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Mathematics. — *"Homogeneous linear differential equations of order two with given relation between two particular integrals."* By Dr. M. J. VAN UVEN. (Communicated by Prof. W. KAPTEYN). (5th communication).

(Communicated in the meeting of April 26, 1912).

The equations (8) and (29) (see 1st comm. p. 393 and 398) show us in the case that the equation $F(x,y,z)=0$ represents a conic (see for the notation: 4th comm. p. 1015):

$$q_1=\frac{c^2z^3H}{(n-1)^2F_z{}^2}=\frac{\Delta z^2}{g^2}=e^{\int I d\tau},$$

where c is put equal to 1.

From this ensues

$$\frac{\dot g}{g}=-\frac{I}{3}.$$

Let us further put:

$$g=a_{33}\zeta^2 z, \quad \ldots \ldots \ldots \quad (72)$$

we then find:

$$\frac{\dot\zeta}{\zeta}=\frac{1}{2}\frac{\dot g}{g}=-\frac{I}{6},$$

or

$$I=-6\frac{\dot\zeta}{\zeta}. \quad \ldots \ldots \ldots \quad (73)$$

The equation (62) (see 4th comm. p. 1015) runs now as follows:

$$I^2=36\frac{\dot\zeta^2}{\zeta^2}=\frac{9}{a_{33}\Delta z^2\zeta^2}(-a_{33}{}^2A_{33}z^2\zeta^4+2a_{33}\Delta z^2\zeta^2-a_{33}\Delta z^2),$$

or making use of the notation (59) (4th comm. p. 1003),

$$4\dot\zeta^2=-\lambda^2\zeta^4+2\zeta^2-1\,; \quad \ldots \ldots \ldots \quad (74)$$

so ζ' is likewise an elliptic function of τ. Its invariant has the same value (68) as that of the function $u=I^2$ (compare (67) [1]) (4th comm. p. 1006).

We can now deduce out of the equation

$$A_{23}x-A_{13}y=\sqrt{-A_{33}g^2+2\Delta gz-\Delta a_{33}z^2}=\sqrt{a_{33}\Delta}\,.\,z\sqrt{-\lambda^2\zeta^4+2\zeta^2-1} \quad (75)$$

(see 4th comm. p. 1005 at the bottom)

$$A_{23}x-A_{13}y=2z\sqrt{a_{33}\Delta}\,.\,\dot\zeta\,. \quad \ldots \ldots \ldots \quad (76)$$

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[1]) In the 4th comm. in the table on p. 1014 and in the enumeration of the cases on p. 1015 $\vartheta_1=e^{i\psi}$ and $\vartheta_2=e^{-i\psi}$ must be replaced by $\vartheta_1=e^{-i\psi}$, $\vartheta_2=e^{+i\psi}$.

As from (73) follows

$$a_{13}x + a_{23}y = a_{33}z\,(\zeta^2-1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

we find with the aid of (76) and (77)

$$\left.\begin{aligned}(a_{13}A_{13}+a_{23}A_{23})\,x &= (\Delta-a_{33}A_{33})\,x = \Delta(1-\lambda^2)\,x = \\ &= \{2a_{23}\sqrt{a_{33}\Delta}\,.\,\dot{\zeta} + a_{33}A_{13}\,(\zeta^2-1)\}\,z, \\ (a_{13}A_{13}+a_{23}A_{23})\,y &= (\Delta-a_{33}A_{33})\,y = \Delta\,(1-\lambda^2)\,y = \\ &= \{-2a_{13}\sqrt{a_{33}\Delta}\,.\,\dot{\zeta} + a_{33}A_{23}\,(\zeta^2-1)\}\,z.\end{aligned}\right\} \quad (78)$$

In this way we have expressed x and y as functions of τ with the aid of the function ζ. It is now still our task to determine ζ as function of τ. Let us now put in

$$9\dot{u}^2 = u^3 - 36u^2 + 324\,(1-\lambda^2)\,u$$

(see 4th comm. p. 1016)

$$u = I^2 = 36v + 12, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

we then find

$$\dot{v}^2 = 4v^3 - \frac{1+3\lambda^2}{3}\,v - \frac{9\lambda^2-1}{27}.$$

By applying the ordinary notation

$$\frac{1+3\lambda^2}{3} = g_2\,, \quad \frac{9\lambda^2-1}{27} = g_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

we then find

$$v = p\,(\tau\,;g_2\,,g_3)$$

and

$$I = \pm\, 6\sqrt{p\,(\tau\,;g_2\,,g_3) + \frac{1}{3}}\,, \quad . \quad . \quad . \quad . \quad (81)$$

so that

$$\frac{\dot{\zeta}}{\zeta} = \mp\sqrt{p\,(\tau\,;g_2\,,g_3) + \frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

Before transforming the p-function of WEIERSTRASS we wish to remind the readers that the roots of $\dot{u}=0$ are

$$u_1 = 0,\ u_2 = 18\,(1+\lambda),\ u_3 = 18\,(1-\lambda),$$

so that for the roots of $\dot{v}=0$ (see (79)) we find

$$v_1 = -\frac{1}{3}\,,\ v_2 = \frac{1+3\lambda}{6}\,,\ v_3 = \frac{1-3\lambda}{6}.$$

We shall now investigate the relative value of these roots in the three cases: II $(+1<\lambda<+\infty)$, IV $(+1>\lambda>0)$, VI $(\lambda=i\lambda')$ (see 4th comm. p. 1014).

Case II: $+1<\lambda<+\infty$

$$v_1 = -\frac{1}{3}, \; v_2 > \frac{2}{3}, \; v_3 < -\frac{1}{3}.$$

The roots are all real. Let us call them in the ordinary way in descending order e_1, e_2, e_3, we then find

$$e_1 = \frac{1+3\lambda}{6}, \; e_2 = -\frac{1}{3}, \; e_3 = \frac{1-3\lambda}{6} \quad . \; . \; . \; . \; . \quad II$$

Case IV: $+1 > \lambda > 0$.

$$v_1 = -\frac{1}{3}, \; \frac{2}{3} > v_2 > \frac{1}{6}, \; \frac{1}{6} > v_3 > -\frac{1}{3}.$$

The roots are here, too, all real and run when arranged:

$$e_1 = \frac{1+3\lambda}{6}, \; e_2 = \frac{1-3\lambda}{6}, \; e_3 = -\frac{1}{3}. \quad . \; . \; . \; . \; . \quad IV$$

Case VI: $\lambda = i\lambda'$.

The roots v_2 and v_3 are now conjugate complex. If we follow the notation generally assumed, we then write:

$$e_2' = -\frac{1}{3}, \; e_1' = \frac{1+3i\lambda'}{6}, \; e_3' = \frac{1-3i\lambda'}{6} \quad . \; . \; . \; . \; . \quad VI$$

When reducing the $\wp$-functions to the elliptic functions of JACOBI we make use of the following formulae of reduction: [1])

$$sn(v) = \sqrt{\frac{e_1-e_3}{\wp(\tau)-e_3}}, \quad cn(v) = \sqrt{\frac{\wp(\tau)-e_1}{\wp(\tau)-e_3}}, \quad dn(v) = \sqrt{\frac{\wp(\tau)-e_2}{\wp(\tau)-e_3}},$$

$$v = \tau\sqrt{e_1-e_3}, \quad k^2 = \frac{e_2-e_3}{e_1-e_3}, \quad k'^2 = \frac{e_1-e_2}{e_1-e_3};$$

$$\wp(\tau; e_1', e_2', e_3') = e_2' + \frac{e_1'-e_3'}{4ikk'} \cdot \frac{cn^2(v)}{sn^2(v) \, . \, dn^2(v)},$$

$$v = \tau\sqrt[4]{(e_2'-e_3')(e_2'-e_1')}, \quad k^2 = \frac{-3e_2' + 2\sqrt{(e_2'-e_3')(e_2'-e_1')}}{4\sqrt{(e_2'-e_3')(e_2'-e_1')}},$$

$$k'^2 = \frac{+3e_2'+2\sqrt{(e_2'-e_3')(e_2'-e_1')}}{4\sqrt{(e_2'-e_3')(e_2'-e_1')}}, \quad kk' = \frac{\sqrt{-9e_2'^2+4(e_2'-e_3')(e_2'-e_1')}}{4\sqrt{(e_2'-e_3')(e_2'-e_1')}}.$$

The expression for $\dot\zeta : \zeta$ becomes in this way:

in case II

$$\frac{\dot\zeta}{\zeta} = \mp\sqrt{\wp(\tau; g_2, g_3)-e_3} = \mp\sqrt{e_1-e_3} \cdot \frac{dn(v)}{sn(v)}$$

in case IV

$$\frac{\dot\zeta}{\zeta} = \mp\sqrt{\wp(\tau; g_2, g_3)-e_3} = \pm\sqrt{e_1-e_3}\,\frac{1}{sn(v)}$$

$$v = \tau\sqrt{e_1-e_3}, \quad k^2 = \frac{e_2-e_3}{e_1-e_3}; \quad k'^2 = \frac{e_1-e_2}{e_1-e_3}$$

[1]) See i. a. M. KRAUSE: Theorie der elliptischen Funktionen (Leipzig, TEUBNER (p. 135, 136, 147, 148).

in case VI

$$\frac{\dot{\zeta}}{\zeta}=\mp\sqrt{p(\tau;e_1',e_2',e_3')-e_2'}=\mp\sqrt{\frac{e_1'-e_3'}{4ikk'}}\cdot\frac{cn(v)}{sn(v).dn(v)},$$

$$v=\tau\sqrt{(e_2'-e_3')(e_2'-e_1')}\ ,\ kk'=\frac{\sqrt{-9e_2'^2+4(e_2'-e_3')(e_2'-e_1')}}{\sqrt{(e_2'-e_3')(e_2'-e_1')}},$$

or, after having expressed the roots $e_1, e_2, e_3, e_1', e_2', e_3'$ in λ:

$$\left.\begin{array}{l}
\text{in case II}\\
\dfrac{\dot{\zeta}}{\zeta}=\mp\sqrt{\lambda}.\dfrac{dn(v)}{sn(v)}\ ;\ v=\tau\sqrt{\lambda}\ ,\ k^2=\dfrac{\lambda-1}{2\lambda}\ ,\ k'^2=\dfrac{\lambda+1}{2\lambda}\ ;\\
\text{in case IV}\\
\dfrac{\dot{\zeta}}{\zeta}=\mp\sqrt{\dfrac{1+\lambda}{2}}.\dfrac{1}{sn(v)}\ ;\ v=\tau\sqrt{\dfrac{1+\lambda}{2}}\ ,\ k^2=\dfrac{1-\lambda}{1+\lambda}\ ,\ k'^2=\dfrac{2\lambda}{1+\lambda}\ ;\\
\text{in case VI}\\
\dfrac{\dot{\zeta}}{\zeta}=\mp\sqrt{\dfrac{\sqrt{1+\lambda'^2}}{2}}.\dfrac{cn(v)}{sn(v).dn(v)}\ ;\ v=\tau\sqrt{\dfrac{\sqrt{1+\lambda'^2}}{2}}\ ;\\
\qquad k^2=\dfrac{1+\sqrt{1+\lambda'^2}}{2\sqrt{1+\lambda'^2}}\ ,\ k'^2=\dfrac{-1+\sqrt{1+\lambda'^2}}{2\sqrt{1+\lambda'^2}}\ .
\end{array}\right\}\ (83$$

Let us substitute these expressions in (14), we then find successively

in case II $\zeta_1=\pm\dfrac{i}{\sqrt{\lambda}}\cdot\dfrac{1+cn(v)}{sn(v)}\quad,\quad\zeta_2=\pm\dfrac{1}{\lambda\zeta_1},$

in case IV $\zeta_1=\pm\dfrac{i}{\lambda}\sqrt{\dfrac{1+\lambda}{2}}.\dfrac{cn(v)+dn(v)}{sn(v)}\quad,\quad\zeta_2=\pm\dfrac{1}{\lambda\zeta_1},$

in case VI $\zeta_1=\pm\dfrac{i\sqrt{4(1+\lambda'^2)}}{\lambda'}\cdot\dfrac{dn(v)}{sn(v)}\quad,\quad\zeta_2=\pm\dfrac{i}{\lambda'\zeta_1}.$

Let us now choose

$$l=+6\sqrt{p(\tau;g_2,g_3)+\frac{1}{3}}$$

and for ζ the expressions ζ_1 with the upper sign, we find:

$$\left.\begin{aligned}
&II\quad I=+6\sqrt{\lambda}\,.\,\frac{dn(v)}{sn(v)}\;;\; v=\tau\sqrt{\lambda}\;,\; k^2=\frac{\lambda-1}{2\lambda}\;,\; k'^2=\frac{\lambda+1}{2\lambda}\;;\\
&\qquad \zeta=\frac{+i}{\sqrt{\lambda}}\cdot\frac{1+cn(v)}{sn(v)}\;,\; \dot{\zeta}=-i\,.\,\frac{\{1+cn(v)\}dn(v)}{sn^2(v)}\,.\\
&IV\quad I=+6\sqrt{\frac{1+\lambda}{2}}\cdot\frac{1}{sn(v)}\;;\; v=\tau\sqrt{\frac{1+\lambda}{2}}\,,\\
&\qquad k^2=\frac{1-\lambda}{1+\lambda}\;,\; k'^2=\frac{2\lambda}{1+\lambda}\;;\\
&\zeta=+\frac{i}{\lambda}\sqrt{\frac{1+\lambda}{2}}\cdot\frac{cn(v)+dn(v)}{sn(v)}\;,\; \dot{\zeta}=\frac{-i\,(1+\lambda)}{2\lambda}\cdot\frac{cn(v)+dn(v)}{sn^2(v)}\,.\\
&VI\quad I=+6\sqrt{\frac{\sqrt{1+\lambda'^2}}{2}}\cdot\frac{cn(v)}{sn(v).dn(v)},\; v=\sqrt{\frac{\sqrt{1+\lambda'^2}}{2}}\,,\\
&\qquad k^2=\frac{1+\sqrt{1+\lambda'^2}}{2\sqrt{1+\lambda'^2}}\;,\; k'^2=\frac{-1+\sqrt{1+\lambda'^2}}{2\sqrt{1+\lambda'^2}}\\
&\qquad \zeta=\frac{\sqrt[4]{4(1+\lambda'^2)}}{\lambda'}\cdot\frac{dn(v)}{sn(v)}\;,\; \dot{\zeta}=-\frac{\sqrt{1+\lambda'^2}}{\lambda'}\cdot\frac{cn(v)}{sn^2(v)}\,.
\end{aligned}\right\}\quad(84)$$

Let us restrict ourselves to real points (x, y) of the conic, then follows from (78) that $\sqrt{a_{33}\Delta}\,.\,\dot{\zeta}$ must always be real.

Case II (in which λ is real) appears only with the hyperbola for which holds $A_{33} < 0$; so we have here

$$a_{33}\Delta=\frac{a_{33}A_{33}}{\Delta}\cdot\frac{\Delta^2}{A_{33}}=\frac{\lambda^2\Delta^2}{A_{33}}<0.$$

From this ensues that in case II we shall find $\dot{\zeta}$ always imaginary, and therefore $\dfrac{\{1+cn\,(v)\}\,dn\,(v)}{sn^2\,(v)}$ is real;

Case IV is found with the hyperbola as well as with the ellipse. As here too λ is real we find

IVa. with the hyperbola $(A_{33} < 0)$ $a_{33}\Delta < 0$, so $\dot{\zeta}$ is imaginary or $\dfrac{cn\,(v)+dn\,(v)}{sn^2\,(v)}$ real;

IVb. with the ellipse $(A_{33} > 0)$ $a_{33} > 0$, so $\dot{\zeta}$ is real and $\dfrac{cn(v)+dn\,(v)}{sn^2\,(v)}$ is purely imaginary.

Also case VI appears with the hyperbola as well as with the ellipse. On account of λ being purely imaginary, thus λ^2 negative, holds:

VIa. for the hyperbola $(A_{33} < 0)$ $a_{33}\,\Delta > 0$, hence $\dot{\zeta}$ real, and $\dfrac{cn\,(v)}{sn^2\,(v)}$ real;

VI*b*. for the ellipse ($A_{33} > 0$) $a_{33}\Delta < 0$, thus $\dot{\zeta}$ purely imaginary and also $\frac{cn(v)}{sn^2(v)}$ purely imaginary.

From the preceding we see that v must move in its complex plane on the sides of the rectangles of the net formed by the lines $v = mK +$ purely imaginary and $v = niK' +$ real.

The value of $\zeta^2 = \frac{g}{a_{33}} = \frac{a_{13}x + a_{23}y}{a_{33}} + 1$ is evidently positive on that side of the polar line $g = 0$ of O with respect to the conic where O lies itself; on the other side ζ^2 is negative. The polar line $g = 0$ of O divides therefore the plane into two parts: in one (in which O lies) ζ is real, in the other ζ is imaginary.

In the points of contact R_1 and R_2 of the tangents out of O to the conic ζ is 0, so $I = \infty$.

In the points at infinity S_1 and S_2 we find that ζ and $\dot{\zeta}$ are both infinite and I is also equal to ∞.

The diameter passing through O ($A_{23}x - A_{13}y = 0$) intersects the conic in two points T_1 and T_2, for which $\dot{\zeta} = 0$, thus $I = 0$.

If we substitute the expressions (84) for ζ and $\dot{\zeta}$ in the formulae (78) we at last arrive at x and y as functions of τ.

With a view to $\sqrt{A_{33}}$ being real or not, we shall deal with the cases of IV and VI separately. Farthermore we shall express λ everywhere in $\delta = \frac{1-\lambda}{1+\lambda}$, thus in the anharmonic ratio of the four points R_1, R_2, S_1, S_2. We shall give the formulae for x only. The expressions for y we can easily find by replacing a_{23} in those for x by $-a_{13}$ and A_{23} by A_{13}.

We then find at last:

$$II \quad x = \frac{1-\delta}{2\delta} \cdot \frac{1+cn(v)}{sn^2(v)} \left[-(1+\delta) \frac{a_{23}}{\sqrt{-A_{33}}} dn(v) + \frac{A_{13}}{-A_{33}} \{1 + \delta\, cn(v)\} \right],$$

$$v = \tau \sqrt{\frac{1-\delta}{1+\delta}};$$

$$IVa \quad x = \frac{1}{2\delta} \cdot \frac{cn(v)+dn(v)}{sn^2(v)} \left[-(1+\delta) \frac{a_{23}}{\sqrt{-A_{33}}} + \frac{A_{13}}{-A_{33}} \{\delta cn(v) + dn(v)\} \right],$$

$$v = \tau \sqrt{\frac{1}{1+\delta}};$$

$$IVb \quad x = \frac{1}{2\delta} \cdot \frac{cn(v)+dn(v)}{i\, sn^2(v)} \left[-(1+\delta) \frac{a_{23}}{\sqrt{-A_{33}}} - \frac{A_{13}}{A_{33}} i \{\delta\, cn(v) + dn(v)\} \right],$$

$$v = \tau \sqrt{\frac{1}{1+\sigma}};$$

$$VIa \quad x = \frac{2\, cn\,(v)}{sn^2\,(v)} \left[\cos\frac{\psi}{2} \cdot \frac{a_{33}}{\sqrt{-A_{33}}} + 2 \cos^2\frac{\psi}{4} \cdot \frac{A_{13}}{-A_{33}}\, cn\,(v) \right],$$

$$v = \frac{\tau}{\sqrt{2\cos\frac{\psi}{2}}}, \quad \sigma = e^{-i\psi}, \psi = i \log \sigma;$$

$$VIb \quad x = \frac{2\, cn\,(v)}{i\, sn^2\,(v)} \left[\cos\frac{\psi}{2} \cdot \frac{a_{33}}{\sqrt{A_{33}}} - 2 \cos^2\frac{\psi}{4} \cdot \frac{A_{13}}{A_{33}}\, i\, cn\,(v) \right],$$

$$v = \frac{\tau}{\sqrt{2\cos\frac{\psi}{2}}}, \quad \sigma = e^{-i\psi}, \psi = i \log \sigma.$$

When point (x,y) describes the conic, the variable v will describe a certain curve in its complex plane. This curve we shall investigate in the five cases mentioned above whilst at the same time we shall indicate how the functions ζ, $\dot{\zeta}$ and I bear themselves during that motion.

Case II. Point O lies in the domain of the conjugate hyperbola; the diameter through O does not intersect the curve, i.e. the points T_1 and T_2 are imaginary. On the contrary the points R_1, R_2, S_1, S_2 are all real.

II	in S_1^∞	on $S_1^\infty R_1$	in R_1	on $R_1 S_2^\infty$	in S_2^∞	on $S_2^\infty R_2$	in R_2	on $R_2 S_1^\infty$	in S_1^∞
v	0	purely imag.	$2iK'$	$2iK'$ + real	$2K+2iK'$	$2K$+p.imag	$2K$	real	0
ζ	∞	pos. real	0	pos. imag.	∞	pos. real	0	pos. imag.	∞
$\dot{\zeta}$		pos. imag.	$+\frac{i}{2}$	pos. imag.	∞	neg. imag.	$-\frac{i}{2}$	neg. imag.	∞
I	∞	neg. imag.	∞	neg. real	∞	pos. imag.	∞	pos. real	∞

Here the curves are sketched which are described by v and I in their respective complex planes.

The points where I turns its direction of motion are arrived at by putting $\dot{I} = 0$. We then find the values of I corresponding to the roots of $\dot{u} = 0$; these are $u_1 = 0$, $u_2 = \infty$, $u_3 = 18\,(1+\lambda)$, $u_4 = 18\,(1-\lambda)$; or $I_1 = 0$, $I_2 = \infty$, $I_3 = 6\sqrt{\frac{1+\lambda}{2}}$,

IVa	in S_1^∞	on $S_1^\infty R_1$	in R_1	on $R_1 T_1$	in T_1	on $T_1 R_2$	in R_2	on $R_2 S_2^\infty$	in S_2^∞	on $S_2^\infty T_2$	in T_2	on $T_2 S_1^\infty$	in S_1^∞
ν	0	real	$2K$	$2K+$p.imag.	$2K+iK'$	$2K+$p.imag.	$2K+2iK'$	$2iK'+$real	$4K+2iK'$	$4K+$p.imag.	$4K+iK'$	$4K+$p.imag.	$4K$
ζ	∞	pos. imag.	0	pos. real	$\frac{\sqrt{1+\lambda}-\sqrt{1-\lambda}}{\lambda\sqrt{2}}$	pos. real	0	pos. imag.	∞	pos. real	$\frac{\sqrt{1+\lambda}+\sqrt{1-\lambda}}{\lambda\sqrt{2}}$	pos. real	∞
$\dot{\zeta}$	∞	neg. imag.	$-\frac{i}{2}$	neg. imag.	0	pos. imag.	$+\frac{i}{2}$	pos. imag.	∞	neg. imag.	0	pos. imag.	∞
I	∞	pos. real	∞	pos. imag.	0	neg. imag.	∞	neg. real	∞	pos. imag.	0	neg. imag.	∞

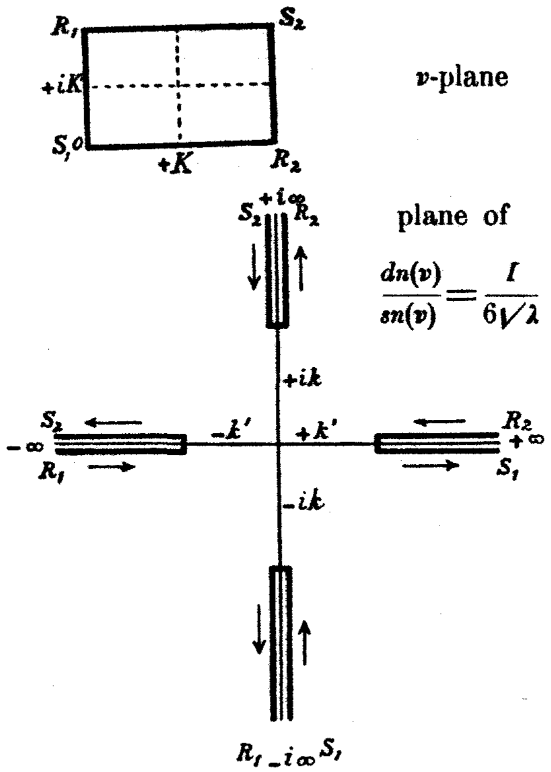

Fig. 1

$I_4 = 6\sqrt{\dfrac{1-\lambda}{2}} = 6i\sqrt{\dfrac{\lambda-1}{2}}$; the quotient $\dfrac{dn\,(v)}{sn\,(v)}$ assumes in those points successively the values 0, ∞, $\pm k'$, $\pm ik$. The corresponding values of v are congruent (*mod.* $2K$ and $2iK'$) with $K+iK'$, 0, K and iK'. (see fig. 1).

Case IVa. Point O lies in the domain between the hyperbola and the asymptotes. The points R_1, R_2, S_1, S_2, T_1 and T_2 are all real; T_1 and T_2 lie both on the same side of the polar line of O as O itself. We shall assume that the polar line intersects that branch on which T_1 lies. The order of the singular points is then S_1, R_1, T_1, R_2, S_2, T_2, S_1.

The values $I_1 = 0$, $I_2 = \infty$, $I_3 = 6\sqrt{\dfrac{1+\lambda}{2}}$, $I_4 = 6\sqrt{\dfrac{1-\lambda}{2}}$ correspond resp. to the values of 0, ∞, ± 1 and $\pm k$ for $\dfrac{1}{sn\,(v)}$, thus to the values of v which are congruent (*mod.* $2K$ and $2iK'$) resp. with iK', 0, K and $K+iK'$. (see fig. 2).

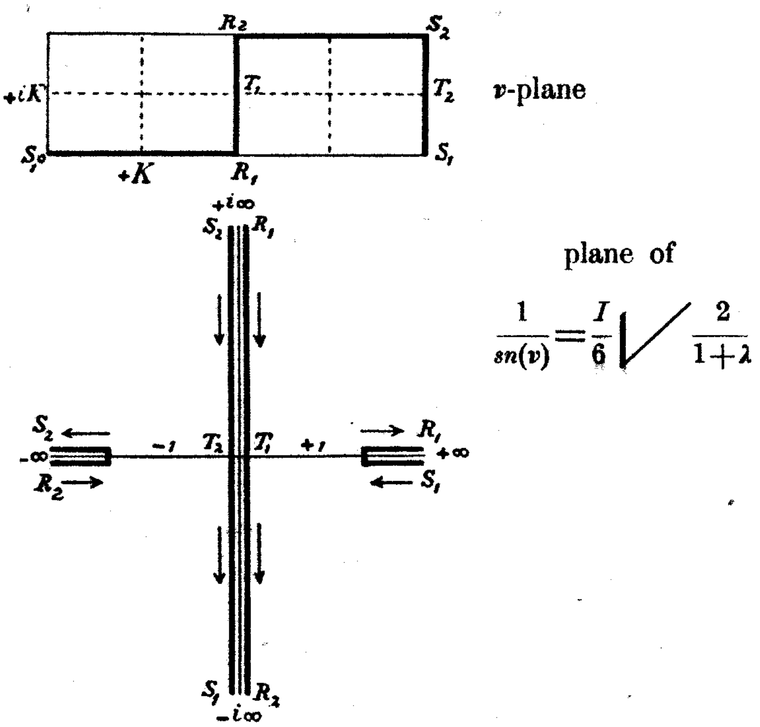

Fig. 2.

Case IVb. Point O lies inside the ellipse; T_1 and T_2 are real, R_1, R_2, S_1 and S_2 are imaginary.

IV*b*	in T_1	on T_1T_2	in T_2	on T_2T_1	in T_1
v	iK'	iK' + real	$2K+iK'$	iK' + real	$4K+iK'$
ζ	$\frac{\sqrt{1+\lambda}+\sqrt{1-\lambda}}{\lambda\sqrt{2}}$	pos. real	$\frac{\sqrt{1+\lambda}-\sqrt{1-\lambda}}{\lambda\sqrt{2}}$	pos. real	$\frac{\sqrt{1+\lambda}+\sqrt{1-\lambda}}{\lambda\sqrt{2}}$
$\dot{\zeta}$	0	neg. real	0	pos. real	0
I	0	pos. real	0	neg. real	0

The points where the motion of I changes its sign are according to what was found in IVa the points for which $\frac{1}{sn\,(v)} = \pm k$, thus $v = K + iK'$ (*mod.* $2K$ and $2iK'$) (see fig. 3).

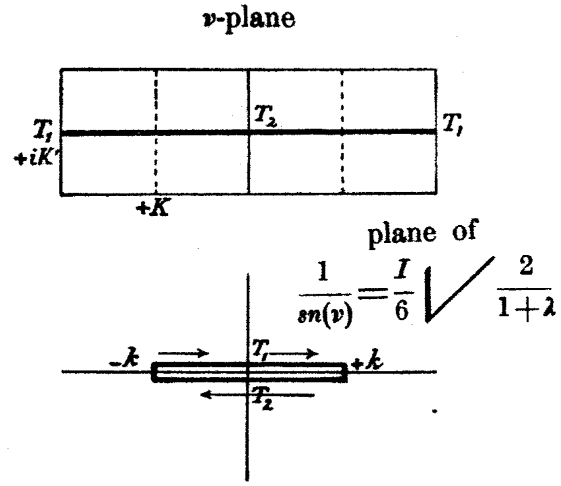

Fig. 3.

Case VIa. Point O lies on the concave side of the hyperbola: S_1, S_2, T_1 and T_2 are real, R_1 and R_2 are imaginary. Let T_1 be the point of intersection of the diameter through O lying on the same side of the polar line as O itself.

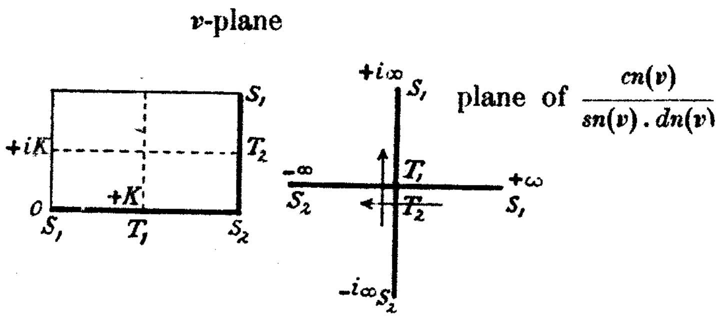

Fig. 4.

The values $I_1 = 0$, $I_2 = \infty$, $I_3 = 6\sqrt{\frac{1+i\lambda'}{2}}$, $I_4 = 6\sqrt{\frac{1-i\lambda'}{2}}$ correspond here respectively to the values 0, ∞, $\pm\sqrt[4]{\frac{1+i\lambda'}{1-i\lambda'}}$ and $\pm\sqrt[4]{\frac{1-i\lambda'}{1+i\lambda'}}$ for $\frac{cn(v)}{sn(v) \, . \, dn(v)}$, thus to the values of v which are congruent (*mod.* $2K$ and $K+iK'$) with $K, 0, \frac{1}{2}(K+3iK'), \frac{1}{2}(K+iK')$ (see fig. 4).

Case VIb. Point O lies outside the ellipse; R_1, R_2, T_1, and T_2 are real, S_1 and S_2 are imaginary. The point of intersection T_1 may lie on the same side of the polar line as O itself.

For the particular values of I and the corresponding values of v we can refer to VI*a*. (see fig. 5).

VIa	in S_1^∞	on S_1T_1	in T_1	on $T_1S_2^\infty$	in S_2^∞	on $S_2^\infty T_2$	in T_2	on $T_2S_1^\infty$	in S_1^∞
u	0	real	K	real	$2K$	$2K$+p.imag.	$2K+iK'$	$2K$+p.imag.	$2K+2iK'$
λ'	∞	pos. real	$+\frac{\sqrt{-1+\sqrt{1+\lambda'^2}}}{\lambda'}$	pos. real	∞	pos. imag.	$+i\frac{\sqrt{1+\sqrt{1+\lambda'^2}}}{\lambda'}$	pos. imag.	∞
μ'	∞	neg. real	0	pos. real	∞	neg. real	0	pos. real	∞
l	∞	pos. real	0	neg. real	∞	neg. imag	0	pos. imag.	∞

VIb	in R_1	on R_1T_1	in T_1	on T_1R_2	in R_2	on R_2T_2	in T_2	on T_2R_1	in R_1
u	$K+iK'$	K+p.imag.	$K+2iK'$	K+p.imag.	$K+3iK'$	$3iK'$+real	$2K+3iK'$	$3iK'$+real	$3K+3iK'$
λ'	0	neg. real	$-\frac{\sqrt{-1+\sqrt{1+\lambda'^2}}}{\lambda'}$	neg. real	0	neg. imag.	$-i\frac{\sqrt{1+\sqrt{1+\lambda'^2}}}{\lambda'}$	neg. imag.	0
μ'	$+\frac{i}{2}$	pos. imag.	0	neg. imag.	$-\frac{i}{2}$	neg. imag.	0	pos. imag.	$+\frac{i}{2}$
l	∞	pos. imag.	0	neg. imag.	∞	neg. real	0	pos. real	∞

Before investigating the cases of degeneration III and V we shall occupy ourselves for a moment with the relation (53) (4[th] comm. p. 1011), existing between I and I^2. In the case of the conic it takes the shape of (65) (4[th] comm. p. 1018). The curve it represents is as can be expected symmetrical with respect to the X-axis ($X = I^2$). To simplify the reasoning we shall translate the curve $\Phi(X, Y) \equiv \Phi(I^2, I) = 0$ parallel to the X-axis and we shall decrease it and that by the formulae of transformation

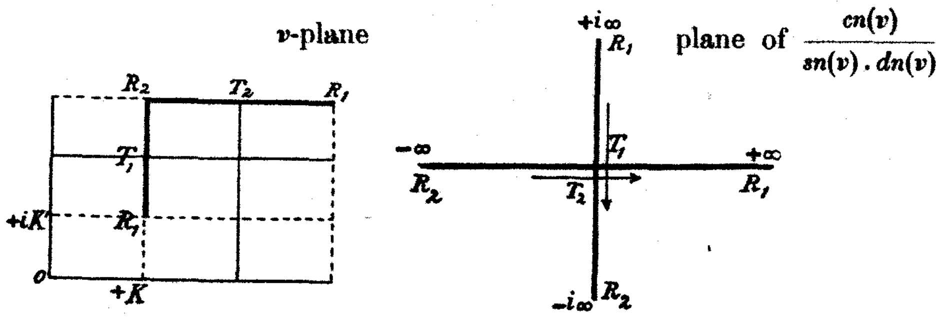

Fig. 5.

$$I^2 - 18 = 36\xi,$$
$$\dot{I} = 6\eta.$$

The equation of the curve transformed in this manner runs as follows:

$$\Phi(\xi, \eta) \equiv \xi^2 - \eta^2 - \frac{\lambda^2}{4} = 0;$$

the curve is therefore a rectangular hyperbola. In the cases II and IV the ξ-axis is the real axis, in case VI the η-axis is the real axis. Each point of the conic $F(x, y) = 0$ corresponds to *one* point of this rectangular hyperbola whilst to *one* point of $\Phi = 0$ two points of $F = 0$ are conjugated. The points for which $I = 0$ have as absciss $\xi = -\frac{1}{2}$. The line $\xi = -\frac{1}{2}$ does not intersect the curve Φ in case II, but it does in the cases IV and VI. The point at infinity on $\xi + \eta = 0$ represents the points S_1 and S_2; the point at infinity on $\xi - \eta = 0$ represents the two points R_1 and R_2. The points T_1 and T_2 are represented by the points of intersection of $\Phi = 0$ with $\xi = -\frac{1}{2}$. The images of the points T_1 and T_2 are in case VI united in the point of intersection of $\xi = -\frac{1}{2}$ with the branch of $\Phi = 0$ lying under the ξ-axis. The images of T_1 and T_2 are always points where the motion changes its sign along the curve Φ.

Now we have to investigate the cases of degeneration.

Case III^a. $\lambda = +1$, $\sigma_1 = 0$, a_{11} and a_{21} not disappearing at the same time.

The point O lies on one of the asymptotes, without coinciding with the centre. So this position occurs with the hyperbola only.

Here equation (71) holds, in which is put $\tau_0 = 0$,

$$I = \pm \frac{6}{\sin \tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71')$$

Equation (62) (4th comm. p. 1015) passes, on account of the relation

$$a_{33}A_{33} = \Delta,$$

and with the aid of (72), into

$$I^2 = \frac{9}{\zeta^2}(-\zeta^4+2\zeta^2-1) = -\frac{9(\zeta^2-1)^2}{\zeta^2},$$

from which ensues, in connection with (71),

$$\frac{6}{\sin\tau} = \pm\frac{3i(\zeta^2-1)}{\zeta},$$

or

$$\zeta = \pm\frac{i(1\pm\cos\tau)}{\sin\tau}.$$

We choose for ζ:

$$\zeta = +i\frac{1-\cos\tau}{\sin\tau} = +i\,tg\frac{\tau}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

and find in this manner

$$\dot{\zeta} = +\frac{i}{2}\sec^2\frac{\tau}{2}.$$

Now the equations (76) and (77) are incompatible. If they depended on each other we should have $A_{33} = 0$, which has not been supposed to be the case.

Equation (77) now runs:

$$a_{13}x + a_{23}y = -a_{33}z\sec^2\frac{\tau}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Bringing this equation into connection with $F(x, y, z) = 0$, we find

$$\left.\begin{aligned} 2A_{13}\sec^2\frac{\tau}{2}\,.\,x &= \left\{a_{22}a_{33}\sec^4\frac{\tau}{2} - a_{23}^2\left(2\sec^2\frac{\tau}{2}-1\right)\right\}z \\ 2A_{23}\sec^2\frac{\tau}{2}\,.\,y &= \left\{a_{11}a_{33}\sec^4\frac{\tau}{2} - a_{13}^2\left(2\sec^2\frac{\tau}{2}-1\right)\right\}z \end{aligned}\right\} \quad . \quad (87)$$

These formulae can be used unless either A_{13} or A_{23} is zero. Therefore we will mention also the expressions for x and y for the case $A_{13} = 0$. Then we have $a_{23} = 0$ on account of $a_{13}A_{13} + a_{23}A_{23} = 0$. We then find immediately out of (86) the expression for x, out of the second equation (87) in which A_{23} is replaced by $a_{12}a_{13}$ the expression for y. So the solution is:

$$a_{13}x = -a_{33}z\,.\,\sec^2\frac{\tau}{2}$$

$$2a_{12}a_{13}\sec^2\frac{\tau}{2}\,y = \left\{a_{11}a_{33}\sec^4\frac{\tau}{2} - a_{13}^2\left(2\sec^2\frac{\tau}{2}-1\right)\right\}z.$$

Case III[b]. $a_{13} = a_{23} = 0$.

The point O coincides with the centre.

Now we have

$$I = 0.$$

The expressions for x and y are of the form:

$$x = (ae^{i\tau} + a'e^{-i\tau})\,z,$$
$$y = (\beta e^{i\tau} + \beta' e^{-i\tau})\,z.$$

In order to have $F \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + a_{33}z^2 = 0$, we must put:

$$\alpha = \sigma(-a_{12} + \sqrt{-A_{33}}), \quad \alpha' = \sigma'(-a_{12} - \sqrt{-A_{33}}),$$
$$\beta = \sigma a_{11} \quad , \quad \beta' = \sigma' a_{11}$$

with the condition

$$\sigma\sigma' = \frac{a_{33}}{-4a_{11}A_{33}}.$$

In the case of the real ellipse we have $A_{33} > 0$ and $\frac{a_{33}}{a_{11}} < 0$. We then can put:

$$\sigma = \sigma' = \frac{1}{2}\sqrt{\frac{-a_{33}}{a_{11}A_{33}}}.$$

So we find

$$\left.\begin{aligned} x &= \frac{1}{2}\sqrt{\frac{-a_{33}}{a_{11}A_{33}}}\cdot\{-a_{12}(e^{i\tau}+e^{-i\tau}) + i\sqrt{A_{33}}\cdot(e^{i\tau}-e^{-i\tau})\}\,z = \\ &= \sqrt{\frac{-a_{33}}{a_{11}A_{33}}}\cdot(-a_{12}\cos\tau - \sqrt{A_{33}}\sin\tau)\,z, \\ y &= \frac{1}{2}\sqrt{\frac{-a_{33}}{a_{11}A_{11}}}\cdot a_{11}(e^{i\tau}+e^{-i\tau})\,z = \sqrt{\frac{-a_{11}a_{33}}{A_{33}}}\cdot\cos\tau\,.\,z. \end{aligned}\right\} \quad (88)$$

We can use the same expression if we have to deal with a hyperbola not intersecting the x-axis. For then $A_{33} < 0$ and $\frac{a_{33}}{a_{11}} > 0$, so $\sigma = \sigma'$ real. We prefer to write $-\sqrt{-A_{33}}\,.\,sh\,(i\tau)$ for $\sqrt{A_{33}}\,.\,\sin\tau = -i\sqrt{-A_{33}}\,.\,\sin\tau$ and $ch\,(i\tau)\cos$ for τ. Then real points of the hyperbola correspond to purely imaginary values of τ.

If the hyperbola does intersect the x-axis we have $A_{33} < 0$ and $\frac{a_{33}}{a_{11}} < 0$, so $\sigma = \sigma'$ imaginary.

We then put $\sigma = -\sigma' = \frac{1}{2}\sqrt{\frac{a_{33}}{a_{11}A_{33}}}$ and get in this manner

$$\left.\begin{aligned} x &= \frac{1}{2}\sqrt{\frac{a_{33}}{a_{11}A_{33}}}\cdot\{-a_{12}(e^{i\tau}-e^{-i\tau}) + \sqrt{-A_{33}}(e^{i\tau}+e^{-i\tau})\}\,z = \\ &= \sqrt{\frac{a_{33}}{a_{11}A_{33}}}\cdot\{-a_{12}\,sh(i\tau) + \sqrt{A_{33}}\,ch(i\tau)\}\,z, \\ y &= \frac{1}{2}\sqrt{\frac{a_{33}}{a_{11}A_{33}}}\cdot a_{11}(e^{i\tau}-e^{-i\tau})\,z = \sqrt{\frac{a_{11}a_{33}}{A_{33}}}\,.\,sh(i\tau)\,.\,z. \end{aligned}\right\} \quad (88)'$$

Here also τ must describe in its complex plane the imaginary axis. For $a_{11}=0$, we get $(2a_{12}x+a_{22}y)\,y+a_{33}=0$.
A solution of this is given by

$$x=\frac{-1}{2a_{12}}\left(a_{22}e^{i\tau}+a_{33}e^{-i\tau}\right),$$
$$y=e^{i\tau}.$$

Here also only purely imaginary values of τ come in consideration, as might be expected.

The second case of degeneration (IV) presents itself for $\lambda=0$, i. e. $\delta_1=+1$. Here we must distinguish three subdivisional cases, viz.

IV^a. $a_{33}=0$: the point O lies on the conic,

IV^b. $A_{33}=0$: the conic is a parabola,

IV^c. $a_{33}=0$ and $A_{33}=0$: the point O lies on the parabola.

Case IV^a. Here we have (70*a*) (4th comm. p. 1017); substitution of $\tau_0=0$ furnishes

$$I=+3\sqrt{2}\,.\,th\frac{\tau}{\sqrt{2}},\quad \ldots\ldots\quad (70'a)$$

so

$$\dot{I}=\frac{3}{ch^2\dfrac{\tau}{\sqrt{2}}}$$

Now the equations (62) and (63) (4th comm. p. 1015) teach us

$$g=a_{13}x+a_{23}y=\frac{2\Delta\dot{I}z}{3A_{33}}=\frac{2\Delta z}{A_{33}}\cdot\frac{1}{ch^2\dfrac{\tau}{\sqrt{2}}}$$

$$A_{23}x-A_{13}y=\sqrt{-A_{33}g^2+2\Delta gz}=\sqrt{\left(-\frac{4\Delta^2}{A_{33}^2}\cdot\frac{1}{ch^4\dfrac{\tau}{\sqrt{2}}}+\frac{4\Delta^2}{A_{33}^2}\cdot\frac{1}{ch^2\dfrac{\tau}{\sqrt{2}}}\right)}\,.\,z$$

$$=\frac{2\Delta z}{A_{33}}\cdot\frac{sh\dfrac{\tau}{\sqrt{2}}}{ch^2\dfrac{\tau}{\sqrt{2}}};$$

so we get

$$\left.\begin{aligned} x&=\frac{2z}{A_{33}ch^2\dfrac{\tau}{\sqrt{2}}}\left(A_{13}+a_{23}\sqrt{A_{33}}\,.\,sh\frac{\tau}{\sqrt{2}}\right)\\ y&=\frac{2z}{A_{33}ch^2\dfrac{\tau}{\sqrt{2}}}\left(A_{23}-a_{13}\sqrt{A_{33}}\,.\,sh\frac{\tau}{\sqrt{2}}\right)\end{aligned}\right\}\quad\ldots\quad(89)$$

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In the case $A_{33} < 0$ we prefer to write $i\sqrt{-A_{33}} \,.\, sh\frac{\tau}{\sqrt{2}} =$
$= +\sqrt{-A_{33}} \,.\, \sin\frac{i\tau}{\sqrt{2}}$ for $\sqrt{A_{33}} \,.\, sh\frac{\tau}{\sqrt{2}}$ and $\cos\frac{i\tau}{\sqrt{2}}$ for $ch\frac{\tau}{\sqrt{2}}$.

So, whilst the formulae (89) are specially suitable for the ellipse we do better in using for the hyperbola

$$\left.\begin{aligned} x &= \frac{2z}{A_{33}\cos^2\frac{i\tau}{\sqrt{2}}}\left(A_{13} + a_{23}\sqrt{-A_{33}} \,.\, \sin\frac{i\tau}{\sqrt{2}}\right) \\ y &= \frac{2z}{A_{33}\cos^2\frac{i\tau}{\sqrt{2}}}\left(A_{23} - a_{13}\sqrt{-A_{33}} \,.\, \sin\frac{i\tau}{\sqrt{2}}\right) \end{aligned}\right\} \quad . \quad . \quad (89')$$

Consequently the real points of the hyperbola correspond to purely imaginary values of τ.

Case IV^b. Putting $\tau_0 = 0$, (70*b*) (4th comm. p. 1017) we find

$$I = -3\sqrt{2} \,.\, th\frac{\tau}{\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70'b)$$

and therefore

$$\dot{I} = -\frac{3}{ch^2\frac{\tau}{\sqrt{2}}} .$$

So the formulae (62) and (63) now give

$$g = a_{13}x + a_{23}y + a_{33}z = -\frac{3a_{33}z}{2\dot{I}} = \frac{a_{33}z}{2}\,ch^2\frac{\tau}{\sqrt{2}},$$

i.e.

$$a_{13}x + a_{23}y = \frac{a_{33}z}{2}\left(ch^2\frac{\tau}{\sqrt{2}} - 2\right)$$

and

$$A_{23}x - A_{13}y = \sqrt{2\Delta gz - \Delta a_{33}z^2} = \sqrt{a_{33}\Delta}.z.sh\frac{\tau}{\sqrt{2}},$$

so we find

$$\left.\begin{aligned} x &= \left\{\frac{a_{23}\sqrt{a_{33}\Delta}}{\Delta}\,sh\frac{\tau}{\sqrt{2}} + \frac{a_{33}A_{13}}{2\Delta}\left(ch^2\frac{\tau}{\sqrt{2}} - 2\right)\right\} z, \\ y &= \left\{\frac{-a_{13}\sqrt{a_{33}\Delta}}{\Delta}\,sh\frac{\tau}{\sqrt{2}} + \frac{a_{33}A_{23}}{2\Delta}\left(ch^2\frac{\tau}{\sqrt{2}} - 2\right)\right\} z. \end{aligned}\right\} \quad . \quad . \quad (90)$$

Case IV^c. Here we have

$$I = \pm 3\sqrt{2}$$

The equation

$$F \equiv a_{11}x^2 + 2a_{12}xy + \frac{a_{12}^2}{a_{11}} y^2 + 2a_{13}x + 2a_{23}y = 0$$

or

$$(a_{11}x + a_{12}y)^2 + 2a_{11}(a_{13}x + a_{23}y) = 0$$

passes by the substitution

$$\left.\begin{aligned} a_{11}x + a_{12}y &= 2a_{11}\xi \\ a_{13}x + a_{23}y &= -2a_{11}\eta \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (91)$$

into

$$\eta = \xi^2,$$

a solution of which (see 2^{nd} comm. p. 590) is

$$\xi = e^{-\frac{\tau}{\sqrt{2}}} \quad , \quad \eta = e^{-\tau\sqrt{2}} \quad . \; . \; . \; . \; . \; . \quad (92)$$

Out of (91) and (92) we deduce

$$\left.\begin{aligned} x &= -\frac{2a_{11}}{A_{12}}\left(a_{23}\, e^{-\frac{\tau}{\sqrt{2}}} + a_{12}\, e^{-\tau\sqrt{2}}\right) \\ y &= \frac{2a_{11}}{A_{12}}\left(a_{13}\, e^{-\frac{\tau}{\sqrt{2}}} + a_{11}\, e^{-\tau\sqrt{2}}\right) \end{aligned}\right\} \quad . \; . \; . \; . \quad (93)$$

These formulae are always applicable, as the supposition $A_{12} = 0$ would imply the degeneration of the parabola.

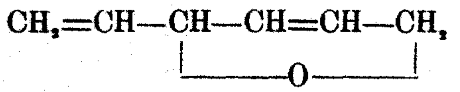
Chemistry. — *"On some internal unsaturated ethers".* By J. W. LE HEUX. (communicated by Prof. VAN ROMBURGH).

(Preliminary communication).

(Communicated in the meeting of April 26, 1912).

By the action of formic acid on mannitol FAUCONNIER obtained a mixture of formic esters of this hexavalent alcohol, which submitted to dry distillation, yielded among other products a liquid of the composition C_6H_8O, boiling at 107°—109°.

VAN MAANEN (Dissertation, Utrecht 1909) who investigated this substance and mentions it as a liquid boiling at 107° proposed as the most probable structural formula:



As the mode of formation of this substance does not give a complete insight into its structural formula, Prof. VAN ROMBURGH proposed to me to prepare the various possible oxides of hexadiene by other

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