$7\,\sigma_4 + 5\,\sigma_3 + 3\,\sigma_2 + \sigma_1 = 5\,(n+1). \quad . \quad . \quad . \quad . \quad (11)$$

By elimination of σ_4 we find

$$17\,\sigma_3 + 20\,\sigma_2 + 9\,\sigma_1 = (n+1)\,(52-7n). \quad . \quad . \quad . \quad (12)$$

So that it appears that n amounts at most to SEVEN.

6. We shall now further consider the case $n=2$. From $\Sigma(m-1)^2 = 3$ follows at once, that (P^3) possesses *three* singular points of the second order $B_k\,(k=1,2,3)$. The curves (B_k) associated to them are conics, which contain involutions (P',P''); the lines p on which those pairs are situated, pass through a point C_k.

The existence of *three* singular straight lines of the second order ensues analogously from $\Sigma(\mu-1)^2 = 3$; the points P, which with the pairs on b_k form triangles of involution, lie on a line c_k; the sides of those triangles envelop a conic $(b_k)^2$.

From (8) we further find $\alpha = 6$; consequently there are *six* singular pairs (A,a).

The correspondence (P,p) is quadratic; B_k are its fundamental points, b_k its fundamental lines.

To an arbitrary line r is associated a curve ϱ^6, which has *nodes* in the three points B and in the point associated to r in the quadratic correspondence. The pairs (P',P'') on this *quadrinodal* curve form the only involution of pairs that can exist on a curve of genus *two*; the straight lines p envelop a *conic* [1]).

If r contains a singular point A, ϱ^5 degenerates into the line a and a ϱ^4; the latter will further degenerate as it must possess four nodes, consequently is composed of two conics.

On a *singular line* a lie two coincidences of the involution $I^2 \equiv (P', P'')$; they are at the same time coincidences of the (P^3). The *curve of coincidences* γ is of *the third order*, so a must contain another coincidence. Let it be $Q' \equiv Q$; Q' forms a triangle of involution Δ with A and a point Q'' of a, but moreover a Δ with Q and a point Q^* lying outside a. Consequently Q' is a singular point viz. a point B, for the pairs A, Q'' and Q, Q^* do not lie on *one* line.

The curve ϱ^5 belonging to a consists first of a itself and a conic $(B)^2$; the completing curve must also have arisen from singular points. No second point B lies on a, for this line would then contain four points of the curve of coincidences. Hence two more singular points of the first order lie on a, A^*, and A^{**}. Each *singular line a* contains therefore *two* points A and *one* point B. If a^* cuts the line a in S, then A^{**} and S form a pair of the involution lying in a; so that AA^*S is a triangle of involution. Hence A is the point of intersection of the singular lines a^*, a^{**}.

7. The connector of two singular points A_k and A_l is not always a singular line a. Let A_l lie on a_k, A_k then forms with A_l and another point Q of a_k a triangle Δ, so that A_kQ is the line a_l. If A_l lies on a_k, a_l passes consequently through A_k.

Let us now consider the line that connects the centres of involution belonging to C_1 and C_2. It contains a pair of points forming a triplet with B_1, and a pair that is completed into a triplet by B_2. Hence it is a *singular line b*, we call it b_3. The axis of involution c_3 belonging to it, is apparently the line B_1B_2; the three lines c form the triangle $B_1B_2B_3$.

In the transformation (P,P') c_3 corresponds with the figure composed of b_3 and the conics $(B)_1^2$, $(B_2)^2$. With γ_3 it has in common the coincidences lying in B_1 and B_2, its third point of intersection with γ^3 lies apparently in b_3c_3. The singular line b_3 is transformed by (P,P') into a figure of the fifth order; to this belongs b_3 itself and the line c_3 twice. As no point B lies on b_3 it must connect

[1]) The quadrinodal curves ϱ^5 I have treated in "*Ueber Curven fünfter Ordnung mit vier Doppelpunkten*" (Sitz. ber. der Akad. d. Wiss. in Wien, vol. CIV, p. 46—59).

two points A; the correponding lines a form the completing figure.
The conic $(B_1)^2$ has in common with γ^3 the two coincidences of I^2 lying on it and the coincidence of the (P^3) lying on B. As it cannot apparently contain a coincidence of an other I^2 it must pass through B_2 and B_3, while it touches γ^3 in B_1.

8. A conic is transformed by (P,P') into a figure of the tenth order. For the conic $(B_1)^2$ it consists of twice $(B_1)^2$ itself, the conics $(B_2)^2$, $(B_3)^2$ and two lines a; it bears consequently *two* points A, which we shall indicate by A_1 and A_1^*. As these points each form a triangle of involution with B_1 and another point of $(B_1)^2$, the lines a_1 and a_1^* pass through B_1.

Analogously we shall indicate the singular lines which meet in B_2 and in B_3, by a_2, a_2^* and a_3, a_3^*; the points A_2 and A_2^* are then situated on $(B_2)^2$; A_3 and A_3^* on $(B_3)^2$.

On a_1 two more points A are lying; one of them belongs to $(B_2)^2$, the other to $(B_3)^2$; we may indicate them by A_2^* and A_3^*.

If we act analogously with the remaining points A and lines a, then the sides a_1, a_2, a_3 of the triangle $A_1^*A_2^*A_3^*$ will pass through B_1, B_2, B_3, and the same holds good concerning the sides a_1^*, a_2^*, a_3^* of the triangle $A_1A_2A_3$.

In connection with the symmetry, which is involved by the quadratic correspondence (P,p), the lines b_1, b_2, b_3 contain respectively the pairs A_1, A_1^*; A_2, A_2^*; A_3, A_3^*. The triangle of the lines b has C_1, C_2, C_3 as vertices; analogously c_1, c_2, c_3 are the sides of B_1,B_2,B_3.

The *six* points A, and the *three* points B form with the *six* straight lines a a *configuration* $(9_2, 6_3)$ B [1]), the points A with the straight line a and the straight lines b the reciprocal configuration $(6_3, 9_2)$ B.

9. That the involution (P^3) discussed above exists, may be proved as follows.

We consider the *congruence* formed by the twisted cubics φ^3, which pass through *two* given points G, G^* and has as bisecants *three* given lines g_1, g_2, g_3 [2]) By h_{kl} and h^*_{kl} we indicate the transversals of g_k, g_l, which may be drawn out of G and G^*.

Let us now consider the net of cubic surfaces Ψ^3, which pass

1) A configuration $(9_2, 6_3)$ A consists of two triplets of lines p_1, p_2, p_3; q_1, q_2, q_3 and the 9 points $(p_k q_l)$.

2) This congruence has been inquired into by analytic method by M. STUYVAERT ("*Etude de quelques surfaces algébriques* . . . " Dissertation inaugurale Gand, Hoste, 1902).

through g_1, g_2, g_3 and G^* and have a node in G. The base of this net consists of the 6 lines $g_1, g_2, g_3, h_{12}, h_{23}, h_{31}$; they form a degenerate twisted curve of the 6^{th} order with 7 apparent nodes. Every two Ψ^3 have moreover in common a twisted cubic, which passes through G and G^* and meets each of the lines g_k twice; these curves φ^3 consequently form the above mentioned congruence.

Through an arbitrary point passes a pencil (Ψ^3), hence *one* φ^3. On an arbitrary line l the net determines a cubic involution of the second rank; through the neutral points of this I^2_3 passes a curve φ^3, which has l as bisecant. The congruence $[\varphi^3]$ is therefore *bilinear*.

Through a point S of g_1 pass ∞^1 curves φ^3, they lie on the hyperboloid H^2, which is determined by S, G, G^*, g_2, g_3. All the curves φ^3 lying on H^2, pass moreover through the point S', in which H^2 again cuts the line g_1.

To $[\varphi^3]$ belongs the figure formed by h_{12} and a conic of the pencil which is determined in the plane (G^*g_3) by the intersections of g_1, g_2, h_{12}, and the point G^*. There are apparently 5 analogous pencils of conics besides.

Let us now consider the surface Λ formed by the φ^3, which meet the line l. Through each of the two points of intersection of l and H^2 passes a φ^3, cutting g_1 in S. From this ensues that the three lines g_k are double lines of Λ. The lines h_{kl}, h^*_{kl} lie on Λ, for l for instance meets a conic of the pencil indicated in the plane (G^*g_3), and this pencil forms with h_{12} a φ^3.

We determine the order of Λ by seeking for its section with the plane (Gg_1). To it belong 1) the line g_1, which counts twice, 2) the conic in that plane, which rests on l and is completed by h_{23}^* into a φ^3, 3) the lines h_{12} and h_{13}, which are component parts of two degenerate φ^3, of which the conic rests on l. From this ensues that Λ is of the *sixth order*.

10. If the congruence $[\varphi^3]$ is made to intersect with a plane φ, a cubic involution (P^3) arises, which has the intersections of the lines g_k, h_{kl}, and h_{kl}^* as singular points. With the intersection B_k of g_k correspond viz. the intersections of the φ^3, which cut φ already in B_k; they lie as we saw on the intersection $(B_k)^2$ of the hyperboloid H belonging to B_k. To the intersection A_1 of h_{23} corresponds the I^2 on the intersection a_1 of the plane (G^*g_1), originating from the pencil of conics in that plane, etc.

On $(B_1)^2$ lie the intersections of g_1, g_2, g_3, h_{23} and h_{23}^*, viz. the points B_1, B_2, B_3, A_1 and A_1^*; on the intersection a_1 of the plane

$(G^* g_1)$ we find the intersections B_1, A_3^+ and A_2^- of g_1, h_{12}^- and h_{13}^+.

To the points P of the line l lying in φ correspond the pairs of points P', P'' lying on the curve of the fifth order, which φ has moreover in common with the surface A^6; this curve passes through the points A_k, A_k' and has the points B_k as nodes.

So we find a cubic involution possessing the same properties as the cubic involution (P^3) *considered before.*

11. We are now going to consider the case that the plane φ is laid through a straight line c. resting on g_1, g_2, g_3 and cutting these lines in the points B_1, B_2, B_3. The three hyperboloids H determined by these points have the line c in common besides a conic φ^2 through G, G^*, resting on c, g_1, g_2 and g_3 and forming with c a curve of the $[\varphi^3]$. For the conics passing through G, G^* and cutting g_1, g_2, g_3, form a surface of the fourth order, cut by c in a point not lying on one of the lines g. The three hyperboloids mentioned cut φ along three lines b_1, b_2, b_3, meeting in a point C not lying on c, where φ^2 intersects the plane φ again.

The curves $[\varphi^3]$ passing through B_1, meet φ in the pairs of points P', P'', of an involution on b_1. So B_k are now *singular points* of the *first order*. C too is a *singular point* now; for the figure (φ^2, c) has all the points of c in common with φ, so that each pair of c corresponds to C.

The conic $(B_1)^2$ of the general case has been replaced here by the pair of lines (b_1, c); on b_1 lie now the singular points A_1, A_1^*.

The singular points and lines now form a configuration $(10_3, 10_3)$, viz. the well-known configuration of DESARGUES. For in the lines b_1, b_2, b_3, passing through C, the triangles $A_1 A_2 A_3$ and $A_1^* A_2^+ A_3^*$ are inscribed, the pairs of corresponding sides $a_1^+, a_1; a_2^*, a_2; a_3^*, a_3$ of which meet in the collinear points B_1, B_2, B_3.

From the curve ϱ^5, which in the general case corresponds to a line r, the line c falls away, in connection with this the *curve of coincidences* γ^3 passes into a *conic*

On the ϱ^4 with *one* node D, now associated to r, exists only *one* involution of pairs; the points P', P'', which form triangles of involution with the points of r, lie therefore on the lines p passing through D; consequently $n = 1$.

This involution differs from the (P^3) described by REYE only in this respect that the singular point C does not correspond to the pairs of an I^2 on c, as *all* the points of c have been associated to C.

12. Another (P^3) differing in this respect from the involution of REYE, is found as follows. We consider two pencils of conics, which have a common base-point E; the remaining base-points we call F_1, F_2, F_3 and G_1, G_2, G_3. If each conic through E, F_k is brought into intersection with each conic through E, G_k, a (P^3) is acquired, possessing a singular point of the fourth order in E, and singular points of the second order in F_k, G_k [1])

If, however, the points G_k lie on the rays EF_k, then the degenerate conics (EF_1, F_2F_3) and (EG_1, G_2G_3) have in common the line $h_1 = F_1G_1$ and the point $H_1 = (F_2F_3, G_2G_3)$, now H_1 is a singular point corresponding to *all* the points of h_1; consequently it is in the same condition as the point C mentioned above. There are now two more similar points still, $H_2 = (F_1F_3, G_1G_3)$ and $H_3 = (F_1F_2, G_1G_2)$.

While with an arbitrary situation of the points F and G, a ϱ^7 corresponds to a straight line r, which ϱ^7 passes four times through E and twice through F_k, G_k, this curve degenerates now into the three lines $h_k = F_kG_k$ and a ϱ^4, which has a node in the third vertex D of the triangle of involution, of which r is a side. On this ϱ^4, P' and P'' are now again collinear with D, so that $n = 1$.

If G_1 is placed on EF_1 and G_2 on EF_2, a special case of a (P^3) is found, where $n = 2$. The curve ϱ^7 now loses only the straight parts h_1 and h_2, consequently becomes a ϱ^5 having nodes in E, F_3, G_3 and D; on this quadrinodal ϱ^5, (P', P'') form again the involution of pairs, so that n appears to be 2. The singular points of the second order are E, F_3, G_3, the singular points of the first order are $F_1, F_2, G_1, G_2, H_1, H_2$; but the last two have respectively been associated to *all* the points of h_1 and h_2, while to each of the first four a quadratic involution corresponds.

13. In the case $n = 3$ we have the relations

$$\Sigma (m-1)^2 = 8 \quad \text{and} \quad \alpha + \Sigma m^2 = 28.$$

The first holds in three ways, for

$$8 = 2 \times 2^2 = 2^2 + 4 \times 1^2 = 8 \times 1^2.$$

But the first solution must be put aside at once. For by (P, P') a line r would be transformed into a ϱ^6; for the connector of two singular points of the 3rd order ϱ^6 would have the two corresponding curves $(B)^3$ as component parts; but then there would be no figure corresponding to the remaining points of the line in question.

The third solution too must be rejected, as, for 8 singular points of the second order $\alpha + 8 \times 2^2 = 28$; so $\alpha = -4$ would be found.

1) See my paper, referred to above, in volume XIII of these Proceedings (p.p. 90 and 91). The notation has been altered here.

For the further investigation there remains consequently the combination of *one* singular point of the 3^{rd} order, and *four* singular points of the 2^{nd} order; we shall indicate them by C and B_k ($k=1,2,3,4$). In addition to this we have moreover *three* singular points of the first order A_k.

Then there are further *three* singular lines of the 1^{st} order, a_k, *four* singular lines of the second order and *one* singular line of the third order.

The curve $(C)^3$ belonging to C has in C a node, which is at the same time node of the curve of coincidence γ^4. The two curves have in C six points in common; so also six points outside C; to them belong the two coincidences of the I^2 lying on $(C)^3$; the remaining four can only lie in the points B.

As $(C)^3$ forms part of the curve ε^4 (§ 5), belonging to C, a singular line a_1 passes through C. With γ^4, a_1 has in common the coincidences of the I^2 lying on it, and the two coincidences lying in C; consequently a_1 cannot contain any of the points B. By the transformation (P,P') it is transformed now into a figure of the 6^{th} order, of which $(C)^3$ and a_1 itself form a part; so the figure consists further of the singular lines a_2 and a_3, belonging to two singular points A_2, A_3 lying on a_1.

The singular line a_2 is transformed by (P,P') into a_2, and a figure of the 5^{th} order, arising from singular points on that line. As a_2 does not pass through C and as it must contain, besides the coincidences of the I^2, situated on it, two more coincidences which can only lie in points B, we conclude that it bears two points B_1, B_2 and the point A_1. From this ensues at once, that a_3 too passes through A_1, and contains the points B_3, B_4.

We consider C, B_1, B_2, A_3 as base-points of a pencil (φ^2) of conics; C, B_3, B_4, A_2 as base-points of a second pencil (ψ^2). If each φ^2 is made to intersect with each ψ^2, a (P^3) will arise, having singular points in C, B_k, A_k (see § 12). If to each φ^2 is associated the ψ^2, which touches it in C, then the pencils rendered projective by it, generate the figure $(C)^3 + a_1$; from this it is evident that $(C)^3$ does not only contain the points B_k, but also the singular point $A_1 = (B_1B_2, B_3B_4)$.

It is easy to see now, that A_3B_1, A_3B_2, A_2B_3, and A_2B_4 are the singular lines of the 2^{nd} order. For the φ^2 formed by A_3B_1 and CB_2 is cut by (ψ^2) in a I^2 on A_3B_1 and a series of points (P) on CB_2; so CB_2 is the axis of the involution (p',p'') belonging to A_3B_1.

As the axes of the involutions (p', p''), determined by the four singular lines of the 2^{nd} order pass through *one* point C, the centres of the involutions (P', P'') lying on the conics $(B_k)^2$ will analogously be collinear.

The line on which they lie contains four pairs (P', P''), which form each a triangle of involution with one of the points B_k; from this we conclude that it is the singular line of 3rd order, which (p^3) must have.

14. Let now $n = 4$. As to a line r a ϱ^7 must correspond, no singular point of the 3rd order $S^{(3)}$ can occur beside a singular point of the 4th order $S^{(4)}$ (see § 13). A simple investigation shows that only two cases are possible, viz. (1) *one* point $S^{(4)}$ with *six* points $S^{(2)}$ or (2) *three* points $S^{(3)}$, with *three* points $S^{(2)}$ and one point $S^{(1)}$.

The *first case* appears on further investigation to be realised by the (P^3) mentioned at the beginning of § 12 [1]) To the singular point of the 4th order, E, belongs a rational curve $(E)^4$, which passes also through the remaining singular points F_k, G_k $(k = 1, 2, 3)$. Singular lines of the 2nd order are F_kF_l and G_kG_l; the axes of involutions (p', p'') belonging to them we find in EF_m and EG_m.

As these six axes meet in E, the singular line of 4th order will contain the centres of the involutions I^2 on the conics $(F_k)^2, (G_k)^2$.

In the *second case* there are *three* singular points $C_k^{(3)}$, *three* points $B_k^{(2)}$, *one* point A, and, analogously, *three* lines $c_k^{(3)}$, *three* lines $b_k^{(2)}$, *one* line a.

With the curve of coincidences γ^6, which possesses nodes in C_k, $(C_1)^3$ has in common the 2 coincidences of the I^2 lying on it, and six points in C_1; the remaining 7 points of intersection must lie in singular points, consequently $(C_1)^3$ passes also through C_2, C_3, and B_k.

On $(C_k)^3$ lies therefore a point P, which forms a Δ with C_k and B_1; hence $(B_1)^2$ passes through C_k.

The line a is transformed by (P, P') into itself and a figure of the 6th order, so, either into the three conics $(B_k)^2$ or into two curves $(C_k)^3$. But the second supposition is to be cancelled, because a would contain 6 coincidences in that case, two of its I^2 and four in the two points C. Consequently the points B_1, B_2, B_3 lie on the singular line a.

Analogously the singular lines b_1, b_2, b_3 meet in A.

Every singular line c_k passes through a point C_k and completes $(C_k)^3$ into a ε^4.

The curve of the 3rd class $(c_1)_3$ belonging to c_1 has c_2, c_3, b_k as tangents (and c_1 as bitangent).

[1]) See also my paper, referred to above, in volume XIII, p. 90, 91.

The curve $(b_1)_2$ touches the three c_k (and b_1).

To a conic corresponds in the correspondence (P, P') a curve of order 14; it consists for the conic β_3^2 passing through C_1, C_2, C_3, B_1, B_2, of three curves $(C_k)^3$, of $(B_1)^2, (B_2)^2$ and a singular line. As β_3^2 is the curve of involution of the involution (p', p''), which is determined by that line, it is a singular line of the 3rd order, consequently a line c.

15. For $n = 5$ a further investigation produces only a (P^3) with *six* singular points of the 3rd order and as many singular lines of the 3rd order. Through each of those points C_k passes one of those lines, c_k. A combination of the curve $(C_k)^3$ with the curve γ^6 makes it clear that the first curve also passes through the remaining points C.

To the conic γ_6^2 passing through C_1, C_2, C_3, C_4, C_5 corresponds a figure of the 16th order, composed of the 5 curves $(C_k)^3$, $k \neq 6$, and a singular line, c_6. So γ_6^2 is the curve of involution belonging to c_6.

This (P^3) may be produced by a *net* of *cubic curves* with base-points C_k. All the curves determined by a point P form a pencil, of which the missing base-points form with P a triplet of the involution [1]).

16. For $n = 6$ we find as the only solution of the relations (10) and (11) $\sigma_4 = 3$, $\sigma_3 = 2$, $\sigma_1 = 4$. But this is to be rejected. For a conic would have to be transformed by (P, P') into a figure of the 18th order. To the conic passing through 3 points $B^{(4)}$ and 2 points $B^{(3)}$ would correspond the figure composed of 3 curves $(B)^4$ and 2 curves $(B)^3$, which is already of the 18th order.

For $n = 7$ we find no solution at all.

The results obtained are united in the following table

n	σ_1	σ_2	σ_3	σ_4	σ
1	10				10
2	6	3			9
3	3	4	1		8
4		6		1	7
4	1	3	3		7
5			6		6

1) This (P^3) is a plane section of a bilinear congruence of twisted cubics indicated by VENERONI (Rend. Palermo, XVI, 210) and amply discussed by STUYVAERT (Bull. Acad. Belgique, 1907, p. 470).

From the relation (9) ensues moreover $\sigma = 6$. In all the (P^3) occur therefore *six* groups, of which the three points P have coincided into *one*; in the (p^3) belonging to them *six* groups with united lines p.

Physics. *Further Experiments with Liquid Helium. I. The* HALL-*effect, and the magnetic change in resistance at low temperatures.* IX *The appearance of galvanic resistance in supra-conductors, which are brought into a magnetic field, at a threshold value of the field".* By H. KAMERLINGH ONNES. Communication No. 139*f* from the Physical Laboratory at Leiden.

(Communicated in the meeting of February 28, 1914).

§ 1. *Introduction, first experiments*. In my last paper upon the properties of supra-conductors, and in the summary of my experiments in that direction which I wrote for the Third International Congress of Refrigeration in Chicago (Sept. 1913, Leiden Comm. Suppl. N°. 34*b*), I frequently referred to the possibility of resistance being generated in supra-conductors by the magnetic field. There were, however, reasons to suppose that its amount would be small. The question as to whether the threshold value of the current might be connected with the magnetic resistance by the field of the current itself becoming perceptible could be answered in the negative, as we had then no reason to think of a law of increase of the resistance with the field other than proportional to it, or to the square of it, and the law of increase of the potential differences at currents above the threshold value could not be reconciled with either supposition. A direct proof that in supra-conductors only an insignificant resistance was originated by the magnetic field was found in the fact that a coil with 1000 turns of lead wire wound within a section of a square centimetre at right angles to the turns round a space of 1 c.m. in diameter remained supra-conducting, even when a current of 0.8 ampère was sent through it. The field of the coil itself amounted in that case to several hundred gauss, and a great part of the turns were in a field of this order of magnitude, without any resistance being observed. The inference was natural, that, even if we should assume an increase with the square of the field, the resistance would probably still remain of no importance even in fields of 100 kilogauss. In my publication (see Report, Chicago, Suppl. N°. 34*b*) I restricted my conclusion about the resistance in the magnetic field to a limit of 1000 gauss, and I also remarked that when it came to making use of the supra-conductors for the construction of strong magnets without iron, it would be necessary in the first place to investigate what resistance the magnetic field would

64