as well as with silk, wool, cotton and cellulose the order of the three following dyestuffs was: crystal-violet, "neufuchsin", patent blue.

The same order, however, is noticed in the distribution of these dyestuffs between water and alcohol. Here again is shown the great analogy between the absorption of the dyestuff in fibres and the transition of the colouring matter into another solvent, which leads to the assumption that the absorbed dyestuff is present as a solid solution in the fibre.

We, therefore, conclude that the dye absorption in fibres is mainly a phenomenon of solid solution and that the assumption of a surface adsorption is in many cases unnecessary and should, therefore, be discarded.

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Mathematics. — *"On loci, congruences and focal systems deduced from a twisted cubic and a twisted biquadratic curve".* I. By Prof. HENDRIK DE VRIES.

(Communicated in the meeting of September 28, 1912).

1. In the Proceedings of the Meeting of this Academy on Saturday Sept. 30, 1911, p. 259, Mr. JAN DE VRIES has investigated the locus of the points sending to three pairs of straight lines crossing each other three complanar transversals, and in the Proceedings of the Meeting of Nov. 25, 1911, p. 495, Mr. P. H. SCHOUTE has made the same investigation for the points sending to $(n+2)_2$ pairs of straight lines crossing each other $(n+2)_2$ transversals lying on a cone of order n. In the following pages one of the three pairs of lines will be replaced by a twisted cubic, the two others by a quartic curve of the first kind. Through a point P one chord a of k^3 passes and two chords b of k^4 pass; we ask after the locus of the points P for which the line a and the two lines b lie in *one* plane.

We imagine a chord a of k^3. Through an arbitrary point P of this chord pass two chords b_1, b_1^* of k^4 and in the plane ab_1 lies one chord b_2 which does not meet b_1 on k^4 itself, in ab_1^* one such-like chord b_2^*; if for convenience sake we call the points of intersection of b_2 and b_2^* with a both Q, then in this way to each point P two points Q correspond. However, it is clear that to each point Q also two points P correspond, so that on a a (2,2) correspondence arises with four coincidences, and for these it is evident that the triplet $a+2b$ is complanar. However, it is easy to see that the four coincidences coincide two by two; for, if we call one of the two chords b through such a point b_1, then the other is b_2, but if we call the latter b_1*, then $b_1 = b_2*$, so that really the coincidences coincide two by two. Furthermore it is easy to point out that in general the two coincidences do not fall in the points of intersection of a and k^3; for, both chords b through such a point will in general not lie with a in *one* plane.

So out of these considerations follows that a intersects the demanded locus outside k^3 in two more points; *if therefore we point out that k^3 is a nodal curve, then we have proved that the demanded locus is a surface Ω^6 of order* 6. Now through a point P of k^3 pass two chords b and in the plane through these lie two chords a; so each point of k^3 is a nodal point for the surface.

2. We again determine the order of Ω^6 by considering a chord b_1 of k^4. Through a point P of b_1 passes one a and in the plane ab_1 lies one b_2; if the latter intersects b_1 in Q then to each point P one point Q corresponds. Inversely through Q passes one b_2, but in the plane b_1b_2 lie three chords a; so on b_1 we find now a (1,3) correspondence with four coincidences, and these do not coincide two by two. For, through each coincidence passes one a and one b, but of course these cannot be exchanged. Neither does a single coincidence fall on k^4; for through a point of intersection P of b, and k^4 passes one a and the line connecting the two remaining points of intersection of plane ab_1 and k^4 does of course in general not pass through P. So a chord of k^4 cuts Ω^6 outside k^4 in four points more; *therefore k^4 is for Ω^6 a single curve.*

This last result has something unexpected, for if we regard k^4 by itself we arrive at quite a different result. Through a point P of k^4 passes one a and in an *arbitrary* plane through this lie three chords b through P; so that each point of k^4 regarded by itself satisfies the given question an infinite number of times; if however we also take into consideration the points outside k^3, then we find according to the above mentioned a surface Ω^6 for which k^4 is only a single curve.

That k^4 is just a single curve is made clearer by the following consideration. The curve k^4 is the section of two quadratic surfaces Φ_1, Φ_2, and the plane of the two chords b_1, b_2 is at the same time the plane through P and the line of intersection s of the two polar planes π_1 π_2, of P with respect to Φ_1 and Φ_2; if now P falls exactly on k^4, then π_1, π_2 become tangential planes in P to Φ_1, Φ_2, so their line of intersection s becomes the tangent t in P to k^4; among all the planes through P only those through t come into consideration, and as now the plane through t and the chord a through P is determined unequivocally, and as in this plane only two chords b lie, point P counts only once.

3. Through k^4 pass four quadratic cones whose vertices we shall call $T_1 \ldots. T_4$. These vertices too behave themselves somewhat irregularly with respect to the question put originally, for an arbitrary plane e.g. through the line a passing through T_1 contains always two chords b, so that also the four vertices of the cones regarded by themselves satisfy the given question an infinite number of times; *nevertheless these points are for Ω^6 only single points.*

This can be proved most easily with the aid of the edges of the tetrahedron $T_1 \ldots. T_4$. Let us consider e.g. $T_1 T_2$ and let us regard k^4 as the intersection of the two cones having T_1 and T_2 as vertices. All points P of $T_1 T_2$ have with respect to the first cone only one polar plane π_1, viz. the plane $T_1 T_3 T_4$, and likewise with respect to the second cone only one polar plane π_2, viz. $T_2 T_3 T_4$; the line of intersection $T_3 T_4$ is therefore the line s for all points P of $T_1 T_2$, or in other words the planes Ps (or $Pb_1 b_2$) for all points of $T_1 T_2$ form a pencil of planes around the edge $T_3 T_4$. The question is to find the points P of $T_1 T_2$, for which the chord a of k^3 passing through P lies in the plane Ps and to this end we have but to intersect each plane Ps by k^3, by means of which we find in each suchlike plane three chords a forming altogether a scroll Ω^4 of order four with k^3 as a nodal curve and s as a single directrix. For, through a point of s only one chord a passes, whilst in a plane through s three of suchlike chords are lying, and through a point of k^3 evidently two chords a pass intersecting s. Now this scroll Ω^4 intersects $T_1 T_2$ in four points, but to these T_1 and T_2 themselves do not belong, because no reason whatever can be given why of the three chords a in the plane $T_1 T_3 T_4$ e.g. just one should pass through T_1; so we find on $T_1 T_2$ four points of intersection besides the two vertices of the cones, and as the latter of course likewise belong to the surface they count once on $T_1 T_2$ and therefore likewise in general.

If we determine the points of intersection of Ω^6 with the chord a through T_1, then we find that the two points which this chord has outside k^3 in common with the surface (§ 1) coincide with T_1, which with a view to the preceding means that a touches the surface in T_1. We endeavour also to acquire on this special chord a the (2,2) correspondence of § 1, which is easily done and where we have but this to remark, that in the plane $b_1 b_2$ as well as in the plane $b_1^* b_2^*$ the four points of k^4 lie two by two on two lines through T_1. If

33

now the point of intersection P of b_1 and a is to coincide with the point of intersection Q of b_2 and a then the four points of k^4 in the plane ab_1b_2 must form a complete quadrangle with P and T_1 as two of the three diagonal points, and this is only possible if the line T_1P, thus a, lies on a special cone of order two, which will in general not be the case. In an arbitrary plane through T_1 lie namely four points of k^4, forming a complete quadrangle; one of the three diagonal points is T_1, the two other ones lie in $T_2T_3T_4$ and evidently describe here when the plane varies a conic through T_2, T_3, T_4. If now a happened to lie on the cone projecting this conic out of T_1, then two coincidences of the (2,2) correspondence would lie on the conic and the two others in T_1; in every other case however all four coincidences must coincide in T_1, and so a must touch the surface Ω^6 in T_1.

4. We now proceed to determine the points of intersection of Ω^6 with an entirely arbitrary line l. To that end we allow a point P to travel along the line l and we investigate how often the chord a passing through P lies in plane Ps. According to § 3 the chords a issuing from the points P of l form a scroll of order four with nodal curve k^3 and single directrix l; the lines s belonging to the points P of l form a regulus and the planes Ps envelope a developable of class 3. If namely point P describes the line l then the two polar planes π_1 and π_2 of P with respect to Φ_1 and Φ_2 (comp. § 2) revolve around the two lines l_1, l_2 conjugated to l, and crossing each other in general; *thus the line s describes a regulus with l_1 and l_2 as bearers.*

Now the surface enveloped by the planes Ps. We imagine an arbitrary point O in space, we choose a point P on l, we determine the corresponding line s and we find the point of intersection Q of the plane Os with l; in this manner to each point P one point Q corresponds. If reversely we wish to know how many points P correspond to Q, we draw the line connecting O and Q and we intersect it by the regulus of the lines s just found; through each of the two points of intersection passes one line s whose corresponding point P lies on l, so that to one point Q two points P correspond. Between the points P and Q on l there exists a (1, 2) correspondence; for the three coincidences the plane Ps passes through O; *so the planes Ps belonging to the points of a line l envelope a developable of class three.*

We now add to the figure an arbitrary plane α and we determine the section of this plane with the scroll of order four, formed by the chords of k^3 resting on l, as well as with the developable just found of class three; the former is a rational curve of order four with three nodes in the points of intersection of α and k^3 and a single point in the point of intersection of α and l, the second a rational curve of class 3 with a double tangent.

Through an arbitrary point of the curve of order four passes one chord a, intersecting l in P, and through P passes one plane Ps, so that in this way to each point of the curve k^4 of order four one tangent of the curve k_3 of class three corresponds, whilst in the same way we can see that to a node of k^4 two different tangents of k_3 correspond. In the same easy way we can convince ourselves that to each tangent of k_3 one point of k^4 corresponds and to the double tangent two different ones; so the result is that there exists a $(1, 1)$ correspondence between the points of k^4 and the tangents of k_3; the question now is how many coincidences this correspondence possesses.

Let us take a point P on k^4 and let us determine the corresponding tangent t of k_3, cutting k^4 in four points Q; reversely through one point Q pass three tangents t, and to each of these one point P corresponds; so between the points P and Q exists a $(3, 4)$ correspondence and, as the bearer is rational, the number of coincidences is seven. One of these must necessarily be the point of intersection of l and α; for, through this point taken as point P of l, passes a chord a and likewise a plane Ps cutting α of course according to a line passing through P, however without it being necessary for a to lie in the plane Ps. So we have here a coincidence in the plane α to which no incidence of a into the plane Ps corresponds; if we set this case apart six coincidences remain which are each the consequence of a point of intersection of l and Ω^6.

For the sake of completeness we add to the preceding that the regulus of the rays s belonging to the points P of l contains the four vertices of the cones $T_1, \ldots, T_4$ (comp. § 3); for T_1 has as polar plane with respect to Φ_1 as well as to Φ_2 the plane $T_2T_3T_4$, so inversely the two polar planes of the point of intersection of l with this plane pass through T_1, and so does therefore their line of intersection s.

The developable of the planes Ps is of class three, so through each point P of l itself three planes Ps must pass; indeed two rays s of the regulus cut l and to these two points P of l correspond; so through l pass two planes Ps and these must for each point of l be added to the plane passing through that point but not through l.

33*

5. As we have seen before k^4 is for the surface Ω^6 a single curve, k^3 a nodal curve, and the surface cannot contain other nodal curves for, if a point O is to be a double point, then through this point either more than one chord a or more than two chords b must pass; the former is only possible for the points of k^3, the latter only for those of k^4, and these two curves we have already investigated. *On the other hand the surface contains a number of single lines crossing each other, as many as twenty*; the chords of k^3 namely form a congruence of rays (1,3), those of k^4 one (2,6), and these congruences have according to the theorem of HALPHEN $1.2 + 3.6 = 20$ rays in common. Through a point P of such a ray passes one chord a, one chord b coinciding with a and one chord b more; so it is a single point for Ω^6. Two of these lines cannot possibly intersect each other outside k^3, for in that case two chords a would pass through one point, which is impossible; it is not impossible for them to intersect on k^3, but this requires a peculiar situation of k^3 and k^4 with respect to each other, which we will not presuppose.

An arbitrary plane through one of the twenty lines cuts Ω^6 besides in this line still according to a curve of order five which has with the line in common its two points of intersection with k^3 but not those with k^4, because the latter are but single points for the surface. However besides the two points of intersection on k^3 the curve must have three points more in common with the line, in which points the indicated plane must therefore touch the surface; *so the surface Ω^6 possesses an infinite number of threefold tangential planes, which are arranged in twenty pencils of planes, around the twenty lines of the surface as axes.*

A surface of order 6 is determined by $\frac{7.8.9}{1.2.3} - 1 = 83$ points or in general single conditions; we shall investigate for how many single conditions k^3, k^4, and the twenty lines of the surface count. The curve k^3 must be a nodal curve; so we try to construct a surface of order 6 having k^3 as a nodal curve. In an arbitrary plane α we assume eighteen points quite arbitrarily; we determine the three points of intersection of α with k^3, and we construct a plane curve of order 6 having these last three points as double points and at the same time containing the 18 points above mentioned; as a double point counts for three single data and a curve of order 6 is determined by $\frac{1}{2} . 6 . 9 = 27$ points, we have in α just enough data to determine the curve of order six.

In a second plane β we assume arbitrarily only 12 points, and we add to these the six points of intersection with the curve lying in α; then we can also find in β a curve of order 6 which must lie on the surface. Finally in a third plane γ we have now of course to assume arbitrarily only 6 points and then the surface is determined; for every arbitrary fourth plane cuts the three curves lying in α, β, γ together in 18 single points, and k^3 in three points which must be double points, by which the section of the surface to be constructed is determined. Besides k^3 we therefore want $18 + 12 + 6 = 36$ points to determine the surface; so the condition that k^3 is a nodal curve is equivalent to $83 - 36 = 47$ single conditions.

If k^4 is to lie on the surface of order six, then we have to take care that it must have twenty-five points in common with the surface; so k^3 as a double curve and k^4 as a single curve absorb $47 + 25 = 72$ single conditions, so that but $83 - 72 = 11$ conditions are left. Now a common chord of k^3 and k^4 has with every surface of order six passing twice through k^3 and once through k^4 in its points of intersection with both curves exactly six points in common with this; thus by distributing the eleven points which are left among eleven of the twenty common chords, we can be sure that also these eleven chords will come to lie on the surface. However, we know that on our surface Ω^6 all the twenty common chords lie; so we can state the following theorem: *the twenty common chords of k^3 and k^4 lie on a surface Ω^6 of order 6 passing twice through k^3 and once through k^4; it is the locus of all the points of space for which the triplet of chords $a + 2b$ is complanar.*

6. The first polar surface of an arbitrary point O of space with respect to Ω^6 is a surface Π_1^5 of order five passing once through k^3; the complete section with Ω^6, which must be of order thirty, breaks up into k^3 counted twice and a residual section r^{24} of order twenty-four, from which ensues immediately that *the apparent circuit of Ω^6 out of an arbitrary point of space on an arbitrary plane is a curve of order twenty-four.*

The curve r^{24} has as is easy to see twelve points in common with k^3. The second polar surface of O, viz. a surface Π_2^4 of order four, does not contain k^3, so it intersects it in twelve points; these are the points which k^3 and r^{24} have in common. If namely we connect O with an arbitrary point P of r^{24}, then OP is a tangent in P of Ω^6; now if P lies on k^3 then OP touches in P one of the sheets of Ω^6 passing through k^3, but in consequence of this on the line OP lie united in P three points of Ω^6, and therefore two of Π_1, and one of Π_2. Each of these twelve points counts for three coinciding points of intersection of Ω^6 with its two polar surfaces; for, if we intersect

$k^3 + r^{24}$, the section of Ω^6 and Π_1, by Π_2, then every point of intersection with k^3 counts for two, with r^{24} for one; therefore each of the twelve points under discussion counts for three. As the complete number of points of intersection of the three surfaces is $6.5.4 = 120$, outside k^3 there are $120 - 3 . 12 = 84$. It is wellknown that the tangents in these points to r^{24} pass through O: *thus the apparent circuit of Ω^6 possesses eighty-four cusps.*

To determine the class of Ω^6 and with it of the circumscribed cone, resp. the apparent circuit, we assume a second point O', and we construct the first polar surface Π_1'; this, too, passes through k^3 and intersects the curve r^{24} just found in 120 points of which twelve however lie on k^3, and count singly, because r^{24} is a single section of Ω^6 and Π_1, and k^3 is again a single curve of Π_1; so outside k^3 the three surfaces have $120 - 12 = 108$ points in common, *so that the class of Ω^6 amounts to* 108.

By applying the PLÜCKER formula $\nu = \mu(\mu - 1) - 2\delta - 3\varkappa$ to the apparent circuit, we find

$$2\delta = \mu(\mu - 1) - \nu - 3\varkappa = 24 . 23 - 108 - 3 . 84$$

or

$$\delta = 96.$$

The projecting cone out of O contains therefore 96 *double edges, the apparent circuit* 96 *nodal points.*

The PLÜCKER equation dualistically related:

$$\mu = \nu(\nu - 1) - 2\tau - 3\iota,$$

applied to the apparent circuit furnishes us with

$$2\tau + 3\iota = \nu(\nu - 1) - \mu = 108 . 107 - 24 = 11532,$$

whilst the third formula: $\iota - \varkappa = 3(\nu - \mu)$ furnishes for ι

$$\iota = 84 + 3(108 - 24) = 336;$$

so we find $2\tau = 11532 - 3 . 336 = 10524$, or $\tau = 5262$.

Now however we have to remember that the planes through O and the twenty lines of Ω^6 are threefold tangential planes of the cone, that their traces are therefore threefold tangents of the apparent circuit and that therefore they count together for sixty double tangents. If we subtract these from the entire number 5262, *then for the apparent circuit remain* 5202 *real double tangents completed by* 20 *threefold ones.*

A cusp in the apparent circuit is generated by a principal tangent (a tangent with contact in three points) of the surface passing through O; these principal tangents form a congruence, of which according to the above mentioned the first characteristic (number of rays through a point) is eighty-four. The second characteristic indicates the number of rays in a plane; in order to find this we have but to determine the number of inflexions of a plane section of Ω^6. We have already seen that this plane section is of order 6 and of class 24, and that it contains 3 double points, whilst the number of cusps is 0; from this ensues easily that the number of inflexions is 54, the number of double tangents 192; *the congruence of the principal tangents of* Ω^6 *has therefore the characteristics* 84 *and* 54, *those of the double tangents* 5202 *and* 192.

7. Through each point P of Ω^6 passes a plane π, in which are situated one chord a of k^3 and two chords b of k^4; we wish to study the surface which is enveloped by those planes π. The class of this surface can be determined in different ways; we shall deduce this number in the first place by asking how many planes π pass through a chord a of k^3. Through the point of intersection A_1 of a with k^3 passes one plane π which in general however does not pass through a, and the same holds for the second point of intersection A_2. Besides these two points a has still but 2 points S_1, S_2 in common with Ω^6, and through *these* passes a plane π containing a; for S_1 e.g. is a point of Ω^6 exactly for this reason that the chord a lies with two chords b of k^4 in a plane π. So to each of the two points S_1, S_2 a plane π through a corresponds.

However planes π can also pass through a without it being necessary for the point of intersection P of the triplet $a + 2b$ to lie exactly on a itself. If we make a plane α to rotate round a, it contains in each position 2 more chords a and 6 chords b, forming a complete quadrangle. The two chords a describe the two quadratic cones by which k^3 is projected out of the two points A_1, A_2, the diagonal points of the complete quadrangle describe a twisted curve possessing in each plane α three points apart from the points lying on a itself and which are nothing but S_1, S_2; so the diagonal points form a twisted curve k^5 of order 5 resting in 2 points S_1, S_2 on a, (and containing evidently the four vertices $T_1, \ldots, T_4$, § 3). Let us consider a point of intersection of this k^5 with one of the just mentioned quadratic cones, we then have evidently obtained a point of Ω^6 and at the same time a plane π through a. Now k^5 intersects each cone in ten points, but among these are S_1 and S_2; so outside a lie only sixteen points of intersection and if we again add S_1 and S_2, counted once, we then find *that the surface enveloped by the planes* π *bearing a triplet* $a + 2b$ *is of class eighteen.* We shall indicate it by Ω_{18}.

As easily we can determine the class of Ω_{18} by means of a chord

b of k^4. If it cuts k^4 in B_1, B_2, we must bear in mind that these points according to § 2 are for the surface Ω^6 single points only, from which ensues that through those points only one plane π passes which comes in consideration if we make, as is done here, a point P to describe the surface and if we ask after the surface to be enveloped by the planes π; this one plane however does not pass in general through b. Besides B_1, B_2 b has with Ω^6 four more points S in common; through each of these evidently passes a plane π containing b.

However, there are of course now again planes π through b, whilst point P lies outside b. A plane β through b contains three chords a and these describe when β rotates round b a scroll of order four with k^3 as a nodal curve and b as a single directrix (§ 3). The plane β contains moreover 6 chords of k^4, of which however one coincides with b, so that one diagonal point lies on b and two outside b. These describe when β rotates round b a twisted curve of order four, resting in B_1, B_2 on b; if namely β touches k^4 in B_1 or B_2, it is easy to see that one of the two diagonal points lying in general outside b coincides with the point of contact. This curve of order four intersects the just mentioned scroll of order four in sixteen points, to which however belong B_1 and B_2 as these lie in b and therefore on the scroll too; if we set these aside, because they do not satisfy the question, fourteen are left, and these added to the four points on b, which do satisfy the question, give us again the number 18.

We can also determine by the way followed here the eighteen tangential planes of Ω_{18} through an entirely arbitrary line l. The chords of k^3 resting on l lie again on a surface of order four, and the diagonal points of the complete quadrangles in the planes λ through l lie on a curve of order five resting in two points on l; for, the chord a of k^3 which we discussed above is for k^4 an arbitrary line, so it contains as many diagonal points as in the general case. The curve and the surface intersect each other now in twenty points, but to these belong the two points of intersection of the curve with l, which do not satisfy the question; so there are again eighteen left.

8. An arbitrary plane through one of the twenty common chords of k^3 and k^4 contains beside this chord, representing an a as well as a b, one chord b more, cutting the other outside k^4, and therefore it is a plane π to be counted once; so through each of the twenty chords pass an infinite number of tangential planes of Ω_{18}, from

which ensues that *the twenty common chords of k^3 and k^4 are single lines of Ω_{18}*.

The plane π issuing from a point of k^3 contains two chords a and so it counts twice as tangential plane of Ω_{18}, whilst reversely it is easy to see that Ω_{18} can have no other double tangential planes than these; for, in such a plane must either lie two chords a, which leads to the curve k^3, or more than two chords l, which is the case for the points of k^4, but as for the latter only the plane through the tangent and the chord a comes into consideration (§ 2), the last possibility disappears and only the points of k^3 are left. *The double tangential planes of Ω_{18} are therefore the planes π corresponding to the points of k^3; they envelope a developable Δ_9 of class* 9.

In order to find this number we look for all the double tangential planes passing through an arbitrary point B_1 of k^4. Such a plane then must contain a chord of k^4 passing through B_1 intersecting k^3, and it can thus be obtained for instance by intersecting k^3 by the cubic cone projecting k^4 out of B_1, which furnishes 9 points of intersection, or inversely by intersecting k^4 by the cubic cone projecting k^3 out of the vertex B_1, which furnishes 12 points of intersection, of which three however coincide with B_1 and must be taken apart. If now we call A such a point of intersection lying on k^3 then really through this point passes one double tangential plane of Ω_{18} containing point B_1; so the class of the developable is nine.

Through a point A of k^3 pass likewise 9 tangential planes of Δ_9; for one of these points A itself is the point from which start the two chords b of k^4, in the eight other planes on the other hand the chords b start from an other point; from this ensues that through A pass altogether ten chords of k^4 which start from the point of k^3 and which at the same time lie in the tangential planes of Δ_9 corresponding to those points; *the locus of those chords is a surface Ω^{20} of order twenty for which k^3 is a tenfold curve.*

For, an arbitrary chord of k^3 meets in each of its 2 points of intersection with k^3 ten generatrices of the scroll to be found, and is intersected outside k^3 by no chords of k^3.

In a tangential plane of Δ_9 lie also two chords b intersecting k^3, viz. in point A to which that tangential plane corresponds; let us also ask after the locus of these chords b. Through each point of k^3 pass two, through each point of k^4 nine, because (see above) the cubic cone projecting k^4 out of that point is intersected by k^3 in nine points; let us now determine the points of intersection of the scroll to be found with a chord b_1 of k^4, then of these in each of the two points of intersection of b_1 with k^4 lie nine united. If furthermore we make a plane β to rotate round b_1 then the chord b_2 in that plane, which cuts b_1 outside k^4, describes a scroll having six points in common with k^3; through each of these passes a chord b_2 which cuts k^3 and b_1; *the scroll to be found is therefore a surface* Ω^{24} *of* order $2 \times 9 + 6 = 24$. *It has* k^3 *as a nodal curve and* k^4 *as a ninefold curve.*

9. The surface Ω^{20} found in the preceding § possesses no other manifold curve than k^3. Each scroll of order n contains namely a nodal curve which is cut by a generatrix in $n-2$ points, because a plane through a generatrix contains as residual section a curve of order $n-1$, and of the $n-1$ points of intersection of this curve with the generatrix only one acts as a point of contact, so that all the remaining ones are due to a nodal curve. Now a plane through a generatrix of Ω^{20} contains a residual section of order nineteen with two ninefold points on k^3; these together form eighteen points of intersection of the generatrix with the nodal curve, so that the latter is complete with k^3 only. *On the other hand the surface contains twenty double generatrices, viz. the common chords of* k^3 *and* k^4, as is easy to see, *and these same lines are double generatrices of* Ω^{24}.

The surface Ω^{24} contains besides the nodal curve k^3 and the ninefold curve k^4 still a new nodal curve which is cut by each generatrix in five points: for, a plane through a generatrix contains a residual section of order twenty-three with two eightfold points on k^4 and a single point on k^3, forming together seventeen points; so the generatrix must contain five points more of an other nodal curve. And indeed, if we make a plane to rotate round a generatrix b_1, it then possesses in each position still one chord b_2 of k^4 not meeting b_1 on k^4; this chord describes a regulus intersected by k^3 in six points, of which one however coincides with the point of intersection of b_1 and k^3; through the remaining five passes every time one generatrix of Ω^{24} meeting b_1 outside k^3 and k^4, thus in a point of the new nodal curve.

We can find the order of this new nodal curve with the help of the theory of the permanency of the number. We conjugate an arbitrary generatrix of Ω^{24} which we call g to all others which shall then be called h, and in this way we find ∞^2 pairs of lines gh to which we will apply in the first place SCHUBERT's formula:

$$\varepsilon\sigma = 2 \,.\, \varepsilon\beta - 2 \,.\, \varepsilon g \; ^{1)}.$$

The letter ε indicates the condition that two rays g and h of a

[1]) SCHUBERT: "Kalkül der abz. Geom.", p. 60, N⁰. 22.

pair lie at infinitesimal distance without intersecting each other, σ on the other hand indicates that they intersect each other without coinciding; the combination $\varepsilon\sigma$ therefore indicates the number of pairs the two components of which lie at infinitesimal distance and cut each other at the same time. This can take place in our case as follows. We know that the double tangential planes of k^4 are simply the tangential planes of the four quadratic cones cutting each other in k^4; k^3 has with these four cones twenty-four points in common and through such a point pass evidently two generatrices satisfying the condition $\varepsilon\sigma$ and forming together one pair satisfying this condition. *These generatrices are the torsal lines of Ω^{24} and their points of intersection with k^3 are the cusps.* The surface Ω^{24} contains however also twenty double generatrices, viz. the common chords of k^3 and k^4, and these too must evidently be regarded as satisfying the indicated condition; the number $\varepsilon\sigma$ is therefore $= 20 + 24 = 44$.

The symbol εg indicates the number of pairs of rays which coincide and where g (or h, which is of course the same) intersects a given line; now that given line intersects the surface in twenty-four points: so εg is twenty-four. We thus find:

$$2 \,.\, \varepsilon\beta = \varepsilon\sigma + 2 \,.\, \varepsilon g = 44 + 48 = 92,$$

so

$$\varepsilon\beta = 46.$$

The symbol β indicates the condition that the two rays of a pair intersect a ray of a given pencil, thus the symbol $\varepsilon\beta$ indicates the condition that those two rays lie moreover at infinitesimal distance without intersecting each other; so the quantity $\varepsilon\beta$ indicates in our case evidently exactly the class of a plane section of Ω^{24}. If now we remember that such a section contains in general no cusps, we then find for the number of double points:

$$2\delta = 24 \,.\, 23 - 46 = 552 - 46 = 506,$$

so:

$$\delta = 253.$$

Now we know of these 253 double points the following: 1. the three points of intersection with k^3; 2. the four points of intersection with k^4, each of which is a ninefold point and therefore absorbs $\frac{1}{2} \,.\, 9 \,.\, 8 = 36$ double points; 3. the points of intersection with the twenty double generatrices, so together $3 + 4 \,.\, 36 + 20 = 167$; *the order of the new nodal curve is therefore* $253 - 167 = 86$.

A plane curve of order twenty-four can possess at most $\frac{1}{2} \,.\, 23 \,.\, 22 = 253$ double points, just the number of our case: Ω^{24} *is therefore a rational surface.*

We control this result by using a second formula of SCHUBERT viz [1]).

$$\sigma p + \varepsilon g + \varepsilon\beta = g_p + gh + h_p,$$

where σp indicates the number of pairs whose components without lying at infinitesimal distance intersect each other, whilst the point of intersection lies in a given plane, thus evidently in our case the order of the complete nodal curve, however taken twice, because each ray can be a g as well as an h, and therefore each pair of rays satisfying the condition σp counts for two pairs; g_p designates the number of pairs where the line g passes through a given point, a number which is evidently zero in our case, because all our rays belong to a surface and can therefore not pass through a point taken arbitrarily; for the same reason we find h_p zero. On the other hand gh designates the number of pairs where g intersects a given line l_1 and h a given line l_2, a number which in our case evidently amounts to $24 . 24 = 576$, because l_1 is intersected by twenty-four generatrices g, l_2 by twenty-four generatrices h, and each line of one group can be joined to each of the other. As $\varepsilon g = 24$, $\varepsilon\beta = 46$, σp becomes $576 - 24 - 46 = 506$, and as the order of the nodal curve is half of it, we find back the quantity 253.

In the formula:

$$\sigma e + \varepsilon g + \varepsilon\beta = g_e + gh + h_e, \text{[2]})$$

which is dualistically opposite to the last but one, σe indicates the number of pairs of rays whose components intersect each other and whose plane passes through a given point. Now, too, each pair we find is counted double, because each ray can be g as well as h; so $\frac{1}{2}\sigma e$ is the class of the developable, enveloped by the double tangential planes of Ω^{24}. The quantity g_e indicates the number of pairs where the ray g lies in a given plane, and h_e indicates the same for h; both numbers are in our case evidently zero; and from this ensues $\sigma e = \sigma p = 506$, *so that the class of the doubly circumscribed developable of Ω^{24} amounts to* 253.

For the sake of completeness we shall discuss in short the surface formed by the chords of k^3 resting on k^4. Through any point of k^4 passes one, so that k^4 is a single curve: through any point of k^3 on the other hand eight pass, because the quadratic cone projecting k^3 out of that point is intersected by k^4 in eight points; so k^3 is an eightfold curve. From this ensues again that an arbitrary chord

[1]) l.c. p. 60, N⁰. 23.
[2]) l.c. p. 60, N⁰. 24.

of k^3 intersects the demanded surface in each of its two supporting points with k^3 in eight points and no more, because two chords of k^3 cannot intersect each other outside k^3; *the demanded surface is thus of order sixteen, and it has k^3 as an eightfold curve, k^4 as a single curve.* That k^3 is the only manifold curve follows again out of the circumstance that two chords of k^3 can meet each other only on the curve itself; *on the other hand the twenty common chords of k^3 and k^4 are again double generatrices.* As an eightfold point counts for $\frac{1}{2} . 8 . 7 = 28$ double points, the complete number of double points of a plane section is $3.28 + 20 = 104$; a plane curve of order sixteen can however contain at most $\frac{1}{2} . 15 . 14 = 105$ double points; *so the surface is of genus* 1.

10. Through a point P of space pass two chords b of k^4 situated in the plane π through P and the line of intersection s of the two polar planes of P with respect to the two quadratic surfaces Φ_1, Φ_2 (§ 2) intersecting each other in k^4; we shall conjugate this plane π as focal plane to P and we shall discuss the focal system that is formed in this way. *Each point of space has then one focal plane* (so $a = 1$ [1])), *with the exception of the points of k^4 having ∞^1 focal planes, viz. all the planes containing the tangent in that point.*

In order to find inversely the number β of the foci P of an arbitrary plane π, we intersect that plane with Φ_1 and Φ_2; this gives rise to two conics k_1^2, k_2^2, and with respect to these we take the polar lines p_1, p_2 of an arbitrary point P of π. The polar planes of P with respect to Φ_1, Φ_2 then pass through p_1, p_2 and the line s conjugated to P contains the point of intersection of p_1 and p_2; if s is thus to be situated in plane π, then p_1 and p_2 must coincide, and this takes place only for the vertices of the polar triangle which k_1^2 and k_2^2 have in common; *so β is* $= 3$.

The third characteristic quantity, γ [1]), indicating how often a focus P lies on a given line, whilst at the same time the focal plane π passes t h r o u g h that line, is found as follows. When P describes the line l the two polar planes rotate round the two lines l_1, l_2 conjugated to l with respect to Φ_1, Φ_2; their line of intersection s describes a regulus with l_1, l_2 as bearers, and passing through the vertices of the four doubly projecting cones of k^4; this regulus intersects l in 2 points, through which every time one line s passes, and the foci conjugated to these lines lie on l as is in fact the case for a l l lines s of the regulus; for t h e s e two foci however the focal plane $\pi = Ps$ passes through l; *so* $\gamma = 2$.

[1]) STURM, "Liniengeometrie" I, p. 78.

Through the points P of space the polar planes π_1, π_2 with respect to Φ_1, Φ_2 are conjugated one by one to each other; so we can regard the lines s as the lines of intersection of conjugated planes of two collinear spaces, and we then find immediately *that the lines s form* [1] *a tetrahedral complex, for which the tetrahedron of the four vertices of the cones of k^4 is the surface of singularity,* in such a sense that each arbitrary ray through one of the vertices or in one of the faces of that tetrahedron is a complex ray, whilst in general the tetrahedral complex being quadratic a point has but a quadratic complex cone, a plane a quadratic complex curve. As namely the two polar planes of the vertex of a cone coincide in the opposite face of the tetrahedron, each line in this face can be regarded as a ray s, and as of a line l through T_1 e. g. the two conjugated lines lie in $T_2 T_3 T_4$, inversely the two polar planes of the point of intersection of those conjugated lines pass through l, so that l is a complex ray s. The complex cone of a point P in $T_2 T_3 T_4 = \tau_1$ breaks up into two pencils, one with vertex P and lying in τ_1, the other with vertex P and lying in a certain plane through P and T_1; and likewise the complex curve in a plane through T_1 degenerates into 2 points, viz. T_1 itself and a certain point in the line of intersection of that plane and τ.

A ray s being the line of intersection of the polar planes π_1, π_2 of a certain point P with respect to Φ_1, Φ_2, inversely through an arbitrary ray s two planes π_1, π_2 must pass having the same pole P; if however a line lies in a plane, then the conjugated line passes through the pole of that plane; thus for s the *two* conjugated lines s_1, s_2 must pass through P and must intersect each other in P; *so we can also define the rays s as those rays of space whose two conjugated lines with respect to Φ_1, Φ_2 intersect each other.* In this we have also a means to determine the focus of an arbitrary ray s; we have but to find the point of intersection of s_1 and s_2.

The rays s conjugated to the points of an arbitrary line l form a regulus as we have seen above; those conjugated to the points of a ray s must thus form according to the preceding a quadratic cone, and this is evidently the complex cone for the focus P of s, by means of which a construction for that cone has been found; we take the ray s conjugated to P, we allow a point to describe that ray and we determine for each position the two polar planes; the line of intersection of these describes the complex cone when the point describes the ray s. Just as the regulus for a line l, so each complex cone contains the vertices of the four doubly projecting cones;

[1] STURM l. c. p. 342.

and as the two conjugated lines of a ray s lie likewise on the complex cone of the focus P, they themselves are again rays s.

The complex curve lying in a plane α we find by regarding the two poles A_1 and A_2 of α. The conjugated lines l_1, l_2 of the lines l of α pass respectively through A_1 and A_2, and are conjugated by the rays l one by one to each other, so that two projective nets of rays are formed ; the locus of the points of intersection of rays conjugated to each other is a twisted cubic through A_1 and A_2, and furthermore through the four vertices of the cones $T_1 \ldots T_4$; for, the two conjugated lines of the line of intersection of α with $T_2 T_3 T_4$ are $A_1 T_1$, $A_2 T_1$. The rays s conjugated to the points P of that twisted cubic as foci lie in α and envelope the complex curve; and as each line of the plane $T_2 T_3 T_4$ can be taken as a ray s conjugated to e.g. T_1, so also the line of intersection with α, the complex conic will touch the four surfaces of the tetrahedron.

Botany. — "*On the demonstration of carotinoids in plants*" (First communication): *Separation of carotinoids in crystalline form.* By Prof. C. VAN WISSELINGH. (Communicated by Prof. MOLL).

(Communicated in the meeting of September 28, 1912).

Many of the chemical, physical, and microscopical investigations on the yellow and red colouring matters of the vegetable kingdom which are grouped under the name carotins [1]) or carotinoids [2]) bear witness to great care and originality. They have, however, not all led to similar results. Especially the microscopical investigation has led to very divergent results which sometimes seriously conflict with those obtained by chemical and physical means. The investigators might be divided into two groups; one is inclined to consider all carotinoids identical; believing that the differences observed are not of a chemical nature. The other group distinguishes several carotinoids.

T. TAMMES [3]) is especially a representative of the first group. After investigating a fairly large number of plants, she comes to the conclusion that the yellow to red colouring matter of plastids, in green, yellow variegated and etiolated leaves, in autumn leaves, in flowers, fruits and seeds, in diatoms, green, blue, brown and red

1) CZAPEK, Biochemie der Pflanzen, 1. p. 172.

2) M. TSWETT, Über den makro- und mikrochemischen Nachweis des Carotins, Ber. d. d. bot. Ges. 29. Jahrg., Heft 9, 1911, p. 630.

3) T. TAMMES, Über die Verbreitung des Carotins im Pflanzenreiche, Flora, 1900, 87. Bd. 2. Heft, p. 244.