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Mathematics. — *"On loci, congruences and focal systems deduced from a twisted cubic and a twisted biquadratic curve".* II. Communicated by Prof. HK. DE VRIES.

(Communicated in the meeting of Oct. 26, 1912).

11. We found in § 1 [1]) a surface Ω^6 as locus of the points P for which the chord a of k^3 and the t w o chords b of k^4 are complanar; in the plane of those three chords then lies a ray s of the tetrahedral complex discussed in the preceding § [2]), so that the rays s corresponding to the points P of Ω^6 form a congruence contained in the complex; we wish to know this congruence better.

Through an arbitrary point P of space pass six rays of the congruence, thus $\mu = 6$; for a l l rays s through that point form a quadratic cone, the complex cone (§ 10), and the foci corresponding to the edges of this cone lie on the ray s of P; this intersects Ω^6 in 6 points and the rays s conjugated to t h e s e pass through P. The number μ is called the order of the congruence.

Exceptions we find only for the points of k^4 and in the 4 cone vertices. If P lies on k^4 then the conjugated line s is the tangent in P, which now belongs itself to the complexcone of P, for it is generated as line of intersection of the two polar planes of P itself with respect to Φ_1, Φ_2, which planes coincide with the tangential planes to the two quadratic surfaces. The tangent s to k^4 is now however at the same time tangent to Ω^6 and it contains therefore besides the point of contact only 4 points of Ω^6; thus besides the tangent only 4 rays of the congruence pass through P, from which ensues that the tangent itself counts double.

The four cone vertices bear themselves quite differently. To T_1 e.g. are conjugated as rays s all the lines of the plane $T_2T_3T_4 = \tau_1$, which plane intersects Ω^6 in a curve k^6 of order 6 containing T_2, T_3, T_4 as single points, the points of intersection with k^3 on the other hand as nodal points; to each point of the curve a ray s through T is conjugated, so that through T_1 pass an infinite number of rays of the congruence forming a cone. This cone can be determined more closely as follows. As of an arbitrary line s_1 in τ_1 the two conjugated lines pass through T_1, the ray s_1 corresponding to the points of that ray s_1 form a quadratic cone; now s_1 intersects the curve k^6 in 6 points, thus the q u a d r a t i c cone must intersect the cone t o b e f o u n d in 6 edges.

Let us consider the point of intersection of s_1 with the edge T_2T_4

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[1]) See *Proceedings* of Oct. 26th, 1912, p. 495.

[2]) l. c. p. 509.

of the tetrahedron. The two polar planes of this point now pass not only through T_1, but also through T_2, because the point itself lies now not only in τ_1 but also in $\tau_2 = T_1 T_3 T_4$; so the quadratic cone contains the edge $T_1 T_2$ and of course for the same reason $T_1 T_3$ and $T_1 T_4$. These same edges lie also on the cone to be found and that as fourfold ones, which is easy to see when we consider e.g. the line $T_2 T_4$. This line intersects k^6 in T_2, T_4 and in four points more; to T_2 all lines of τ_2 are conjugated and thus also particularly all lines of τ_2 through T_1, so that this plane (and for the same reason the two other tetrahedral planes through T_1) separate themselves from the cone; however, for each of the 4 remaining points of intersection the conjugated ray s is determined and identical with $T_1 T_2$, so that this line is indeed for the cone under discussion a fourfold edge. So the quadratic cone and the cone under discussion have in common:

1. the three fourfold edges of the latter, 2. the 6 rays s, conjugated to the points of intersection of s_1 with k^6, thus altogether $3 \times 4 + 6 = 18$ edges; so the cone under discussion is of order nine. If finally we see that this cone possesses three double edges too, formed by the rays s conjugated to the three nodal points of k^6 lying in k^3, we can comprise our results as follows:

For the congruence of the rays s corresponding to the points of Ω^6 the four cone vertices are singular points, as through these points pass instead of 6, as in the general case, ∞^1 rays of the congruence; these form at each of those 4 points in the first place three pencils situated in the three tetrahedral planes through that point, and in the second place a cone of order nine with three double edges and three fourfold edges, the latter coinciding with the three tetrahedral edges through that point.

The cone of order nine must intersect the tetrahedral plane $\tau_4 = T_1 T_2 T_3$ in nine edges, four of which lie united in $T_1 T_2$, four others in $T_1 T_3$, so that only one is left; the latter is to be regarded as the line s more closely conjugated to point T_4, and it will change its position if k^6 changes its form, and passes through T_4 in an other direction.

The complete nodal curve of the surface of tangents of k^4 consists of four plane curves of order four lying in the four tetrahedral planes and every time with 3 vertices of that tetrahedron as nodes; let us now regard in particular the nodal curve lying in τ_1. Through a point P of this pass two tangents of k^4 representing the two chords b through that point; the line connecting the two contact points passes through T_1 and is an edge of the doubly projecting cone having this point as vertex, and from this all follows easily that that edge of the cone is the line s conjugated to the point P of the nodal curve. The nodal curve now intersects k^6 in 24 points, of which 6 however coincide two by two with T_2, T_3, T_4; the lines s conjugated to the 18 remaining ones are the lines of intersection of the cone of order nine with the doubly projecting cone at the vertex T_1.

The surface of tangents of k^4 is of order eight, it contains the 4 just mentioned plane curves of order 4 as nodal curves and the four cone vertices as fourfold points; it intersects Ω^6 in a curve of order 48 having the cone vertices as fourfold points, the 24 points of intersection with k^3 and the 4 times 18 points of intersection on the 4 nodal curves as nodal points. For an arbitrary point of this curve a chord a of k^3 and 2 chords b of k^4 are complanar; *one* of these two chords b however is a tangent of k^4. For one of the 24 nodal points on k^3 the same holds, as is easy to see; *for each of the* 4×18 *remaining nodal points on the other hand a chord* a *of* k^3 *is complanar to* 2 tangents *of* k^4.

12. We now determine the second characteristic number, the class ν of the congruence formed by the rays s conjugated to the points of Ω^6, i. e. the number of rays of the congruence in an arbitrary plane. The locus of all foci of all the rays s lying in an arbitrary plane α is according to § 10 a twisted cubic through the four cone vertices; this intersects Ω^6 in 18 points, but to these belong the four cone vertices. To each of the 14 remaining ones one ray s is conjugated, lying in the assumed plane; to a cone vertex on the other hand all rays of the opposite tetrahedral face are conjugated, and therefore also the line of intersection of that face with α, so that if we like we can say that in each plane lie 18 congruence rays, among which, however, then always appear the lines of intersection with the four tetrahedral planes. So we prefer to say that *in an arbitrary plane lie* 14 *congruence rays and that from the complete congruence the* 4 *fields of rays situated in the four tetrahedral planes separate themselves.*

In § 8 we found that the double tangential planes of the surface Ω_{18} discussed in § 7 of class 18 envelop a developable Δ_9 of class 9; they are nothing else than the focal planes of the points of k^3. The lines s they contain belong to the congruence we are discussing, and these rays count double in the congruence because k^3 is for Ω^6 a nodal curve; let us find the locus of these double rays.

If a point P describes the curve k^3, then each of its two polar planes

π_1, π_2 with respect to Φ_1, Φ_2 envelops the reciprocal figure of a cubic curve, i. e. a developable of class 3, and the tangential planes of these two developables are conjugated through the points P one by one to each other; for, a tangential plane π_1 of the first developable has only one pole P and this again only one polar plane with respect to Φ_1. Now the lines s are the lines of intersection of the conjugated tangential planes of the two developables; they form a scroll the order of which appears to be 6. Let us namely assume a line l; through a point P of this line pass 3 tangential planes π_1 of the first developable, and to these three planes π_2 are conjugated; if these intersect the line l in three points Q, then to one point P three points Q are conjugated, but of course inversely too; through each of the 6 coincidences passes one line s, so the line l intersects the demanded surface in 6 points.

For each of the three points of intersection of k^3 with one of the four tetrahedral faces the corresponding line s passes through the opposite vertex; the four cone vertices are therefore threefold points of the surface. Moreover the surface possesses a nodal curve cut by each generatrix in $6-2=4$ points and which proves to be of order 10; the four cone vertices are as points of intersection of 3 generatrices of the surface also threefold points of the nodal curve.

The order of the nodal curve we determine again as in § 9 with the aid of SCHUBERT's formula:

$$2 \,.\, \varepsilon\beta = \varepsilon\sigma + 2 \,.\, \varepsilon g,$$

by conjugating each generatrix of the scroll as ray g to all others as rays h. The symbol εg, the number of coinciding pairs where g intersects an arbitrary line, is 6, viz. equal to the order of the surface; the question is now how great is $\varepsilon\sigma$, the number of pairs gh of which the components lie at infinitesimal distance and intersect each other; these are evidently the torsal lines of the surface. We shall show that their number is 8.

The rays s conjugated to the points of a line l describe a regulus through the four cone vertices (§ 4) and so they cross each other all, then too when they lie at infinitesimal distance; they can intersect each other only when line l is itself a ray s (§ 10); however, they then intersect each other all and that in the same point, viz. the focus of s. If thus two rays s corresponding to two points of k^3 are to intersect each other, then their connecting line must be a ray s; and if moreover these rays are to lie at infinitesimal distance then the line connecting the points must be a tangent of k^3; so the question is simply this how many tangents of k^3 are rays of the tetrahedral complex. Now according to one of the theorems of HALPHEN a complex of order p and a scroll of order n have pn generatrices in common; the tetrahedral complex is quadratic, the surface of tangents of k^3 is of order 4 and so the number of common rays is 8; so $\varepsilon\sigma = 8$. From this ensues $2 . \varepsilon\beta = 8 + 2 \times 6 = 20$, $\varepsilon\beta = 10$. Now $\varepsilon\beta$ represents the class of a plane section of our scroll of order 6; by applying the first PLÜCKER formula for plane curves (§ 9) we thus find $\sigma = 10$, a number we can control with the aid of

$$\sigma p + \varepsilon g + \varepsilon\beta = gh \quad (\S\ 9); \text{ viz.}$$
$$\sigma p + 6 + 10 = 36$$
$$\sigma p = 20,$$

and this is twice the order of the nodal curve as we proved in § 9. Summing up we thus find: *The nodal rays of our congruence form a scroll of order* 6 *with* 8 *torsal lines and therefore also* 8 *pinch points lying on a nodal curve of order* 10 *which is intersected by each generatrix in* 4 *points and having the vertices of the four doubly projecting cones of* k^4 *as threefold points.*

§ 13. We shall inquire in this § into the scroll of the rays s of our congruence, resting on an arbitrary line l and in particular on a ray s. All rays s intersecting l form a congruence (2,2); for the quadratic complex cone with an arbitrary point of space as vertex intersects l in 2 points, so that through that point 2 rays of the congruence pass; and an arbitrary plane contains of the complex cone of the point of intersection with l likewise 2 rays, so that in an arbitrary plane lie likewise 2 rays of the congruence. An exception is made by the points *on* l, which are vertices of quadratic cones of rays of the congruence and the planes *through* l containing an infinite number of rays of the congruence, which evidently envelop a conic because two of them pass through any point of the plane. Among these planes are four, which are distinguished from the others, because the conic which they bear breaks up into a pair of points, and dualistically related to these are 4 points on l whose quadratic cone breaks up into a pair of planes; the *planes* are those through l and the 4 cone vertices, the *points* are the points of intersection of l with the four tetrahedral planes. In the plane lT_1 e.g. according to § 10 all the rays through T_1 belong to the complex, so the conic in this plane must degenerate into T_1 and one other point; or expressed in other words: of the two rays of the congruence through a point of this plane one passes through the fixed point T_1, so the second must also pass through a fixed point.

This point is a certain point of the line of intersection of the plane lT_1 with the face τ_1 lying opposite T_1; for, for an arbitrary point P of τ_1 the complex cone breaks up into the pencil with vertex P lying in τ_1 and a pencil with vertex P lying in a certain plane through P and T_1, and inversely for a plane through T_1, so e.g. our plane lT_1, the complex conic breaks up into point T_1 and a second point lying on the line of intersection of that plane with τ_1 (§ 10). So the four singular points *on* l are therefore nothing else but the points of intersection with the four tetrahedral planes.

Two congruences according to the theorem of HALPHEN possess in general only a finite number of common rays; however, the congruence discussed above and the one deduced out of the points of Ω^6 possess an infinite number, therefore a scroll; for *all* complex rays s cutting l belong to the former, and every time 6 of these through a point of l belong according to § 4 to the second; the two congruences have thus a scroll in common for which the line l is a sixfold line. As furthermore according to § 12 there lie in each plane 14 rays of the second which as rays s cutting l also belong to the former (*and therefore, as we now discover, envelop a conic*) the scroll to be found is a Ω^{20} of order twenty and with a nodal curve which by each plane through the sixfold line l is cut in $\frac{1}{2} . 14 . 13 = 91$ points not lying on l.

If a point P describes a line l, then the corresponding line s describes a regulus through the 4 cone vertices (§ 4); and if we wish to construct for that same point P the complex cone, then according to § 10 we must determine the lines s which correspond to the points of the line s conjugated to P; *from this ensues that the regulus formed by the lines corresponding to the points P of l is the locus of the points P whose conjugated rays form the congruence of the rays s which intersect l. And so furthermore from this ensues that the curve k^{12} of order twelve along which that regulus and Ω^6 intersect each other, is the locus of the points P whose conjugated lines s form the just found surface Ω^{20}.*

Each generatrix s of the regulus contains 6 points of Ω^6 or therefore of k^{12}; the corresponding lines s are the six generatrices of Ω^{20} issuing from the focus conjugated to the generatrix P of the regulus on the sixfold line l. The curve k^{12} admits 6 nodal points, viz. the points of intersection of the regulus with k^3; the line s of the regulus through such a point intersects Ω^6 in two coinciding points, from which ensues that through the point P on l conjugated to that line s really only 5 rays s' pass instead of 6; one of these, however, viz. the one corresponding to the nodal point of k^{12}, is a generatrix of the surface of the double rays of the congruence (§ 12), and thus evidently a double generatrix of Ω^6. So: *the* 6 *points of intersection of l with the surface of the double rays of the congruence deduced from Ω^6 are double generatrices of Ω^{20}.*

The curve k^{12} passes through the 4 cone vertices and the lines s corresponding to them fill the tetrahedral faces lying opposite; so we can ask how Ω^{20} bears itself with respect to those faces. We now have separated in § 12 of the complete congruence deduced from Ω^6 the four fields of rays in the tetrahedral faces; if we thus follow k^{12} through the vertex T_1, then to all points on either side of T_1 every time a completely determined ray intersecting l is conjugated; by this also in τ_1 one ray is determined, so that Ω^{20} has simply one of its generatrices in τ_1 and therefore this plane as an ordinary tangential plane. The cone vertices on the contrary are themselves singular points of Ω^{20}. Our curve k^{12} namely cuts τ_1 in 12 points lying on a conic and at the same time on the section k^6 of Ω^6 with τ_1, to which belong the three cone vertices T_2, T_3, T_4; the rays s corresponding to these lie, it is true, according to the above, respectively in τ_2, τ_3, τ_4, but they do not pass through T_1 (if let us say s conjugated to T_2 had to pass e.g. through T_1 it would have to pass for the same reason through T_3 and T_4), however the rays corresponding to the remaining 9 points of intersection do; so in the plane $T_1 l$ nine generatrices of Ω^{20} pass through T_1; *they are the lines of intersection of this plane with the cone of order nine, on which lie according to* § 11 *the rays s which are conjugated to the points of intersection of Ω^6 with τ_1*. The same holds of course for the planes through the remaining vertices and l.

In such a plane the conic which must be touched by the 14 generatrices of Ω^{20} degenerates, as we have seen at the beginning of this §, into a pair of points; *so in each of these four planes not only nine generatrices pass through a cone vertex, but also the five remaining ones pass through another fixed point, lying in the opposite face. The vertices are thus for the nodal curve of Ω^{20} $\frac{1}{2} \cdot 9 \cdot 8 = 36$-fold points, the other points $\frac{1}{2} \cdot 5 \cdot 4 = 10$-fold points.* If we add these $36 + 10$ points to the 45 points generated by the intersection of the two groups of 5 and respectively 9 generatrices lying in a plane through a cone vertex and l, we find back the 91 points of the beginning of this §.

If we add to the figure, as we are now studying it, another arbitrary line m, then to this also belongs a regulus through the 4 cone vertices cutting the regulus conjugated to l in a curve of order four through the vertices; this biquadratic curve has with Ω^6 twenty-four points in common among which again the cone vertices; if we set these apart for reasons more than once mentioned, then there remain twenty; the rays s corresponding to these rest on l as well as on m, i.e. the rays resting on l form a surface Ω^{20}. This in order to control the result.

§ 14. We shall now try to determine the order of the nodal curve of Ω^{20}, which is according to the preceding equal to 91, augmented by the number of points unknown for the present, with which that curve rests on l; this number is connected with other numbers which we must also calculate to be able to find the former, and to this a deeper study is necessary of Ω^{20}, as well as of the figures which are in relation with this surface.

A scroll possesses in general a certain number of pinch points and torsal lines, and those of Ω^{20} can be divided into two kinds which bear themselves very differently in the following considerations. To the first kind we reckon the torsal lines whose pinchpoint lies on l but whose torsal plane does not pass through l; to the second kind the dualistically opposite, thus those whose pinchpoint does not lie on l (thus on the nodal curve to be investigated), but whose torsal plane for it does pass through l.

A third kind might be a combination of the two others, torsal-lines, whose pinchpoint lies on l and whose torsal plane passes through l; we shall however show that these do not appear on Ω^{20}.

We can get some insight in the appearance of these torsal lines if we return to the regulus and the curve k^{12} of the preceding §; k^{12} contains the foci of all generatrices s of Ω^{20}, and the regulus is the locus of all the rays s, which are conjugated to the points P of l. Moreover lie in a plane through l fourteen generatrices of Ω^{20} and the foci of these lie on a cubic curve through the four cone vertices. Let us now consider the generatrices of the regulus and the curve k^{12}. A generatrix s_r of the regulus intersects Ω^{6} in six points and these lie on k^{12}, for k^{12} is the intersection of Ω^{6} with the regulus; the rays s corresponding to these six points are the generatrices of Ω^{6}, which pass through a same point P of l, viz. the focus of s_r. If however s_r has two coinciding points in common with k^{12}, then two of the six generatrices through P coincide, and this can happen in two ways. The curve k^{12} has namely 6 nodal points (viz. on k^{3}), and through each of these passes a line s_r which has with k^{12} besides the nodal point only four points in common; of the six generatrices of Ω^{20} through the focus P of s_r two coincide and that in a double generatrix of Ω^{20}, the number of which, as we know, (§ 13) amounts to 6. Those double lines can be regarded as "full coincidences" in the sense of SCHUBERT, i. e. as coinciding lines whose point of intersection as well as whose connecting plane is indefinite; so they satisfy the definition we have given above of torsal lines of the first kind.

In the second place now however an s_r can touch the curve k^{12}; in this case the two coinciding rays s conjugated to the point of contact form a "single coincidence", i. e. two coinciding rays whose point of intersection and whose connecting plane both remain definite; the point of intersection lies on l, the connecting plane however does not pass in general through l, for then it would be necessary that in the point where s_r touches the curve k^{12} at the same time also one of the cubic curves through the vertices were to touch that curve, which can of course in general not be the case; so we find torsal lines of the first kind. However, if there really were torsal lines of the third kind, then there would have to be among the points of contact of the rays s_r with k^{12} also some where at the same time a cubic curve were to touch k^{12}; these particular points of contact would then give rise to the torsal lines of the third kind.

The cubic conjugated to a plane λ through l may have with k^{12} two coinciding points in common; in this case two generatrices lying in the same plane λ coincide. This happens in the first place for those planes λ whose conjugated cubic passes through one of the six nodal points of k^{12}, and so we find again the nodal lines of Ω^{24}; this, however, also takes place if a cubic touches k^{12}, and then we find a torsal line of the second kind; for the two rays s conjugated to the point of contact coincide whilst their connecting plane λ remains definite. Their point of intersection lies in general not on l, because the point of contact of k^{12} with the cubic is in general not a point of contact of k^{12} with a generatrix s_r of the regulus; for those points however where that might be the case we would find torsal lines of the third kind.

We calculate the complete number of points, where a line of the regulus has two coinciding points in common with k^{12}, with the aid of the formula of SCHUBERT:

$$\varepsilon = p + q - g$$ [1])

which relates to a set of ∞^1 pairs of points. We can now indeed obtain such a set by conjugating on each line of the regulus each of the six points k^{12}, regarded as a point p, to the five others, which are then named q; each line of this kind bears then thirty pairs, because each of the six points of k^{12} lying on it can be conjugated succes-

[1]) SCHUBERT l. c. p. 44.

sively as point p to the five others (which are then called q), and the whole number is ∞^1. The quantity p in the formula points to the number of pairs, where the point p lies in a given plane; now this plane intersects k^{12} in twelve points, which we can all regard as points p; through each of these passes one line of the regulus containing still five other points of k^{12}, which we shall call q; it is then clear that there are 60 pairs pq whose component p lies in a given plane. The symbol q has the same meaning as p, in this case for the points q; however, as in our case each point of k^{12} can be a p as well as a q, the quantity q is also $= 60$. Finally the letter g indicates the number of pairs whose connecting line intersects a given line; now that given line intersects only two lines of the regulus, on each of which 30 pairs pq are situated; g is therefore 60, and in this way we find for ε, the number of coincidences,

$$\varepsilon = 60 + 60 - 60 = 60.$$

So there are sixty lines of the regulus containing two coinciding points of k^{12}; 6 of them correspond to the double generatrices of Ω^{20}, but a closer investigation shows us that these must be counted double; the remaining forty-eight are tangents of k^{12} and correspond to torsal lines: *so* Ω^{20} *contains forty-eight torsal lines of the first kind.*

The formula $\varepsilon = p + q - g$, or written as: $p + q = g + \varepsilon$, is namely deduced by assuming a system of ∞^1 pairs of points p,q and by projecting these out of a line l. If a plane λ through l contains p points p, we can connect the points q conjugated to these by planes with l, so that p planes are conjugated to λ; if inversely a plane λ contains q points q, then to this plane q others are conjugated, and thus is generated a correspondence (p,q) with $p + q$ coincidences, which are evidently furnished by means of the coincidences of the pairs of points themselves (ε) and by the pairs of points whose connecting line intersects l.

Let us now apply this to our case. A plane λ intersects k^{12} in twelve points p; to each of these the five points q are conjugated lying with p on a generatrix of the regulus, so that to λ sixty other planes are conjugated. A plane λ through a nodal point D of k^3 however contains of k^{12} besides D only ten more points, which give rise to fifty planes; so the ten remaining ones must be furnished by D itself. Now the generatrix of the regulus through D intersects Ω^6 besides in D only in four points more, the planes through these and l count double in the correspondence, because D itself counts double in the plane lD, but this furnishes only four planes counting double, or eight single ones; so the two missing ones must coincide with the plane lD, i. e. lD is a double plane counting double (and likewise a fourfold "branchplane") Q. E. D. In § 17 we shall see a confirmation of the considerations given here.

§ 15. In order to be able to point out the eventual existence of the torsal lines of the third kind, we must include a new auxiliary surface in our consideration, which we deduce from the tetrahedral complex. All complex rays lying in one and the same plane envelop a conic which also touches the four tetrahedral planes, and indeed in § 13 we have already drawn attention to the fact that the fourteen generatrices of Ω^{20} lying in a plane λ through l are the tangents of a conic; the auxiliary surface which we must introduce to find the torsal lines of the third kind is the locus of these conics, thus the locus of the complex conics lying in the planes λ through l. In each plane λ lies one and through each point of λ pass two of these conics, as is easy to prove. For, let us imagine an arbitrary plane λ and an arbitrary point P on l, then λ intersects the complex cone of P in two rays s, and these are the tangents out of P to k^2 lying in λ; therefore if k^2 is to pass through P then the two tangents out of P must coincide, and this takes place in the two tangential planes through l to the complex cone. *The locus to be found is therefore an Ω^4 with double line l.*

If a surface possesses a double line it is an ordinary phenomenon that only a part is efficient, the rest parasitical; so applied to our case that through certain points of l two real conics go, through others two conjugated imaginary ones, and through the limiting points between both groups two coinciding ones; for the surface we have here under discussion those limiting points are the points of intersection of l with the four tetrahedral planes. Let us namely assume the point of intersection s_1 of l with τ_1. The complex cone of S_1 breaks up into two planes, viz. τ_1 and a plane through S_1 and T_1 cutting τ_1 along a line s_1 through S_1; s_1 is nothing else but the generatrix which Ω^{20} has in common with τ_1. Now the tangential planes through l to this degenerated cone coincide in the plane ls_1, which bears a complex conic touching τ_1 in S_1 (with tangent s_1); this conic is the only one passing through S_1.

Of great importance for our surface Ω^4 are furthermore the planes through l and the four cone vertices. We know i.a. that of the fourteen generatrices of Ω^{20} in the plane lT_1 nine pass through T_1 and the other five through a point $T_1{}^*$ lying in τ_1, and really the complex conic in this plane breaks up into the pair of points T_1, $T_1{}^*$, which means for the surface Ω^4 that it is intersected by the plane lT_1 (except in the nodal line l of course) in the line

$T_1 T_1^*$, counted double, whilst the tangents to this conic degenerated into a double line, thus the complex rays in this plane can only go through T_1 and T_1^*; *the four planes* lT_i $(i = 1, \ldots, 4)$ *touch* Ω^4 *along the four lines* $T_i\, T_i^*$ $(i = 1, \ldots, 4)$, *and the* 8 *points* T_i, T_i^* $(i = 1, \ldots, 4)$, *are nodal points of* Ω^4.

The nodal point T_1^* lies in τ_1 and is characterized by the property that its complex cone breaks up into the plane τ_1 and the plane $T_1^* l$, so that each ray through T_1^* cutting l is a complex ray. Let us assume e.g. the plane T_1^*, T_2^*, T_3^*; this cuts l in a certain point L and according to the preceding the lines LT_1^*, LT_2^*, LT_3^* are complex rays. But if three complex rays lying in one plane pass through the same point, then the complex curve in that plane must degenerate into a pair of points, and this takes place only for the planes through the four cone vertices; so the plane $T_1^*\, T_2^*\, T_3^*$ passes through a vertex, in our notation T_4. And with this we have proved the following property: *the eight nodal points of* Ω^4 *can be divided into two groups of four,* $T_1, \ldots, T_4$ *and* $T_1^*, \ldots, T_4^*$, *and the four tetrahedra having these points as vertices are simultaneously described in and around each other.*

The surface Ω^4 is one of those already found and described by PLÜCKER in his "Neue Geometrie des Raumes", Part 1, § 6, p. 193 etc., on the occasion of his general investigations of quadratic complexes.

We shall now intersect the surfaces Ω^4 and Ω^{20} with each other. The section which must be of order 80 consists in the first place of the line l to be counted twelve times, because l is for Ω^4 a double line and for Ω^{20} a sixfold line; the residual section is thus a curve of order $80 - 12 = 68$. Now there lie in a plane λ through l fourteen generatrices of Ω^{20}, and these touch a conic lying on Ω^4; so the residual section is a curve having with a plane λ through l fourteen points in common. However, we must keep in view that the two surfaces touch each other in every ordinary point which they have in common outside l; so the residual section must be a curve to be counted twice, from which ensues that its order must be 34; as it has outside l with a plane λ only fourteen points in common, it must have with l itself 20 points in common. It then goes 9 times through each of the four points T_i $(i = 1, \ldots, 4)$, and 5 times through each of the four points T_i^* $(i = 1, \ldots, 4)$ because these points are respectively 9- and 5- fold points of Ω^{20} (§ 13) and nodal points of Ω^4; the curve counted d o u b l e has then 18- and resp. 10-fold points, as should.

How does now a point of intersection of the curve found just now with l make its appearence? An a r b i t r a r y point is generated when in the plane λ through that point and l a generatrix of Ω^{20} and a conic of Ω^4 touch each other; so a point on l is generated when in a certain plane λ through l a generatrix of Ω^{20} and a conic of Ω^4 touch each other exactly on l; then through the point of contact, however, pass two coinciding tangents of the conic, thus two coinciding complex rays; or, in a better wording, whilst in an arbitrary plane λ through each of the 14 points of l lying at the same time on generatrices of Ω^{20} two complex rays pass one of which does not belong to Ω^{20}, in the case under discussion the last ray coincides with the former, so that it might look as if here a torsal line of the third kind was generated; but it would have to be possible to show that in the plane through l and such a line only twelve other generatrices of Ω^{20} were situated, or that whilst tending to such a plane two generatrices were tending to each other, for which there is no reason whatever; so we conclude that Ω^{20} does not possess torsal lines of the third kind, and we shall find this conclusion justified in future in different moments.

16. In a plane λ through l lie fourteen generatrices of Ω^{20}; through each of the points L in which these generatrices intersect l five other generatrices pass which in general determine with l 70 different planes; we shall conjugate these to λ. In this manner the planes through l are arranged in a symmetrical correspondence of order 70; we wish to submit the 140 double planes σ of this correspondence to a closer investigation. Such a plane is evidently generated if for a certain point L of l two of the 6 generatrices s through that point lie with l in the same plane; the point L is then evidently at the same time a point of the nodal curve of Ω^{20} lying on l, for this double curve is the locus of the points of intersection of all generatrices lying in a plane λ through l. We shall now, however, show that each suchlike plane as a matter of fact represents two coinciding double planes. Let us assume to that end a plane λ in which two generatrices s_1, s_2 are lying, cutting l in two points L_1, L_2 lying close together. Through each of these last pass five generatrices not lying in λ, and that in such a way that one of the generatrices through L_1 lies in the vicinity of s_2 and inversely, whilst the remaining ones lie two by two in e a c h o t h e r's vicinity. If we allow λ to transform itself gradually into σ, then that one generatrix through L_1 coincides with s_2, and inversely, whilst the remaining ones coincide two by two in four double planes of the second kind

to which σ corresponds as "branch plane" [1]; if we remember that in σ, besides s_1 and s_2, lie only twelve other generatrices of Ω^{20}, then to σ are conjugated $12 \times 5 + 2 \times 4 = 68$ planes not coinciding with σ. The two missing ones *do* coincide with σ, so that σ is really a double double plane.

It is easy to see that the reasoning given here is literally applicable to the six double generatrices, but not to the torsal lines of the first kind, and much less to those of the second. The plane λ through a torsal line of the first kind is, it is true, a double plane σ, but only a single one, for besides that torsal line there are now in σ still 13 other generatrices of Ω^{20} (because namely the torsal plane does not pass through l), and through the pinchpoint pass four generatrices not lying in σ; so to σ are now conjugated $13 \times 5 + 4 = 69$ planes, so that only one coincides with σ. And as for the torsal lines of the second kind, these give no rise whatever to double planes, but only to branch planes. Let us assume again, as above, a plane λ, in which lie two generatrices s_1, s_2 which almost coincide, but in such a way, that their point of intersection lies at finite distance from l. Through L_1 and L_2 pass again every time five generatrices not lying in λ, but now lying neither in the vicinity of s_1 nor of s_2, and when λ transforms itself into the plane through l and the torsal line of the second kind, those ten generatrices coincide two by two; so the torsal plane becomes a fivefold branch plane, but not a double plane.

Let us now draw the conclusion from these considerations. If we assume the double curve of Ω^{20} to have x points in common with l, then our correspondence contains $x + 6$ (namely on account of the double generatrices) double planes counting twice, and 48 (on account of the torsal lines of the first kind, see § 14) double planes counting once, so that the equation exists:

$$2(x + 6) + 48 = 140$$

out of which we find: $x = 40$.

So the double curve of Ω^{20} *rests in* 40 *points on* l *and is therefore of order* $40 + 91 = 131$.

A plane section of Ω^{20} contains however not only 131 double points, but $131 + 6 + 15 = 152$, viz. 6 on the generatrices and a sixfold point on l; so it is of class $20 \times 19 - 2 \times 152 = 76$, so that if we again apply the formula

$$\varepsilon\sigma = 2 \, . \, \varepsilon\beta - 2 \, . \, \varepsilon g$$

we must substitute for $\varepsilon\beta$ the number 76; and as $\varepsilon g = 20$, because the line of the condition g intersects Ω^{20} in 20 points, we find.

[1]) Em. Weyr, "Beiträge zur Curvenlehre," pp. 9, 10.

$$\varepsilon\sigma = 2.76 - 2.20 = 112.$$

This number comprises all pairs of lines of the surface whose components lie at infinitesimal distance and intersect each other, thus the 6 double generatrices, the 48 torsal lines of the first kind and the still unknown number of torsal lines of the second kind: *so Ω^{20} contains 58 torsal lines of the second kind.*

For a congruence is characteristic, besides the number of rays through a point (in our case 6) and in a plane (in our case 15), the number of pairs of rays which belong with an arbitrary line to a pencil, the so-called rank; according to the preceding this number is nothing but our quantity x, thus 40; *the congruence deduced from Ω^{20} is therefore a* (6, 14, 40).

The results found above allow being controlled, by our finding the $4 \times 131 = 524$ points of intersection of the surface Ω^4 with the nodal curve of Ω^{20}. The greatest number of these points we find united in the points T_i and T_i^*, the eight nodal points of Ω^4. A point T_i is a 36-fold point of the nodal curve (§ 13) and counts thus for 72 points of intersection; a point T_i^* is a tenfold point of the curve and counts thus for 20 points of intersection, together $4 \times 92 =$ **368**. In the 40 points where the nodal curve rests on l, the curve meets the double line of Ω^4; so this gives **80** points. In a pinch point of a torsal line of the second kind the nodal curve traverses Ω^4 in a single point of intersection. Let us assume e.g. a plane λ through l and such a torsal line as well as two planes λ_1 and λ_2 on both sides of λ and in the immediate vicinity of λ; then in λ_1 e.g. two generatices of Ω^{20} will nearly coincide, so their point of intersection will almost lie on the conic of Ω^4 lying in this plane; in λ itself this point of intersection really falls exactly on k^2, and in λ_2 the two tangents have become conjugate imaginary; their point of intersection has nevertheless remained real, i. e. the nodal curve naturally continues its course but now lies inside k^2; so it has intersected the surface. As Ω^{20} possesses 58 torsal lines of the second kind we find 58 new points of intersection.

We must finally discuss the 6 double generatrices of Ω^{20} which bear themselves as regards the nodal curve about the same as torsal lines of the second kind do. We must not lose sight of the fact that a double edge d of Ω^{20} is a singular ray for the congruence but not for the complex; so if it intersects l in D, then the complex cone of D shows in no way anything particular; the plane λ through l and d contains thus two different generatrices of that cone, of which d is one. The consequence is that the conic of Ω^4 in λ must touch the line d in some point or other not lying on l, through which point the nodal curve passes, just as with a torsal line of the second kind; and indeed the plane λ through d contains besides d only twelve generatrices of Ω^{20}, intersecting each other mutually in $\frac{1}{2} \cdot 12 \cdot 11 = 66$, and d in 12 points counting double, which amounts together to $66 + 24 = 90$ points of the nodal curve; so one is missing, but is the point of intersection in a closer sense of the two generatrices coinciding in d, and according to the above this cannot lie on l. In passing we learn from this consideration that *the nodal curve of Ω^{20} touches each plane through l and either a torsal line of the second kind or a double edge in twelve points lying either on that torsal line or on that double edge.*

That a double edge, however, does not bear itself altogether as a torsal line follows from a repetition of the above given consideration with the three planes λ_1, λ, λ_2; for now in λ_1 as well as in λ_2 two *real* generatrices of Ω^{20} will lie. Nevertheless the nodal curve has here with Ω^4 not only a contact by two points, but even one by three points, so that the plane of osculation of the nodal curve coincides with the tangential plane of Ω^4, and the nodal curve touches one of the two branches of the section of Ω^4 lying in the tangential plane.

Indeed, it is clear that besides the $368 + 80 + 58 = 506$ points of intersection already found no others are possible than the 6 points on the double edges, which occupy us here; for each point of intersection not lying on l must be the point of contact of a generatrix of Ω^{20} with a conic of Ω^4, so a pinchpoint of a torsal line of the second kind, or of a double edge; as there are 6 of the latter sort in evidence and $524 - 506 = 18$ points missing, each of those six points must be counted three times.

Physiology. — "*The posterior longitudinal fascicle, and the manege movement.*" By Dr. L. J. J. Muskens. (Communicated by Prof. C. Winkler).

(Communicated in the meeting of October 26, 1912).

In a series of experiments in cats by means of different needles a lesion was caused in the cerebro-spinal axis, between the postérior commissure and the vestibular nuclei, avoiding the *N*-vestibularis, of which the lesion invariably causes such vehement rolling movements to the side of lesion, that the observation of the manege-movements is impossible. The microscopical control of the lesion and its results was performed after the method of Marchi.