**Mathematics.** — *"The scale of regularity of polytopes"*. By Dr. E. L. ELTE (Meppel). (Communicated by Prof. P. H. SCHOUTE).

In my dissertation [1]) it was my aim to determine the semiregular polytopes, i. e. the polytopes analogous to the semiregular polyhedra. So this investigation had to be based on a definition of the notion "semiregular polytope". Now ordinarily a semiregular polyhedron is defined as follows: "A semiregular polyhedron has *either* congruent (or symmetric) vertices and regular faces *or* congruent faces and regular vertices. So there are two kinds of semiregular polyhedra which we will call with CATALAN [2]) "semiregular of the first kind" and "semiregular of the second kind"; those of the first kind are enumerated in the following table. For any of these polyhedra this table gives the numbers of vertices, edges, faces and indicates which faces pass through each vertex and which couples of faces pass through each kind of edges. Here $p_n$ denotes a regular polygon with $n$ vertices.

| Notation | N°. | Vertices | Edges | Faces | Faces through a vertex | Faces through the edges | | |
|---|---|---|---|---|---|---|---|---|
| tT | 1 | 12 | 18 | 8 | $1p_3$ , $2p_6$ | $p_6, p_6$ | $p_6, p_3$ | |
| tC | 2 | 24 | 36 | 14 | $1p_3$ , $2p_8$ | $p_8, p_8$ | $p_8, p_3$ | |
| tO | 3 | 24 | 36 | 14 | $1p_4$ , $2p_6$ | $p_6, p_6$ | $p_6, p_4$ | |
| tD | 4 | 60 | 90 | 32 | $1p_3$ , $2p_{10}$ | $p_{10}, p_{10}$ | $p_{10}, p_3$ | |
| tI | 5 | 60 | 90 | 32 | $1p_5$ , $2p_6$ | $p_6, p_6$ | $p_6, p_5$ | |
| CO | 6 | 12 | 24 | 14 | $2p_3$ , $2p_4$ | $p_4, p_3$ | | |
| ID | 7 | 30 | 60 | 32 | $2p_3$ , $2p_5$ | $p_5, p_3$ | | |
| RCO | 8 | 24 | 48 | 26 | $1p_3$ , $3p_4$ | $p_4, p_4$ | $p_4, p_3$ | |
| RID | 9 | 60 | 120 | 62 | $1p_3$ , $2p_4$ , $1p_5$ | $p_4, p_5$ | $p_4, p_3$ | |
| tCO | 10 | 48 | 72 | 26 | $1p_4$ , $1p_6$ , $1p_8$ | $p_6, p_8$ | $p_4, p_8$ | $p_4, p_6$ |
| tID | 11 | 120 | 180 | 62 | $1p_4$ , $1p_6$ , $1p_{10}$ | $p_{10}, p_6$ | $p_{10}, p_4$ | $p_4, p_6$ |
| CS | 12 | 24 | 60 | 38 | $1p_4$ , $4p_3$ | $p, p_4$ | $p_3, p_3$ | |
| DS | 13 | 60 | 150 | 92 | $1p_5$ , $4p_3$ | $p_5, p_3$ | $p_3, p_3$ | |
| $P_n$ | 14 | $2n$ | $3n$ | $n+2$ | $1p_n$ , $2p_4$ | $p_n, p_4$ | $p_4, p_4$ | |
| $AP_n$ | 15 | $2n$ | $4n$ | $2n+2$ | $1p_n$ , $3p_3$ | $p_n, p_3$ | $p_3, p_3$ | |

[1]) "The semiregular polytopes of the hyperspaces", Groningen, 1912.

[2]) "Mémoire sur la théorie des polyèdres", *Journal de l'École Polytechnique*, Cahier 47.

The semiregular polytopes of the second kind are the polar-reciprocal figures of those given in the table with respect to a concentric sphere.

The definition of semiregular polyhedron given above had to be modified in order to make it applicable to polydimensional spaces.

We say that a polyhedron possesses a "characteristic of regularity", if either all the vertices, or all the edges, or all the faces are equal to each other. Equality of vertices signifies that the polyangles formed by the edges concurring in each vertex are congruent (or symmetric); equality of faces consists in the congruency of the limiting polygons. But the equality of edges includes two different parts which can present themselves each for itself: equality in length of the edges and equality of the angles of position of the faces through the edges. So all the polyhedra of the table have edges of the same length but -- with exception of the numbers 6 and 7 — more than one kind of angles of position, whilst quite the reverse presents itself with the corresponding polyhedra of the second kind. If the equality of edges is realized only partially — as in the case of the polyhedra of the table — we speak of a "half characteristic" so that these polyhedra admit 1½ characteristics. By bringing this result in connection with the circumstance that a polyhedron can admit 3 characteristics, the epitheton "semiregular" obtains a *literary* signification. As the polyhedra N°. 6 and N°. 7 of the table possess both the half characteristics of the edges, these polyhedra must be called "$^2/_3$-regular" according to our system.

We remark that the characteristics of a semiregular polyhedron of one of the two kinds are lacking in the corresponding polyhedron of the other. Moreover that we are obliged to observe a quite determinate order of succession in counting the characteristics of a polyhedron of defined kind and, beginning at the commencement, to count successive characteristics only, i. e. in the case of polyhedra of the first kind to take into account successively equality of vertices, equality in length of edges, equality of angles of position round edges, equality of faces, and reversely in the case of polyhedra of the second kind. If this order of succession was not observed e. g. with respect to the two half characteristics of the edges a beam with different length, breadth, and height would appear as a semiregular polyhedron of the first kind on account of equality of vertices and angles of position, whilst a double pyramid formed by the superposition of two faces of two equal regular tetratredra would appear as a semiregular polyhedron of the second kind, to which enunciations fundamental objections can be raised.

Now the *definition* of "degree of regularity" extended to higher spaces runs as follows:

"The degree of regularity of an $n$-dimensional polytope is a fraction with $n$ as numerator and the number $p$ of the successive characteristics of regularity as denominator, this number $p$ being counted in the case of a polytope of the first kind from the vertex end, in the case of a polytype of the second kind from the end of the limiting $n$—1-dimensional polytope."

In my dissertation I have confined myself to polytopes of the first kind, the degree of regularity of which is ½ at least. For the methods employed in unearthing these polytopes I must refer to that memoir.

In discussing my dissertation my promotor Dr. P. H. SCHOUTE remarked that if all the fractions representing possible degrees of regularity of an $n$-dimensional polytope are reduced to the denominator $2n$ the numerators 1 and $2n-1$ will be lacking, on account of the fact that the first and the last characteristic have not been subdivided into two halves; so in this sense my scale contains something superfluous.

Indeed the classification of the polyhedra according to my scale is indicated in the diagram

| 0 | $\frac{1}{6}$ | $\frac{2}{6}$ | $\frac{3}{6}$ | $\frac{4}{6}$ | $\frac{5}{6}$ | 1 |
|---|---|---|---|---|---|---|
| *I* | | *P* | 1—5, 8—15 | 6,7 | | *R* |

where the numbers 1—5, 8—15 at the *midpoint* and 6,7 at the right designate the polyhedra bearing these numbers in the table, whilst *I* and *R* stand for quite irregular and regular polyhedra and *P* either for the beam or for the double pyramid mentioned above, according to the scale corresponding either to polyhedra of the first or to polyhedra of the second kind. Indeed the points of division $\frac{1}{6}$ and $\frac{5}{6}$ are unoccupied and in $S_n$ the analogous characteristic property presents itself with respect to the points of division $\frac{1}{2n}$ and $\frac{2n-1}{2n}$.

It goes without saying that we can take away the superfluity indicated (of the two points of division adjacent on either side to the extremities) *either* by counting each of the two extreme characteristics, that of the vertices and that of the limiting $n$—1-dimensional polytopes, for half a characteristic, *or* — what comes to the same — by counting each of the two extreme characteristics and each of the two halves of the remaining intermediate characteristics for one. So the scale relating to our space passes into

| 0 | $\frac{1}{4}$ | $\frac{2}{4}$ | $\frac{3}{4}$ | 1 |
|---|---|---|---|---|
| $I$ | $P$ | 1—5, 8—15 | 6,7 | $R$ |

where the numbers and the letters have the same meaning as above. An $n$-dimensional polytope of the degree of regularity $\frac{p}{n}$ according to the scale given in my dissertation will be qualified, for $1 \leq p \leq n - 1$, by the degree of regularity $\frac{p-\frac{1}{2}}{n-1}$ according to the new scale, whilst this degree would acquire the same value for both scales in the cases $p = 0$ and $p = n$, i.e. for entirely irregular and for regular polytopes. For in the cases $1 \leq p \leq n-1$ a polytope loses in the first of the two possibilities indicated by *either* and *or* a half characteristic, whilst the total number of available characteristics diminishes by a half at either side which changes the denominator $n$ into $n$—1.

In this paper I wish to take position with respect to the modification of my scale due to Dr. SCHOUTE. Thereby I will have occasion to point out three different moments.

1. Besides for entirely irregular and for regular polytopes the two scales coincide with respect to semiregular polytopes proper. For the supposition

$$\frac{p}{n} = \frac{p-\frac{1}{2}}{n-1}$$

gives

$$2p(n-1) = n(2p-1),$$

i. e. $p = \frac{1}{2}n$ and therefore

$$\frac{p}{n} = \frac{p-\frac{1}{2}}{n-1} = \frac{1}{2}.$$

So, if we arrange the polytopes of space $S_n$ in three groups, for which the degree of regularity is successively smaller than a half, equal to a half and larger than a half the modification proposed brings no alteration in these groups. Otherwise: in passing to the new scale the polytopes with a degree of regularity equal to a half do not stir, whilst — if we use scales of the same length — the others execute a movement enlarging their distance from the centre. So the polytopes with a degree of regularity of at least a half found by me present themselves quite as well if we use the new scale; so in this respect I have not the least objection to accept this new scale.[1]

2. However one may not flatter oneself with the hope, that the new scale shall not contain superfluous points of division with respect to either of the two kinds of polytopes considered for itself. In space $S_4$ already we find with respect to the polytopes of the first kind in this new scale, agreeing with the old one for $n = 3$, the point of division $\frac{5}{6}$ unoccupied. For we have

| $0$ | $\frac{1}{6}$ | $\frac{2}{6}$ | $\frac{3}{6}$ | $\frac{4}{6}$ | $\frac{5}{6}$ | $1$ |
|---|---|---|---|---|---|---|
| $I$ | $P$ | $e_1 S(5)$ | $e_2 S(5)$ | $ce_1 S(5)$ | | $R$ |

where $I$ and $R$ have the same meaning as before, whilst $P$ represents a rectangular parallelotope with edges of four different lengths and $e_1 S(5)$, $e_2 S(5)$, $ce_1 S(5)$ indicate three polytopes deduced from the regular simplex $S(5)$ of $S_4$ in the notation given by Mrs. A. BOOLE STOTT[2]).

3. As the new scale contains no unoccupied points of division in the case $n = 3$ only, it would not be worth while to substitute it for mine, which has the advantage of treating all the groups of limiting elements — vertices, edges, faces, etc. and the limits with the highest number of dimensions — on the same footing, if it did not possess a second advantage, in my opinion of great importance. We will treat this somewhat in detail.

In the determination of the semiregular polytopes of the first kind I consider of any polytope the corresponding "vertex polytope"[3]). In general the vertices of the latter are those vertices of the former joined by edges to a vertex of this original polytope. In an appendix to my dissertation I state the rule, that a polytope with edges of

1) Dr. SCHOUTE requests me to communicate that the primitive idea of this new scale for $S_n$ presented itself to him in an intercourse with F. ZERNIKE, candidate in mathematics and physics at the University of Amsterdam.

2) "Geometrical deduction of semiregular from regular polytopes and space fillings", *Verh. Kon. Akad. v. Wetenschappen*, Amsterdam, 1st series, Vol. XI, nº. 1.

3) Not to be confounded with the polytope of vertex import of Mrs. A. BOOLE STOTT.

the same length [1]) admits one characteristic of regularity more than its vertex polytope, i.e. if the latter is $n$-dimensional and admits the degree of regularity $\frac{p}{n}$, the former must admit the degree of regularity $\frac{p+1}{n+1}$. In this rule the indicated modification of the scale evidently does not bring any alteration. If we build up an $n+1$-dimensional polytope by starting from a given $n$-dimensional vertex polytope, the $n+1$-dimensional polytope will possess all the characteristics of regularity of the $n$-dimensional one, each of these adapted to limiting elements of one dimension higher, and moreover it obtains at the beginning of the series two new halves of characteristics, i.e. equal vertices and edges of the same length. Finally the denominator likewise increases by unity, the new polytope admitting one dimension more than its vertex polytope.

In my dissertation I had to point out an exception to this rule, presenting itself in the case $p = 0$, i.e. when the vertex polytope is irregular. For in that case $\frac{0}{n}$ passes into $\frac{1\frac{1}{2}}{n+1}$ instead of $\frac{1}{n+1}$. So the vertex polytope of the semiregular polyhedra of the table — i.e. "the vertex polygon" here — is an isosceles triangle for the numbers 1—5 and 14, an isosceles trapezium for 8, 9, 15, a scalene triangle for 10, 11, a symmetric pentagon for 12, 13 and therefore the degree of regularity $\frac{0}{2}$ of the vertex polygon has to lead to $\frac{1\frac{1}{2}}{3} = \frac{1}{2}$ in the cases enumerated. This exception now disappears by introduction of the new scale of Dr. Schoute; for according to this scale $\frac{0}{1}$ passes into $\frac{1}{2}$ in these cases.

On account of the latter important advantage of the new scale over the old one I wish to accept the first. Therefore I insert finally a second table in which the polydimensional polytopes with a degree of regularity equal to or surpassing $\frac{1}{2}$ are enumerated with addition of their degree of regularity according to the new scale.

The superscripts $S_n$ represent the number of the $n$-dimensional limits of the polytope. The character of these limits is indicated by notations, the meaning of which is partially clear by itself or by the first table of this paper. Moreover we may state the meaning of the following symbols:

[1]) The latter has been supposed tacitly on p. 129.

| Degree | Notation | $S_0$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ |
|---|---|---|---|---|---|---|---|---|---|
| 4 : 6 | $tC_5$ | 10 | 30 | $(10+20)p_3$ | $5\,O+5\,T$ | | | | |
| 3 : 6 | $tC_8$ | 32 | 96 | $64p_3+24p_4$ | $8\,CO+16\,T$ | | | | |
| 3 : 6 | $tC_{24}$ | 96 | 288 | $96p_3+144p_4$ | $24\,CO+24\,C$ | | | | |
| 4 : 6 | $tC_{600}$ | 720 | 3600 | $(1200+2400)p_3$ | $600\,O+120\,I$ | | | | |
| 3 : 6 | $tC_{120}$ | 1200 | 3600 | $2400p_3+720p_5$ | $120\,ID+600\,T$ | | | | |
| 3 : 6 | | 30 | 60 | $20p_3+20p_6$ | $10\,tT$ | | | | |
| 3 : 6 | | 288 | 576 | $192p_3+144p_8$ | $48\,tC$ | | | | |
| 3 : 6 | | 20 | 60 | $40p_3+30p_4$ | $10\,T+20\,P_3$ | | | | |
| 3 : 6 | | 144 | 576 | $384p_3+288p_4$ | $48\,O+192\,P_3$ | | | | |
| 3 : 6 | | $n^2$ | $2n^2$ | $n^2p_4+2np_n$ | $2n\,P_n$ | | | | |
| 5 : 8 | $S_5{}^2$ | 20 | 90 | $120p_3$ | $30\,T+30\,O$ | $12\,tC_5$ | | | |
| 6 : 8 | $HM_5$ | 16 | 80 | $160p_3$ | $(80+40)\,T$ | $16\,C_5+10\,C_{16}$ | | | |
| 4 : 8 | $S_5{}^1$ | 15 | 60 | $(20+60)p_3$ | $30\,T+15\,O$ | $6\,C_5+6\,tC_5$ | | | |
| 4 : 8 | $Cr_5{}^1$ | 40 | 240 | $(80+320)p_3$ | $160\,T+80\,O$ | $32\,tC_5+10\,C_{16}$ | | | |
| 4 : 8 | $Cr_5{}^2$ | 80 | 480 | $(320+320)p_3$ | $80\,T+200\,O$ | $32\,tC_5+10\,C_{24}$ | | | |
| 7 : 10 | $V_{72}$ | 72 | 720 | $2160p_3$ | $2160\,T$ | $432\,C_5+270\,C_{16}$ | $54\,HM_5$ | | |
| 6 : 10 | $HM_6$ | 32 | 240 | $640p_3$ | $(160+480)\,T$ | $192\,C_5+60\,C_{16}$ | $32\,S_5+12\,HM_5$ | | |
| 8 : 10 | $V_{27}$ | 27 | 216 | $720p_3$ | $1080\,T$ | $216\,C_5+432\,C_5$ | $72\,S_5+27\,Cr_5$ | | |
| 10 : 12 | $V_{56}$ | 56 | 756 | $4032p_3$ | $10080\,T$ | $12096\,C_5$ | $(2016+4032)\,S_5$ | $576\,S_6+126\,Cr_6$ | |
| 8 : 12 | $V_{126}$ | 126 | 2016 | $10080p_3$ | $20160\,T$ | $(4032+12096)\,C_5$ | $4032\,S_5+756\,Cr_5$ | $576\,S_6+56\,V_{27}$ | |
| 6 : 12 | $V_{576}$ | 576 | 10080 | $40320p_3$ | $(30240+20160)\,T$ | $16128\,C_5+7560\,C_{16}$ | $2016\,S_5+2268HM_5$ | $126\,HM_6+56\,V_{72}$ | |
| 8 : 14 | $V_{2160}$ | 2160 | 69120 | $483840p_3$ | $1209600\,T$ | $(241920+967680)C_5$ | $483840S_5+60480Cr_5$ | $138240S_6+6720\,V_{27}$ | $17280\,S_7+240\,V_{126}$ |
| 12 : 14 | $V_{240}$ | 240 | 6720 | $60480p_3$ | $241920\,T$ | $483840\,C_5$ | $483840\,S_5$ | $(69120+138240)S_6$ | $17280\,S_7+2160\,Cr_7$ |

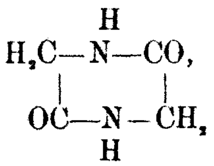
| | | | |
|---|---|---|---|
| T ... | ,, | tetrahedron, | |
| C | ,, | hexahedron (cube), | |
| O | ,, | octahedron, | |
| $C_5$.. | ,, | fourdimensional | five-cell, |
| $C_{16}$ | ,, | ,, | sixteen-cell, |
| $C_{24}$ | ,, | ,, | twinty four-cell, |
| $S_n$ | ,, | $n$-dimensional | simplex, |
| $Cr_n$ | ,, | ,, | cross polytope. |

The cases in which we have to deal with a half characteristic are also indicated in this table. So e.g. the first polytope of the table is limited by equilateral triangles of two different kinds, presenting themselves in the numbers 10 and 20.

*Meppel,* June, 1912.

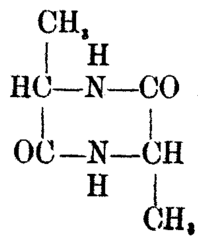
**Chemistry.** — "*Contribution to the knowledge of the direct nitration of aliphatic imino compounds*". By Prof. A. P. N. FRANCHIMONT and Dr. J. V. DUBSKY.

In the January meeting 1907 I had the honour to give a survey of the action of absolute nitric acid on saturated heterocyclic compounds whose ring consists of C and N atoms. This originated in the fact observed and described by Dr. DONK, that the so-called *glycocollanhydride*



in which the group NH is placed between CO and $CH_2$, nitrated with difficulty, with much more difficulty than I had expected because a number of other heterocyclic compounds with rings of five or six atoms in which the group NH is placed in the same manner may be readily nitrated with absolute nitric acid at the ordinary temperature. This was not the case here; only a treatment of the nitrate with acetic anhydride or, as I showed with Dr. FRIEDMANN, of the glycocoll anhydride with acetic anhydride and nitric acid gave a mono- and a dinitroderivative.

With the so-called alanine anhydride



and with the