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Mathematics. "*Continuous one-one transformations of surfaces in themselves*". (5th communication [1]). By Prof. L. E. J. BROUWER.

In CRELLE's Journal, vol. 127, p. 186 Prof. P. BOHL has enunciated without proof the following theorem proved by me (as a particular case of a more general theorem) in vol. 71 of the Mathematische Annalen (compare there page 114):

"*Werden die Punkte einer Kugeloberfläche wieder in Punkte der Kugeloberfläche übergeführt und geschieht diese Ueberführung durch stetige Bewegung, welche den Mittelpunkt nicht berührt, so kehrt mindestens ein Punkt in seine frühere Lage zurück. Unter einer stetigen Bewegung ist hier eine Bewegung verstanden, bei welcher die rechtwinkligen Koordinaten stetige Funktionen der Zeit und der Anfangswerte sind.*"

Now I shall show here in the first place that the theorem enunciated and proved in the first communication on this subject [2]), i. e. that each continuous one-one transformation with invariant indicatrix of a sphere in itself possesses at least one invariant point, may be considered as a particular case of the quoted theorem of BOHL [3]). To that end I shall establish the following theorem:

"*Any continuous one-one transformation α with invariant indicatrix of a sphere in itself can be transformed by a continuous modification* [4]) *into identity*" [5]).

In order to prove this property we choose in the sphere two opposite points P_1 and P_2 determining a net of circles of longitude and latitude and passing by α into Q_1 and Q_2. By means of a continuous series τ of conform transformations of the sphere in itself we can transform Q_1 and Q_2 into P_1 and P_2. Let c be an arbitrary circle of latitude, described in such a sense that P_1 possesses with respect to c the *order* [6]) $+1$, and c' the image of c for $\alpha\tau$, then P_1 possesses also with respect to c' the order $+1$.

[1]) Compare these Proceedings XI, p. 788; XII, p. 286; XIII, p. 767; XIV, p. 300 (1909—1911).

[2]) These Proceedings XI (1909), p. 797.

[3]) This I indicated already shortly Mathem. Ann. 71 (1911), p. 325, footnote *).

[4]) Under a continuous modification of a univalent continuous transformation we understand in the following always the construction of a continuous series of univalent continuous transformations, i. e. a series of transformations depending in such a manner on a parameter, that the position of an arbitrary point is a continuous function of its initial position and the parameter.

[5]) That this theorem wants a proof is shown by the fact that e.g. for a torus it does not hold.

[6]) Compare e. g. J. TANNERY, "*Introduction à la théorie des fonctions d'une variable*", vol. II, p. 438.

Let P be an arbitrary point coinciding neither with P_1 nor with P_2 and passing by $a\tau$ into R, and let Q be the point corresponding in latitude with P and in longitude with R. Then by transforming the different points R continuously and uniformly along circles of longitude into the corresponding points Q we define a continuous series ϱ of univalent continuous transformations of the sphere in itself with the property that of none of the points R the path passes through P_1 or P_2. So an arbitrary curve c' is transformed by ϱ into a curve c'', with respect to which P_1 possesses likewise the order $+1$, so that c'' covers the corresponding circle of latitude c with the *degree*[1] $+1$.

From this ensues that an arc of a circle of latitude connecting an arbitrary point P with the corresponding point Q defines unequivocally for any point P an arc of circle of latitude PQ whose variation with P is uniformly continuous, so that it is possible to construct a continuous series ϱ' of univalent continuous transformations of the sphere in itself, transforming each point Q into the corresponding point P, and thereby the transformation $a\tau\varrho$ into identity. But then $\tau\varrho\varrho'$ is the looked out for continuous series of transformations, transforming a into identity.

We shall say that two transformations *belong to the same class*, if they can be transformed continuously into each other. We then can state the theorem proved just now in the following form :

THEOREM 1. *All continuous one-one transformations with invariant indicatrix of a sphere in itself belong to the same class.*

As the continuous one-one transformations with invariant indicatrix form a special case of the univalent continuous transformations of degree $+1$[2]), the question arises whether perhaps theorem 1 is a special case of the more general property that all the univalent continuous transformations of the same degree of a sphere in itself belong to the same class. We shall see that this is indeed the case; we shall namely show that any univalent continuous representation of degree zero of a sphere μ on a sphere μ' can be transformed by continuous modification into a representation of μ in a single point of μ', and that any univalent continuous representation of degree $n \gtrless 0$ of a sphere μ on a sphere μ' can be transformed by continuous modification into a *canonical representation of degree n*, i. e. into a representation for which $n-1$ non intersecting simple closed curves of μ are each represented in a single point of μ', whilst the n

[1]) Mathem. Ann. 71 (1911), p. 106.
[2]) Mathem. Ann. 71 (1911), p. 106 and 324.

domains determined by these curves are each submitted to a continuous *one-one* representation on μ', and that either all with degree $+1$ or all with degree -1. By means of an indefinitely small modification a canonical representation can be transformed into a simply ramified *Riemann representation*, i.e. into a representation which in the sense of analysis situs is identical to a simply ramified representation of a Riemann surface with n sheets and of genus zero on the complex plane. That all simply ramified Riemann representations belong to the same class, follows, according to a remark made by KLEIN[1]), out of a known theorem of LÜROTH—CLEBSCH.

In order to transform an arbitrarily given univalent continuous representation α of μ on μ' into a representation in a single point, resp. into a canonical representation, we first modify it continuously into a *simplicial approximation*[2]) α', to which we have imparted, by means of eventual subdivisions of the corresponding simplicial divisions of μ and μ', the property that any base triangle of μ covers in μ' either a single base triangle, or a single base side, or a single base point; we then investigate the possibility of finding two base triangles of μ, one positively and the other negatively represented, allowing that we pass from the one to the other by transversing exclusively base sides of μ *not* represented in a single point. If this be the case, μ will possess a positively represented base triangle t_1 and a negatively represented one t_n both represented in the *same* fundamental triangle t' of μ', allowing us to pass from the one to the other by transversing exclusively such base sides of μ, as are represented in the *same* side s_1 of t'. The base triangles $t_2, t_3, \ldots, t_{n-1}$ of μ crossed on this way leading from t_1 to t_n are then also represented entirely in s_1.

Let s_2 and s_3 be the other two sides of t'; by a continuous modification of α' and a suitable farther subdivision of $t_1, t_2, \ldots, t_{n-1}, t_n$, we can generate a representation α'' for which all the triangles $t_1, t_2, \ldots, t_{n-1}, t_n$ are represented entirely in s_2 and s_3, and which possesses still the same property as α', viz. that any base triangle of μ covers in μ' either a single base triangle, or a single base side, or a single base point.

In the same manner as we transformed α' into α'', we transform α'' if possible into α''', and we continue this process until after a

[1]) Compare: "*Ueber Riemann's Theorie der algebraischen Funktionen und ihrer Integrale*". Leipzig, 1882.

[2]) Mathem. Ann. 71 (1911), p. 102.

finite number of steps we have reached a representation $a^{(\nu)}$ no more allowing a suchlike modification.

We now construct on μ all those polygons formed by base sides belonging to $a^{(\nu)}$ which are represented by $a^{(\nu)}$ *in a single point*. These polygons divide μ into a finite number of domains $g_1, g_2, \ldots, g_k$. Each domain g_ν, which by $a^{(\nu)}$ is not represented nowhere dense, admits the property that there is no polygon lying entirely within it or partly within it and partly on its boundary, which is represented by $a^{(\nu)}$ in a single point [1]). Any two base triangles belonging to the same domain g_ν can be connected within g_ν by a path transversing only base sides *not* represented in a single point, so that of the base triangles of g_ν either no one is represented negatively or no one positively.

As each coherent part of the boundary of g_ν is represented on μ' by a single point, μ' is covered by the image of g_ν with a certain *degree* which we will suppose to be positive. Then there are no negative image triangles, but there are in general singular image triangles with two coinciding vertices.

By considering each coherent part $\gamma_{\nu\tau}$ of the boundary of g_ν as a single point $P_{\nu\tau}$, g_ν is transformed into a sphere sp_ν, and we can deduce a simplicial division of sp_ν from the simplicial division of g_ν belonging to $a^{(\nu)}$, by bisecting all those base sides of g_ν which touch the boundary but do not lie in the boundary, dividing by means of these bisecting points each base triangle one side of which lies in the boundary, into a triangle and a trapezium to be considered as a base triangle of sp_ν, and dividing those of the remaining base triangles of which sides have been bisected, into new base triangles corresponding to those bisecting points. The simplicial representation $a^{(\nu)}$ of g_ν on μ' is then at the same time a simplicial representation of sp_ν on μ', whilst by suitable subdivisions of the simplicial divisions

[1]) For, as this property holds for polygons formed by base sides, any base triangle of g_ν possesses at most one base side represented in a single point. Therefore each broken line, lying in a single base triangle and not in a single base side, which is represented in a single point, must necessarily lie entirely in a straight line segment connecting two points of the circumference not coinciding with vertices. So a polygon represented in a single point must either consist exclusively of base sides, or it can transverse only such base sides as are represented in one and the same base side of μ'. In the latter case however the series of the base triangles of μ crossed in this way would have to be represented in that selfsame base side of μ', so that each of the two limiting polygons of this series (of which at most one can be illusory) would be a polygon formed by base sides and represented in a single point of μ'.

of sp_ν and μ' we can effectuate that any base triangle of sp_ν covers in μ' either a single base triangle, or a single base side.

By choosing one of the base sides of sp_ν represented by $\alpha^{(\nu)}$ in a single point, and considering it as a single point and accordingly the two base triangles adjacent to it as line segments, sp_ν passes into an other sphere sp_ν' represented likewise simplicially by $\alpha^{(\nu)}$. In the same way we deduce from sp_ν' an other sphere sp_ν'' if this be possible, and we continue this process until after a finite number of steps we obtain a sphere $sp_\nu^{(r)}$ no more possessing for $\alpha^{(\nu)}$ any singular image triangle.

Let us denote by B and D the two base points of $sp_\nu^{(m-1)}$ identified for $sp_\nu^{(m)}$ and by a and c the two base triangles of $sp_\nu^{(m-1)}$ contracted into line segments for $sp_\nu^{(m)}$. Then the triangles a and c have either only the side BD in common, or moreover a second side, which we may assume to contain the vertex B.

In the first case we represent the third vertex of a, resp. c, by A, resp. C, and the domain covered by a and c together, by d. At least one of the base points B and D, say D, *does not* coincide with a point $P_{\nu\tau}$. We then connect in $sp_\nu^{(m-1)}$ outside d the points A and C by an arc of simple curve β situated in the vicinity of the broken line ADC, and we represent the domain included between β and the broken line ADC, by d'. By means of a continuous series of continuous one-one transformations leaving the points of β invariant and transforming each point of AB and BC into points coinciding with it on $sp_\nu^{(m)}$, we can reduce the domain $d + d'$ with its boundary continuously into the domain d' with its boundary. If we represent by $\alpha_\nu^{(m)}$ an arbitrary univalent continuous representation of $sp_\nu^{(m)}$ on μ', then to the continuous reduction of $d + d'$ to d' corresponds a continuous series of univalent continuous representations of $sp_\nu^{(m-1)}$ on $sp_\nu^{(m)}$ transforming the representation obtained by the identification of B and D, into a continuous one-one correspondence ${}_m\alpha_{m-1}$ in which the points $P_{\nu\tau}$ correspond to themselves, thus also a continuous series of univalent continuous representations of $sp_\nu^{(m-1)}$ on μ', leaving invariant the images of the points $P_{\nu\tau}$, and transforming $\alpha_\nu^{(m)}$ considered as a representation of $sp_\nu^{(m-1)}$ on μ', into that representation $\alpha_\nu^{(m-1)}$ of $sp_\nu^{(m-1)}$ on μ', which follows from $\alpha_\nu^{(m)}$ by means of ${}_m\alpha_{m-1}$.

In the second case we represent the third vertex of a and c by F, choose on the side DF of a, the side DF of c, and the common side BF successively three such points A, C, and G, as in passing from $sp_\nu^{(m-1)}$ to $sp_\nu^{(m)}$ are brought to coincidence, connect A within a rectilinearly with B and G, C within c rectilinearly with B and G, and apply the operation of the first case to the pairs of fundamental triangles ABD and CBD; BCG and FCG; BAG and FAG successively [1]).

By applying this operation successively to $sp_\nu^{(r)}$, $sp_\nu^{(r-1)}, \ldots, sp''_\nu$ and sp'_ν we experience that the representation $a^{(r)}$ of sp_ν on μ' can be transformed by a continuous modification leaving the images of the points $P_{\nu\tau}$ invariant, into a representation a_h of sp_ν on μ', which follows from $a^{(r)}$ by means of a continuous one-one correspondence between $sp_\nu^{(r)}$ and sp_ν. As $sp_\nu^{(r)}$ can be divided into elements each of which is submitted for $a^{(r)}$ to a one-one representation of degree $+1$ on a base triangle of μ', it is clear that sp_ν can be divided into elements each of which is submitted for a_h to a one-one representation of degree $+1$ on a base triangle of μ'. The representation a_h of sp_ν on μ' is therefore a *Riemann representation*, and eventually it may be transformed by an indefinitely small modification leaving the images of the points $P_{\nu\tau}$ invariant, into a simply ramified Riemann representation.

By executing this process of modification for all the values of ν for which it is applicable we arrive at a representation a_e being for any of the spheres $sp_1, sp_2, \ldots, sp_k$ either a simply ramified, positive or negative Riemann representation, or a representation nowhere dense.

In each domain g_ν we approximate the boundary parts $\gamma_{\nu\tau}$ by simple closed curves $\varkappa_{\nu\tau}$ not intersecting each other. Each $\varkappa_{\nu\tau}$ includes with the corresponding $\gamma_{\nu\tau}$ a domain $g'_{\nu\tau}$, and the $\varkappa_{\nu\tau}$ situated in the same domain g_ν include together a domain g'_ν. The domains $g'_{\nu\tau}$ belonging to the same τ form together a domain g''_τ. By means of a continuous series of univalent continuous representations of g_ν on sp_ν we can transform identity into a representation which for g'_ν with the exclusion of its boundaries is a continuous one-one representation on sp_ν, whilst $\varkappa_{\nu\tau}$ and $g'_{\nu\tau}$ are represented in $P_{\nu\tau}$. By doing this for all values of ν we transform a_e into a representation a_l being for each of the domains g'_ν and g''_τ after contraction of its rims into points either a simply ramified, positive or negative Riemann representation, or a representation nowhere dense.

The domains g'_ν and g''_τ, which will be represented henceforth by $\vartheta_1, \vartheta_2, \ldots, \vartheta_u$, are determined on μ by a finite number of simple closed curves not intersecting each other.

[1]) If we dropped the condition of the invariancy of the images of the points $P_{\nu\tau}$ (introduced only for the sake of clearness), this second case might have been treated of course in the same manner as the first.

We choose an arbitrary domain ϑ_ν and suppose in the first place that α_l is for the sphere σ_ν into which ϑ_ν is transformed by contraction of its rims into points, a simply ramified Riemann representation. We then draw on μ' a system of ramification sections belonging to this representation and corresponding to a system of simple closed "ramification curves" on σ_ν. By first leaving the ramification sections on μ' invariant and varying eventually continuously the ramification curves on σ_ν in such a manner that after that they contain no more a point corresponding to a rim of ϑ_ν, and then leaving the ramification curves on σ_ν invariant and contracting the ramification sections on μ' continuously into points, we can transform the representation of σ_ν on μ' determined by α_l continuously into a canonical representation. During this continuous modification the points representing the rims of ϑ_ν vary also in general. Let i_τ be such a rim and $\vartheta_{\nu\tau}$ the residual domain of ϑ_ν on μ determined by i_τ. We then can follow the continuous variation of the image point of i_τ by a continuous series of continuous one-one transformations of μ' in itself to which corresponds a continuous modification of the representation of $\vartheta_{\nu\tau}$ on μ' determined by α_l. By applying this modification to the representations of all the residual domains of ϑ_ν we generate a representation α'_l of μ on μ' into which α_l can be transformed continuously, and which is a canonical representation for σ_ν.

In the second place we suppose α_l to be for σ_ν a representation nowhere dense. Then we can modify the representation of σ_ν on μ' determined by α_l into a representation in a single point. The variation of the image points of the rims of ϑ_ν implied by this modification, can be followed once more in the way described above by a continuous modification of the representation of the residual domains of ϑ_ν, furnishing us with a representation α'_l of μ on μ' into which α_l can be transformed continuously, and which represents σ_ν in a single point.

By executing this operation for all values of ν successively, we get a representation $\alpha_l^{(\lambda)}$ of μ on μ', into which α_l can be transformed continuously, and which represents each of the domains $\vartheta_1, \vartheta_2, \ldots \vartheta_n$ either after contraction of the rims into points canonically, or in a single point. The sphere μ is now divided by a finite number of non intersecting simple closed curves into a finite number of domains $d_1, d_2, \ldots, d_n$ in such a way that for $\alpha_l^{(\lambda)}$ each of these domains is submitted either after contraction of the rims into points to a continuous one-one representation, or to a representation in a single point. Thus the degree of these representations is $0, +1$, or

— 1, according to which we distinguish domains of the first, the second, and the third kind.

If for the representation $\alpha_l^{(\psi)}$, which may be denoted henceforth by α_f, all domains d_ν are of the first kind, we have attained our aim; for then we have transformed α continuously into a representation of μ in a single point of μ'. So we further confine ourselves to the case that among the d_ν there are domains of the second or of the third kind, and we will suppose that there occur moreover domains of the first kind. Then there is certainly a domain $d_\varkappa$ of the first kind adjacent to a domain d_ϑ of the second or third kind. The domain formed by $d_\varkappa$ and d_ϑ together, may be indicated by $d_{\varkappa\vartheta}$, the sphere deduced from $d_{\varkappa\vartheta}$ by contraction of its rims into points, by $\sigma_{\varkappa\vartheta}$. We then can modify the univalent continuous representation of $\sigma_{\varkappa\vartheta}$ on μ' determined by α_f continuously into a continuous one-one representation of $\sigma_{\varkappa\vartheta}$ on μ'. The variation of the image points of those rims of $d_{\varkappa\vartheta}$ which originate from $d_\varkappa$ necessarily implied by this modification, can once more be followed in the manner described above by a continuous modification of the representation determined by α_f of those residual domains of $d_{\varkappa\vartheta}$ which originate from $d_\varkappa$, furnishing us with a representation α'_f distinguishing itself thereby from α_f that a domain of the first kind and a domain of the second (resp. third) kind have been united into a single domain of the second (resp. third) kind.

By repeating this operation as many times as possible we arrive after a finite number of steps at a representation $\alpha_f^{(\sigma)}$, distinguishing itself thereby from α_f that all the domains of the first kind have been absorbed by domains of the second and of the third kind.

If there are for the representation $\alpha_f^{(\sigma)}$, which may be denoted henceforth by α_q domains of the second as well as of the third kind, we consider a domain d_π of the second kind separated by a simple closed curve $i_{\pi\rho}$ from a domain d_ρ of the third kind, and we represent the domain formed by d_π and d_ρ together, by $d_{\pi\rho}$, and the sphere deduced from $d_{\pi\rho}$ by contraction of its rims into points, by $\sigma_{\pi\rho}$. Moreover we represent by P_1 the image point of $i_{\pi\rho}$ for α_q, by P_2 the opposite point of P_1 on μ', and we modify the representation of $\sigma_{\pi\rho}$ determined by α_q into a representation of $\sigma_{\pi\rho}$ in the single point P_2, by diminishing the polar distances measured from P_2 continuously and proportionally to each other to zero. The variation of the image points of the rims of $d_{\pi\rho}$ necessarily implied by this modification, can be followed in the manner described above by a continuous modification of the representation of the residual domains of $d_{\pi\xi}$ determined by α_q, furnishing us with a representation $\alpha_{q'}$ distinguishing itself thereby from α_q that a domain of the second and one of the third kind have been united into a single domain of the first kind: this domain however, if it does not occupy the whole sphere μ, can be absorbed in the manner described above by an adjacent domain of the second or of the third kind, by which process $\alpha_{q'}$ passes continuously into a representation α'_q, distinguishing itself thereby from α_q that a domain of the second and one of the third kind have been absorbed together by a domain of the second resp. of the third kind.

By repeating this operation as many times as possible we arrive after a finite number of steps at a representation $\alpha_q^{(\xi)}$ for which the domains d_ι are either all of the second or all of the third kind. So this representation is a *canonical* one, and we have proved:

THEOREM 2. *All univalent continuous transformations of the same degree of a sphere in itself belong to the same class.*

A proof of the inverse theorem has been given Mathem. Ann. 71, p. 105.

In carrying out the ideas sketched in the second communication on this subject[1]) I experienced that in some points of the course of demonstration indicated there, still a tacit part is played by the Schoenfliesian theory of domain boundaries criticized by me[2]), so that the theorems 1 and 2 formulated p. 295 and likewise the "general translation theorem" founded upon them and enunciated without proof Mathem. Ann. 69, p. 178 and 179, cannot be considered as proved[3]), and a question of the highest importance is still to be decided here.

The "plane translation theorem" stated at the end of the second communication (p. 297) and likewise Mathem. Ann. 69, p. 179 and 180, has meanwhile been proved rigorously by an other method.[4])

[1]) These Proceedings XII (1909), p. 286—297.

[2]) Ccmpare Mathem. Ann. 68 (1910), p. 422—434.

[3]) Already the property of p 288 that the transformation domain constructed in the way indicated there determines at most two residual domains, vanishes for some domains incompatible with the Schoenfliesian theory.

[4]) Compare Mathem. Ann. 72 (1912), p. 37—54.

Chemistry. — *"Extension of the theory of allotropy. Monotropy and enantiotropy for liquids."* By Prof. A. SMITS. (Communicated by Prof. A. F. HOLLEMAN).

The extension meant above concerns the case that the pseudo-binary system exhibits the phenomenon of unmixing in the liquid state.

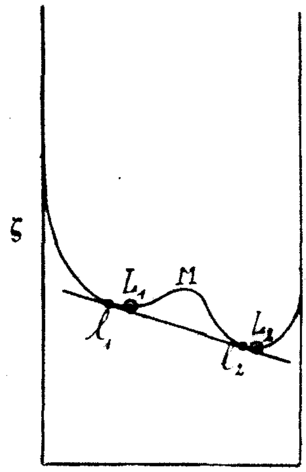

Fig. 1 *X*.

Let the ζ,x-line be schematically represented by fig. 1 at the temperature and pressure at which the phenomenon of unmixing takes place. Then in the first place it is noteworthy that l_1 and l_2 are the coexisting liquid phases of the pseudo-binary system, and that moreover there exist two minimum points L_1 and L_2 representing the liquid phases which may be formed when the system gets in internal equilibrium, and consequently behaves as a unary substance.

The two liquid phases are not miscible, and when they are brought into contact the metastable liquid L_1 will pass into the stable liquid phase L_2, so that this operation means the same thing as seeding the metastable liquid. As fig. 1 shows the *metastable* unary liquid point L_1 lies inside, and the *stable* unary liquid point L_2 outside the region of incomplete miscibility, and now it is of importance to examine what happens when we move toward such a temperature that the critical phenomenon of mixing occurs in the pseudo-binary system. The coexisting phases l_1 and l_2 have drawn nearer and nearer to each other, and finally coincided in the critical mixing-point, and the ζ,x-line has then changed into a curve with only one minimum, as fig. 2 shows.

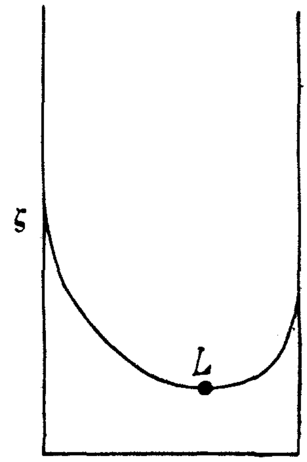

Fig. 2 *X*.

It is now, however, of importance for our purpose to consider the way in which the ζ,x-line has changed its form from that of fig. 1 to that of fig. 2.

It is known that before the points l_1 and l_2 coincide, the maximum