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Mathematics. — *"New researches upon the centra of the integrals which satisfy differential equations of the first order and the first degree."* (Second Part). By Prof. W. Kapteyn.

8. Assuming in the third place

$$a' + c' = i(a+c)$$
$$aa' - cc' = (b - ib')(a+c)$$
$$2b' = 3a + 5c$$

or putting $b = i\beta$

$$2a' = -i(3a - 2\beta + 3c)$$
$$2c' = i(5a - 2\beta + 5c)$$
$$2b' = 3a + 5c.$$

We have

$$q_1 = a' - i(3a + 2b') = -\frac{i}{2}(15a - 2\beta + 13c)$$

$$q_2 = 2a + 3b' - ib = \frac{1}{2}(13a + 2\beta + 15c)$$

$$r_0 = -\frac{i}{6}(86a^2 + 26a\beta + 179ac - 4\beta^2 + 28\beta c + 99c^2)$$

$$r_1 = -\frac{1}{4}(45a^2 - 36a\beta + 84ac + 4\beta^2 - 32\beta c + 39c^2)$$

$$r_2 = -\frac{i}{2}(130a^2 - 6a\beta + 265ac - 4\beta^2 - 8\beta c + 137c^2)$$

$$r_3 = \frac{1}{12}(421a^2 + 116a\beta + 972ac - 12\beta^2 + 120\beta c + 567c^2)$$

and for the coefficients of P_4

$$s_1 = (5a + 2b')r_0 + a'r_1$$
$$2s_2 - 4s_0 = (8b + 2c')r_0 + (4a + 4b')r_1 + 2a'r_2$$
$$3s_3 - 3s_1 = 3cr_0 + (6b + 3c')r_1 + (3a + 6b')r_2 + 3a'r_3$$
$$4s_4 - 2s_2 = 2cr_1 + (4b + 4c')r_2 + (2a + 8b')r_3$$
$$-s_3 = cr_2 + (2b + 5c')r_3.$$

To determine the next condition we introduce the two following polynomia

$$P_5 = t_0x_5 + t_1x^4y + t_2x^3y^2 + t_3x^2y^3 + t_4xy^4 + t_5y^5$$
$$P_6 = u_0x^6 + u_1x^5y + u_2x^4y^2 + u_3x^3y^3 + u_4x^2y^4 + u_5xy^5 + u_6y^6.$$

The coefficients of the first are determined by the relations

$$t_1 = (6a+2b')\,s_0 + a's_1$$
$$2t_2 - 5t_0 = (10b+2c')\,s_0 + (5a+4b')\,s_1 + 2a's_2$$
$$3t_3 - 4t_1 = 4c\,s_0 + (8b+3c')\,s_1 + (4a+6b')\,s_2 + 3a'\,s_3$$
$$4t_4 - 3t_2 = 3c\,s_1 + (6b+4c')\,s_2 + (3a+8b')\,s_3 + 4a'\,s_4$$
$$5t_5 - 2t_3 = 2c\,s_2 + (4b+5c')\,s_3 + (2a+10b')\,s_4$$
$$-\ t_4 = cs_3 + (2b+6c')\,s_4$$

which may always be satisfied, and the coefficients of the second are related to those of the first by the following system

$$u_1 = (7a+2b')\,t_0 + a't_1$$
$$2u_2 - 6u_0 = (12b+2c')\,t_0 + (6a+4b')\,t_1 + 2a't_2$$
$$3u_3 - 5u_1 = 5c\,t_0 + (10b+3c')\,t_1 + (5a+6b')\,t_2 + 3a'\,t_3$$
$$4u_4 - 4u_2 = 4c\,t_1 + (8b+4c')\,t_2 + (4a+8b')\,t_3 + 4a'\,t_4$$
$$5u_5 - 3u_3 = 3c\,t_2 + (6b+5c')\,t_3 + (3a+10b')\,t_4 + 5a'\,t_5$$
$$6u_6 - 2u_4 = 2c\,t_3 + (4b+6c')\,t_4 + (2a+12b')\,t_5$$
$$-\ u_5 = c\,t_4 + (2b+7c')\,t_5.$$

This system is impossible unless

$$5u_1 + (3u_3-5u_1) + (5u_5-3u_3) + 5\,(-u_5) = 0$$

or

$$(35a+10b'+5c)\,t_0 + (5a'+10b+3c')\,t_1 + (5a+6b'+3c)\,t_1 +$$
$$+ (3a'+6b+5c')\,t_3 + (3a+10b'+5c)\,t_4 + (5a'+10b+35c')\,t_5 = 0$$

which may be written

$$A\,t_1 + B\,(2t_2-5t_0) + C\,(3t_3-4t_1) + D\,(4t_4-3t_2) + E\,(5t_5-2t_3) + F\,(-t_4) = 0$$

if

$$A = \frac{5}{3}(7a'+14b+17c')\quad,\quad B = -(7a+2b'+c),$$
$$C = \frac{1}{3}(5a'+10b+19c')\quad,\quad D = -\frac{1}{3}(19a+10b'+5c).$$
$$E = a' + 2b + 7c'\quad,\quad F = -\frac{5}{3}(17a+14b'+7c).$$

Thus, choosing as before $s_0 = 0$, the sought condition takes this form

$$s_1\,[\,a'A + (5a+4b')\,B + (8b+3c')\,C + 3c\,D]$$
$$+ s_2\,[2a'B + (4a+6b')\,C + (6b+4c')\,D + 2c\,E]$$
$$+ s_3\,[3a'C + (3a+8b')\,D + (4b+5c')\,E + c\,F]$$
$$+ s_4\,[4a'D + (2a+10b')\,E + (2b+6c')\,F] = 0.$$

Writing this equation

$$f_1 s_1 + f_2 s_2 + f_3 s_3 + f_4 s_4 = 0$$

and eliminating $a'\ b'\ c'$ we obtain

$$A = \frac{20\,i}{3}(8a+\beta+8c)\ ,\ B = -2\,(5a+3c)\ ,\ C = \frac{4i}{3}(10a-\beta+10c)$$

$$D = -\frac{2}{3}(17a+15c) \quad , E = 4i(4a-\beta+4c), \; F = -\frac{10}{3}(19a+21c)$$

$$f_1 = -10\ (a+c)(13a+10\beta+11c) = 10\ (a+c)\, g_1$$
$$f_2 = \quad 30i\,(a+c)(3a-2\beta+5c) \quad = 10\ (a+c)\, g_2$$
$$f_3 = -10\ (a+c)(31a-2\beta+41c) \quad = 10\ (a+c)\, g_3$$
$$f_4 = -10i\,(a+c)(61a-14\beta+59c) = 10\ (a+c)\, g_4.$$

Now, omitting the factor 10 $(a+c)$, we get

$$g_1 s_1 + g_2 s_2 + g_3 s_3 + g_4 s_4 = 0$$

wherein the values s may be expressed in function of r in this way

$$2s_1 = (16a+10c)\, r_0 - i\,(3a-2\beta+3c)\, r_1$$
$$2s_2 = i\,(5a+6\beta+5c)\, r_0 + 10\,(a+c)\, r_1 - i\,(3a-2\beta+3c)\, r_2$$
$$2s_3 = -2c\, r_2 - i\,(25a-6\beta+25c)\, r_3$$
$$2s_4 = \frac{i}{2}(5a+6\beta+5c)\, r_0 + (5a+6c)\, r_1 + \frac{i}{2}(7a+2\beta+7c)\, r_2 + (7a+10c)\, r_3$$

Substituting these values, and putting

$$G_1 = 13a + 10\beta + 11c \quad , \quad G_2 = 9a - 6\beta + 15c\, ,$$
$$G_3 = 31a - 2\beta + 41c \quad , \quad G_4 = 61a - 14\beta + 59c$$

we find

$$r_0\left[-(16a+10c)\, G_1 - (5a+6\beta+5c)\, G_2 + \frac{1}{2}(5a+6\beta+5c)\, G_4\right]$$
$$+\ ir_1\left[(3a-2\beta+3c)\, G_1 + 10\,(a+c)\, G_2 - (5a+6c)\, G_4\right]$$
$$+\ r_2\left[(3a-2\beta+3c)\, G_2 + 2c\, G_3 + \frac{1}{2}(7a+2\beta+7c)\, G_4\right]$$
$$+\ ir_3\left[(25a-6\beta+25c)\, G_3 - (7a+10c)\, G_4\right] = 0$$

which may be reduced to

$$\frac{1}{2}\, r_0\left[-201\,a^2 - 72\,a\beta - 252ac - 12\beta^2 - 36\beta c - 75c^2\right]$$
$$+\ ir_1\left[-176\,a^2 + 14a\beta - 349ac - 20\beta^2 + 32\beta c - 171c^2\right]$$
$$+\ \frac{1}{2}\, r_2\left[481\,a^2 - 48a\beta + 1108ac - 4\beta^2 - 84\beta c + 667c^2\right]$$
$$+\ ir_3\left[348\,a^2 - 138a\beta + 777ac + 12\beta^2 - 156\beta c + 435c^2\right] = 0$$

Writing this result

$$\frac{1}{3}\, r_0\, T_0 + ir_1\, T_1 + \frac{1}{2}\, r_2\, T_2 + ir_3\, T_3 = 0$$

and assuming

$$r_0 = -\frac{i}{6}R_0\, , \; r_1 = -\frac{1}{4}R_1\, , \; r_2 = -\frac{i}{2}R_2\, , \; r_3 = \frac{1}{12}R_3$$

we obtain

$$R_0 T_0 + 3R_1 T_1 + 3\,R_2 T_2 - R_3 T_3 = 0$$

which after reduction gives finally the condition

$$12\,(a+c)\,(a-2\beta-c)\,(3\,a-2\beta+5c) = 0.$$

This condition breaks up into three others from which the first $a+c=0$ has already been examined in Art. 2.

9. Introducing the second, we must examine the case where

$$\begin{aligned} 2a' &= -\,i\,(3a-2\beta+3c) \\ 2c' &= i\,(5a-2\beta+5c) \\ 2b' &= 3a+5c \\ 2\beta &= a-c \end{aligned}$$

or, remembering that $b=i\beta$

$$\begin{aligned} a' &= -\,i\,(a+2c) \\ c' &= i\,(2a+3c) \\ 2b' &= 3a+5c \\ 2b &= i\,(a-c). \end{aligned}$$

This case has already been met with in Art. 7.

10. Finally we have the relations

$$\begin{aligned} 2a' &= -\,i\,(3a-2\beta+3c) \\ 2c' &= i\,(5a-2\beta+5c) \\ 2b' &= 3a+5c \\ 2\beta &= 3a+5c \end{aligned}$$

which are identical with

$$\begin{aligned} a' &= ic \\ c' &= ia \\ 2b &= 2ib' = i\,(3a+5c). \end{aligned}$$

The differential equation reduces in this case to

$$\frac{dy}{dx} = \frac{-x + icx^2 + (3a+5c)\,xy + iay^2}{y + ax^2 + i\,(3a+5c)\,xy + cy^2}$$

whose general integral may be constructed from the two particular integrals

$$(a+3c)\,(x-iy)^2 + 2i\,(x+iy) + \frac{1}{a+c} = 0$$

and

$$(a+3c)\,(x-iy)^3 + 3\,i\,(x^2+y^2) = 0$$

which are easily found.

This general integral

$$\frac{\{(a+3c)\,(x-iy)^3 + 3i\,(x^2+y^2)\}^2}{\left\{(a+3c)\,(x-iy)^2 + 2i\,(x+iy) + \dfrac{1}{a+c}\right\}^3} = const.$$

may be expanded for small values of x and y in the form

$$x^2 + y^2 + F_3 + F_4 + \ldots = const.$$

which proves again that the origin is a centrum.

11. Resuming we may conclude that where

$$(a + c)^2 + (a' + c')^2 = 0.$$

the differential equation

$$\frac{dy}{dx} = \frac{-x + a'x^2 + 2b'xy + c'y^2}{y + ax^2 + 2b\,xy + cy^2}$$

has a centrum in the origin of coordinates only in the following cases

I. $a + c = 0$ en $a' + c' = 0$

II. $a' + c' = \pm i\,(a + c)$ en $a + b' = 0$

III. $2a' = \pm i\,(a - 2b' + c)$, $2c' = \pm i\,(a + 2b' + c)$, $2b = \pm i\,(a - c)$

IV. $a' = \pm ic$, $c' = \pm ia$, $2b' = 3a + 5c$, $2b = 3a + 5c$

for it is easily seen that in the last three cases everywhere i may be replaced by $-i$.

The results obtained in our former paper show that the origin is also a centrum in the three following cases

V. $a' + c' = 0$, $a' = b$ en $a + b' = 0$

VI. $a' + c' = 0$, $a' = b = 0$

VII. $a' + c' = 0$, $a' = b$, $2b' = 3a + 5c$, $ac + b^2 + 2c^2 = 0$.

We found there one case more viz.

$$a' + c' = 0, \quad a' = b \quad \text{and} \quad a + c = 0$$

but this is included in I.

12. To compare these results with those of Dulac, we will transform our differential equation

$$\frac{dy}{dx} = \frac{-x + a'x^2 + 2b'xy + c'y^2}{y + ax^2 + 2bxy + cy^2} = \frac{-x + Y}{y + X}$$

in his form. This may be done by the substitution

$$h\xi = x + iy \qquad k\eta = x - iy.$$

This gives

$$\frac{hd\xi}{y + X + i(-x + Y)} = \frac{kd\eta}{y + X - i(-x + Y)}$$

where

$$y - ix = -ih\xi\,, \qquad y + ix = ik\eta$$

$$X + iY = -i\,(A - B)\,h^2\xi^2 - 2i\,(C - C')\,hk\xi\eta - i\,(D - E)\,k^2\eta^2$$

$$X - iY = -i\,(D + E)\,h^2\xi^2 - 2i\,(C + C')\,hk\xi\eta - i\,(A + B)\,k^2\eta^2$$

and

$$A = \frac{i}{4}(a+2b'-c), \quad B = \frac{1}{4}(a'-2b-c'), \quad C = \frac{i}{4}(a+c)$$

$$D = \frac{i}{4}(a-2b'-c), \quad E = \frac{1}{4}(a'+2b-c'), \quad C' = \frac{1}{4}(a'+c').$$

Thus we find generally

$$\left[\xi + h(A-B)\xi^2 + 2k(C-C')\xi\eta + \frac{k^2}{h}(D-E)\eta^2\right]d\eta$$
$$+\left[\eta - k(A+B)\eta^2 - 2h(C+C')\xi\eta - \frac{h^2}{k}(D+E)\xi^2\right]d\xi = 0 \qquad (A)$$

and when $C' = 0$ or $a' + c' = 0$

$$\left[\xi + h(A-B)\xi^2 + 2kC\xi\eta + \frac{k^2}{h}(D-E)\eta^2\right]d\eta$$
$$+\left[\eta - k(A+B)\eta^2 - 2hC\xi\eta - \frac{h^2}{k}(D+E)\xi^2\right]d\xi = 0 \qquad (B)$$

where

$$A = \frac{i}{4}(a+2b'-c), \; B = \frac{2(a'-b)}{4}, C = \frac{i(a+c)}{4}$$

$$D = \frac{i}{4}(a-2b'-c), \; E = \frac{2(a'+b)}{4}.$$

If now we compare with (B) the first equation (1) of Art. 1 we have

$$h(A-B) = 1, \qquad 2kC = \mu, \qquad \frac{k^2}{h}(D-E) = \nu$$

$$-k(A+B) = 1, \quad -2hC = \mu, \quad -\frac{h^2}{k}(D+E) = \nu$$

which may be satisfied by taking $k = -h$ and

$$B = 0, \quad E = 0$$

or

$$a' = b = 0.$$

This first equation therefore belongs to our class VI.

In the same way we may infer that

(2) belongs to class V
(3) is a special case of class I
(4) belongs to class VII
(7) is a special case of class I
(9) is a special case of class VI
(11) is a special case of class I.

If now $C' \neq 0$ we compare with (A). This gives for the fifth equation of Art 1

4*

$$h(A-B)=1, \quad 2k(C-C')=0, \quad \frac{k^2}{h}(D-E)=0$$

$$-k(A+B)=0, \quad -2h(C+C')=v, \quad -\frac{h^2}{k}(D+E)=v$$

which may be satisfied by

$$A+B=0, \quad D-E=0, \quad C-C'=0$$

or

$$2a'=i(a-2b'+c), \quad 2c'=i(a+2b'+c), \quad 2b=i(a-c).$$

Thus (5) belongs to class III.

In the same way it is seen that

(6) is a special case of class II
(8) belongs to class IV
(10) is a special case of class III.

The eleven equations given by DULAC are therefore contained in the preceding 7 classes.

Chemistry. — *"On a few oxyhaloids."* By Prof. F. A. H. SCHREINEMAKERS and Mr. J. MILIKAN.

Of the chlorides, bromides, and iodides of the alkaline earths several oxy-salts have already been described; in order to further investigate the occurrence or non-occurrence of these salts, to determine the limits of concentration between which they exist and, if possible, to find other oxyhaloids, different isotherms have now been determined and by means of the "residue method" [1]) the compositions of the solid phases have been deduced therefrom. Here, we will discuss only the solid substances that can be in equilibrium with solution.

The system $CaCl_2$—CaO—H_2O.

Temperature 10° *and* 25°. At both these temperatures occur, besides $CaCl_2 . 6 H_2O$ and $Ca(OH)_2$, as solid phases the oxychlorides:

$$CaCl_2 . 3 CaO . 16 H_2O \text{ and } CaCl_2 . CaO . 2 H_2O$$

the composition of the second salt may be expressed also as:

$$Ca\begin{matrix}\diagup Cl \\ \diagdown OH\end{matrix} . \tfrac{1}{2} H_2O$$

This latter oxychloride has already been found previously by a determination of the isotherm of 25° [2]); the first one was then

[1]) F. A. H. SCHREINEMAKERS. Die heterogenen Gleichgewichte von H. W. BAKHUIS ROOZEBOOM. IIIe 149.

[2]) F. A. H. SCHREINEMAKERS and TH. FIGEE, Chem. Weekbl. 683 (1911).